Computer-assisted-proofs of dynamics in a nonconservative NLS

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Introduction

Consider the nonlinear Schrödinger equation

$$iu_t = \triangle u + u^2, \qquad x \in \mathbb{T} \equiv \mathbb{R}/\tfrac{2\pi}{\omega}\mathbb{Z}. \qquad (1)$$

This NLS does not have gauge invariance, $(e^{i\theta}u)^2 \neq e^{i\theta}u^2$ for generic $\theta \in \mathbb{R}$, and it does not admit a natural Hamiltonian structure.
When restricting (1) to constant initial data one obtains the ODE $i\dot{z} = z^2$, whereby 0 is foliated by homoclinic solutions, with the exception of some finite time blowup solutions.

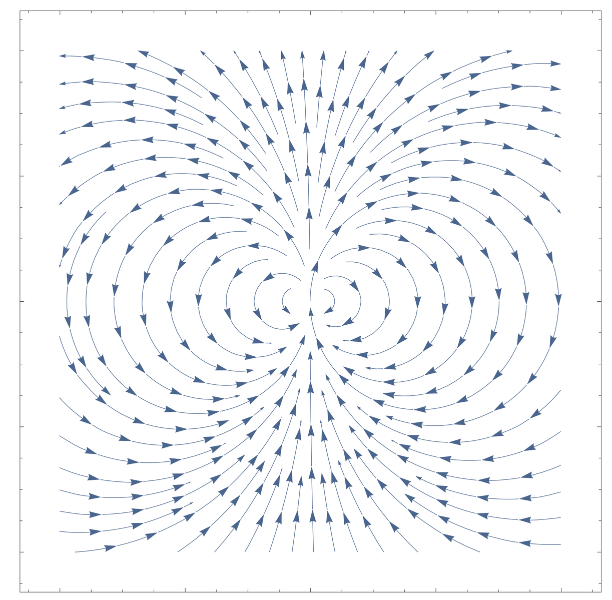

Figure 1: Dynamics of $iz_t = z^2$

Homoclinics & Nonexistence of conserved quantities

From their numerics, Cho et al. conjectured that real initial data to (1) is globally well posed [1]. For close-to-constant real initial data we show this to be the case, and solutions limit to 0 as $t \to \pm\infty$ [2]. Moreover, we prove the following.

Theorem 1 ([2]). *There exists an open set of complex initial data with summable Fourier coefficients whose solutions are homoclinic orbits, limiting to 0 in both forward and backward time.*

Note that if there exists some continuous conserved quantity H, it would necessarily be constant on this open set and equal to $H(0)$. Moreover, if H was analytic then it would have to be globally constant; ie (1) is nonconservative.

Corollary 2 ([2]). *The only analytic functionals conserved under (1) are constant.*

Integrable Dynamics, Periodic Solutions & Blowup

Surprisingly, (1) has an integrable subsystem the space of initial data supported on non-negative Fourier coefficients, akin to the cubic Szegő equation [4].

Theorem 3 ([3]). *The initial data $u_0(x) = \sum_{n\in\mathbb{N}} \phi_n e^{i\omega n x}$, with $\sum_{n\in\mathbb{N}} |\phi_n| < \infty$ has a solution to (1) given by*

$$u(t,x) = \sum_{n\in\mathbb{N}} a_n(t) e^{i\omega n x} \qquad (2)$$

where each function $a_n(t)$ may be solved for explicitly by quadrature.

For an example, we consider monochromatic initial data $u_0(x) = Ae^{ix}$. For this solution, each function $a_n(t)$ is given by $A^n/\omega^{2(n-1)}$, multiplied by a $2\pi/\omega^2$ periodic function.
Given this geometric scaling, one may expect that if the ratio is very small then the solution will converge to a periodic orbit, and if the ratio is very large then the solution will blowup. This is indeed the case.

Theorem 4 ([3]). *Fix $A \in \mathbb{C}$, $\omega > 0$ and initial data $u_0(x) = Ae^{i\omega x}$.*

- *If $A/\omega^2 \leq 3$ then the solution is periodic with period $2\pi/\omega^2$.*
- *If $A/\omega^2 \geq 6$ then the solution blows up in finite time in the L^2 norm.*

The lower value of 3 was obtained by computer assisted proof, and the upper value of 6 was obtained with pen-and-paper. Using non-validated numerics we estimate that the critical dividing line between periodic orbits and blowup is approximately $A/\omega^2 \approx 3.37$.

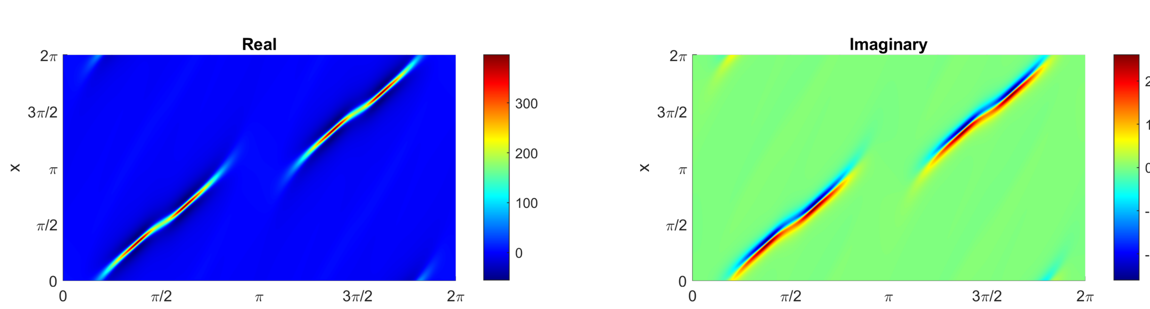

Figure 2: A periodic solution with initial data $u_0(x) = 3e^{ix}$.

