Computational complexity, Newton polytopes, and Schubert polynomials

Anshul Adve *University of California, Los Angeles*, **Colleen Robichaux**† *University of Illinois at Urbana-Champaign*, **Alexander Yong** *University of Illinois at Urbana-Champaign*

Introduction

Schubert polynomials form a basis of all polynomials and appear in the study of cohomology rings of flag manifolds. The nonvanishing problem asks if a coefficient of a Schubert polynomial is nonzero. We give a tableau criterion for nonvanishing, from which we deduce the first polynomial time algorithm. These results are obtained from new characterizations of the *Schubitope*, a generalization of the permutahedron defined for any subset of the $n \times n$ grid.

Nonvanishing Problem

In algebraic combinatorics we often study polynomial families:

$$F_\diamond = \sum_\alpha c_{\alpha,\diamond} x^\alpha = \sum_{s \in S} \mathsf{wt}(s) \in \mathbb{Z}[x_1, \ldots, x_n].$$

Example 1 *With $\diamond = \lambda$, use $F_\lambda = s_\lambda$, the Schur polynomial, and $S = \mathsf{SSYT}(\lambda)$. For instance, $s_{(2,1)} = x_1^2 x_2 + x_1 x_2^2$ since*

$$\mathsf{SSYT}((2,1)) = \left\{ \begin{array}{cc} 1 & 1 \\ 2 & \end{array}, \begin{array}{cc} 1 & 2 \\ 2 & \end{array} \right\}.$$

In this framework, we can discuss the complexity of the **nonvanishing problem**:

Problem 2 *What is the complexity of deciding $c_{\alpha,\diamondsuit} \neq 0$, as measured in the input size of α and $\diamondsuit$?*

In our cases of interest, $c_{\alpha,\diamondsuit} \in \mathbb{Z}_{\geq 0}$ has **combinatorial positivity**, which implies $\mathsf{nonvanishing}(F_\diamondsuit) \in \mathsf{NP}$.

Newton Polytopes

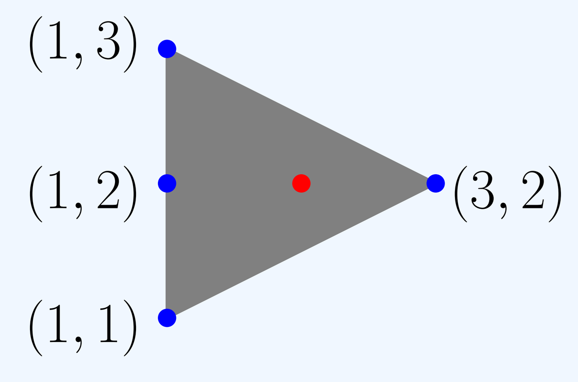
To $F_\diamond$, we can associate its **Newton polytope**:

$$\mathsf{Newton}(F_\diamond) = \mathrm{conv}\{\alpha : c_{\alpha,\diamond} \neq 0\} \subseteq \mathbb{R}^n.$$

C. Monical-N. Tokcan-A. Yong '19 defined that $F_\diamond$ has **saturated Newton polytope** (SNP) if

$$\beta \in \mathsf{Newton}(F_\diamond) \iff c_{\beta,\diamond} \neq 0.$$

Example 3 *Let $f = x_1 x_2^3 + x_1^3 x_2^2 + x_1 x_2^2 + x_1 x_2$. Then f does not have SNP since $x_1^2 x_2^2$ does not appear in the monomial expansion of f.*



SNP combined with a polynomial-size halfspace description of $\mathsf{Newton}(F_\diamond)$ implies $\mathsf{nonvanishing}(F_\diamond) \in \mathsf{coNP}$.

Schubert Polynomials

Schubert polynomials form a $\mathbb{Z}$-basis of $\mathbb{Z}[x_1, x_2, x_3, \ldots]$. They were introduced by A. Lascoux–M.-P. Schützenberger to study the cohomology ring of the flag manifold.

For $w_0 = n\ n-1\ \cdots 2\ 1 \in S_n$,

$$\mathfrak{S}_{w_0}(x_1, \ldots, x_n) := x_1^{n-1} x_2^{n-2} \cdots x_{n-1}.$$

For $w \neq w_0$, apply Newton's divided difference operator

$$\partial_i f = \frac{f - f^{s_i}}{x_i - x_{i+1}}$$

recursively using weak Bruhat order to define $\mathfrak{S}_w$. To each $w \in S_\infty$ there is a unique **code**,

$$\mathsf{code}(w) = (c_1, c_2, \ldots, c_L) \in \mathbb{Z}_{\geq 0}^L,$$

where c_i counts the number of boxes in the i-th row of the **Rothe diagram** $D(w)$ of w.

