Smooth Self-Similar Solutions to the Compressible Euler Equations

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Main Results

- Numerical construction of SMOOTH self-similar solutions
- Numerical contruction of SMOOTH linear perturbations
- Numerical evidence that the singularity formation is UNSTABLE

① Setup

Isentropic Compressible Euler Equations

Density: ρ
Velocity: u
Pressure: p
Dimension: $d \geq 2$
Heat capacity ratio: $\gamma > 1$
Time: t
Spatial coordinate: y

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0 & \text{(Mass Conservation)} \\ \rho \partial_t u + \rho u \cdot \nabla u + \nabla p = 0 & \text{(Momentum Conservation)} \\ p = \frac{\gamma - 1}{\gamma} \rho^\gamma & \text{(Polytropic EoS)} \\ \rho(t, y) > 0 & \text{(No vacuum regions)} \end{cases}$$

Self-Similar Solutions

Implosion of an Isotropic Gas

$$\begin{cases} \rho(t,y) = \left(\frac{1}{T-t}\right)^{\frac{2}{\gamma-1}\left(\frac{r-1}{r}\right)} \hat{\rho}\left(\frac{|y|}{r(T-t)^{1/r}}\right) \\ u(t,y) = \left(\frac{1}{T-t}\right)^{\left(\frac{r-1}{r}\right)} \hat{u}\left(\frac{|y|}{r(T-t)^{1/r}}\right) \end{cases}$$

Blow-up speed: $r > 1$
Blow-up time: T
Radial coordinate: $|y|$

These solutions form a singularity in finite time: density and velocity go to infinity at the origin at time T

Self-Similar Coordinates (τ, Z)

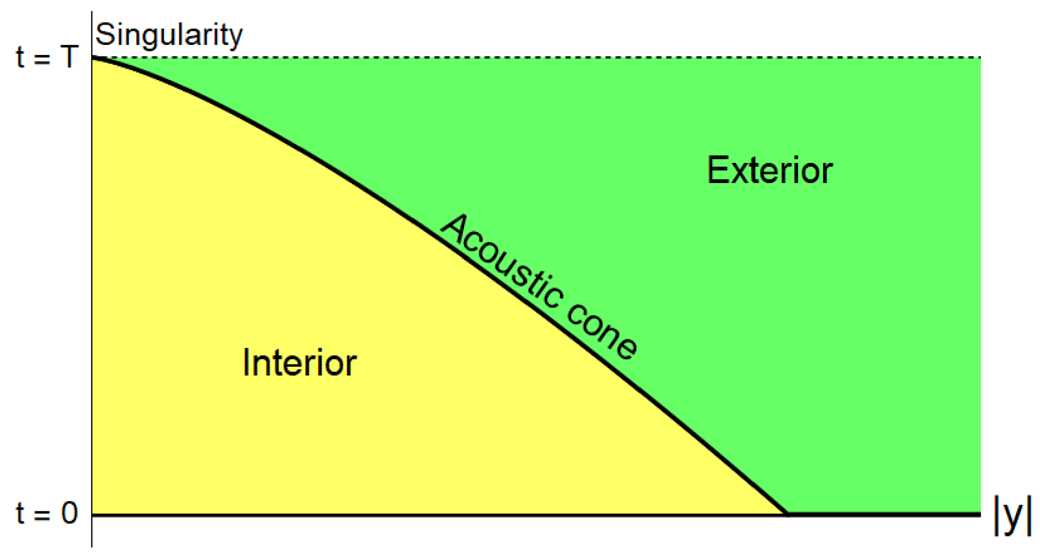
$$Z = \frac{|y|}{r(T-t)^{\frac{1}{r}}} \qquad (T-t) = Te^{-r\tau}$$

- In these coordinates self-similar solutions are static.
- The blow-up time T corresponds with $\tau = \infty$

② Acoustic Cone

Space-Time Diagram of the Implosion Process

The singularity has a backward acoustic cone (similar to the light cone in Special Relativity)



Self-similar solutions extend from the origin ($|y| = 0$) to infitnity ($|y| = \infty$) crossing the acoustic cone of the singularity. When they cross the acoustic cone they lose regularity (some derivatives are not continuous).

Self-Similar Solutions are GENERICALLY Non-Smooth on the cone

③ Stability

Linar Modes

$$\hat{\rho}(\tau, Z) = \hat{\rho}_0(Z) + \epsilon\ \alpha(Z) e^{\Omega \tau} + \mathcal{O}(\epsilon^2) \qquad \Omega \in \mathbb{R}$$
$$\hat{u}(\tau, Z) = \hat{u}_0(Z) + \epsilon\ \beta(Z) e^{\Omega \tau} + \mathcal{O}(\epsilon^2) \qquad |\epsilon| \ll 1$$