Equilibria & Heteroclinic Orbits

By way of computer assisted proofs, we are able to demonstrate existence of nontrivial equilibria, and heteroclinic orbits between these nontrivial equilibria and 0.

Theorem 5 ([2]). *There exist at least two nontrivial equilibria to (1), each of whose linearization has at least one unstable eigenvalue.*

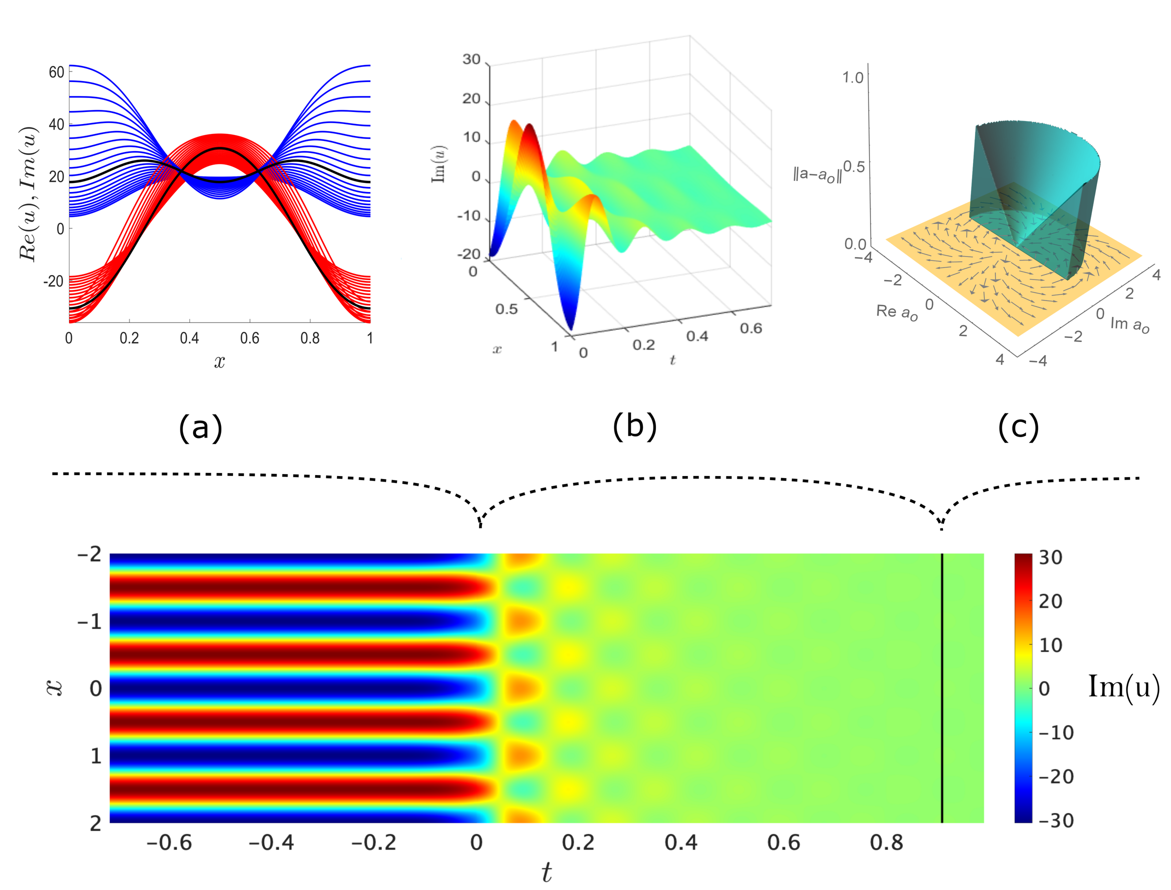

Figure 3: Validated heteroclinic orbit between equilibria

Theorem 6 ([2]). *For each equilibrium $\tilde{u}$ in Theorem 5, there exists a heteroclinic orbit u_a traveling from $\tilde{u}$ to 0, and a heteroclinic orbit u_b traveling from 0 to $\tilde{u}$.*

The heteroclinic u_a is proved using validated numerics in three steps.

(a) Construct a high order approximation of the unstable manifold using the *parameterization method* [5–7].
(b) Use our validated integrator adapted from [8] to propagate these solutions forward in time.
(c) Integrate until the trajectory enters an explicit trapping region (an open set) of solutions which converge to 0.

The heteroclinic u_b follows from the time reversal symmetry of conjugate solutions. In [9] we systematically study the long term behavior of trajectories in the unstable set of an equilibrium.

References

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