C. Monical-N. Tokcan-A. Yong defined the **Schubitope** $\mathcal{S}_D$, a polytope defined for $D \subseteq [n]^2$, and conjectured the following:

Theorem 4 (A. Fink-K. Mészáros-A. St. Dizier, '18)
$\mathcal{S}_{D(w)} = \mathsf{Newton}(\mathfrak{S}_w)$.

Let Schubert be $\mathsf{nonvanishing}(\mathfrak{S}_w)$. The $\mathtt{INPUT}$ is $\mathsf{code}(w) = (c_1, \ldots c_L)$ with $c_L > 0$ and $\alpha \in \mathbb{Z}_{\geq 0}^L$.

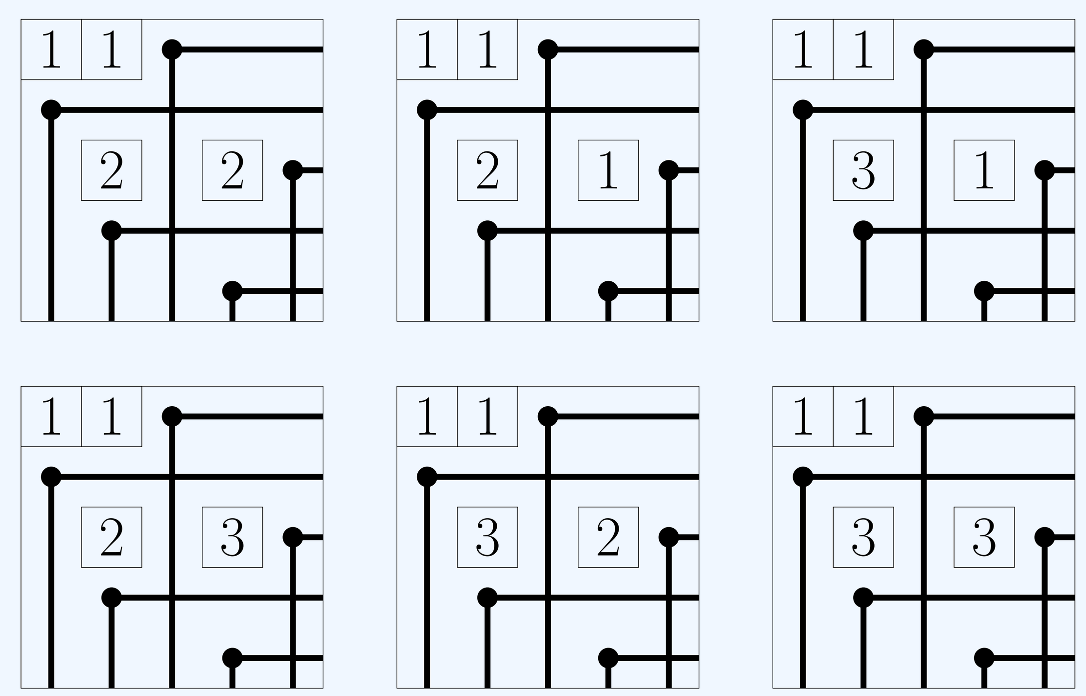
Theorem 5 (A. Adve-C. Robichaux-A.Yong, '21)
Schubert $\in$ P.

Let $\mathsf{Tab}(w, \alpha)$ be the fillings of $D(w)$ with α_k many k's, where entries in each column are distinct, and any entry in row i is $\leq i$. We prove Theorem 5 using Theorem 4 and the following:

Theorem 6 (A. Adve-C. Robichaux-A.Yong, '21)
$c_{\alpha,w} \neq 0$ if and only if $\mathsf{Tab}(D(w), \alpha) \neq \emptyset$.

In general $\#\mathsf{Tab}(D(w), \alpha) \geq c_{\alpha,w}$.

Example 7 *For $w = 31524$, $\bigcup_\alpha \mathsf{Tab}(D(w), \alpha)$:*



Hence, for instance, $c_{(2,1,1),w} > 0$ but $c_{(4),w} = 0$.

Our results give a polynomial time algorithm to check if a lattice point is in $\mathcal{S}_D$. This more general result gives a polynomial time algorithm for any SNP polynomial family whose Newton polytopes are Schubitopes.

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