Stable Mode $(\Omega < 0)$ Unstable Mode $(\Omega > 0)$

Linear Modes are also GENERICALLY Non-Smooth on the acoustic cone

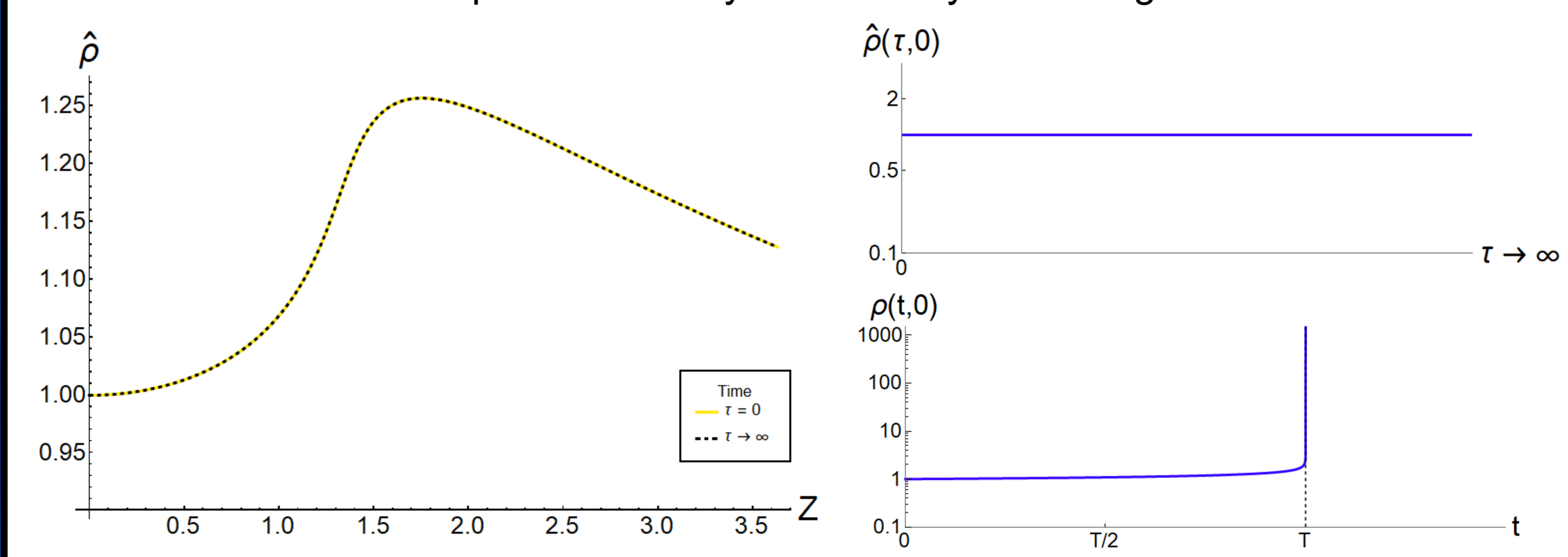
Smooth Self-Similar Solutions & Smooth Perturbations

A semi-numerical exploration of the space of parameters shows that:

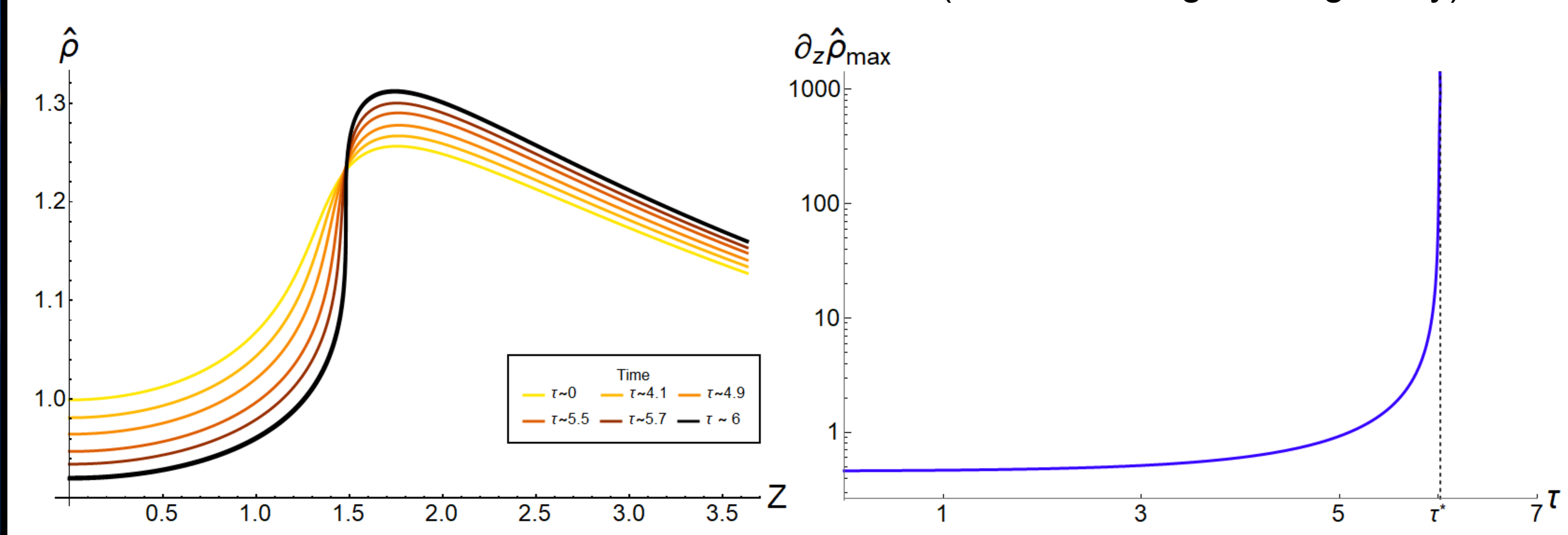
- For fine-tuned values of (d γ r) self-similar solutions are SMOOTH
- For fine-tuned values of Ω linear perturbations are SMOOTH
- Smooth solutions are linearly unstable $(\Omega > 0)$ under smooth perturbations

④ Unstable Singularity

Original Singularity: (no perturbations)
Blow-up of the density and velocity at the origin in finite time



New Singularity: (small perturbations)
Shock formation in finete time $t < T$, finite τ, (before the original singularity)



Our self-similar solutions model a gas that implodes isotropically to the center of symmetry. In finite time, t=T, the concentration process makes that the density and velocity go to infinity. However, small perturbations of these self-similar solutions produce the formation of a shock (density and velocity remain finite but their derivatives go to infinity) before the blow-up at the origin. The structure of the new singularity is different to the original one; for this reason the implosion process associated with smooth self-similar solutions is unstable.

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