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Ideal Preserving Operations for Reaction Networks

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Introduction

Chemical reaction networks model the change in the concentrations of a set of chemical species within a system, e.g., inside a cell. Under the assumption of mass-action kinetics, each chemical reaction network gives rise to a differential system of equations governing the change in species concentrations. Monomial terms appearing in the ideal generated by the differential equations tell us some behavior of the dynamics at steady state. This lead us to the question: when do two networks have the same ideal? Are these ideals monomial? We will introduce network operations which preserve the network associated ideals.

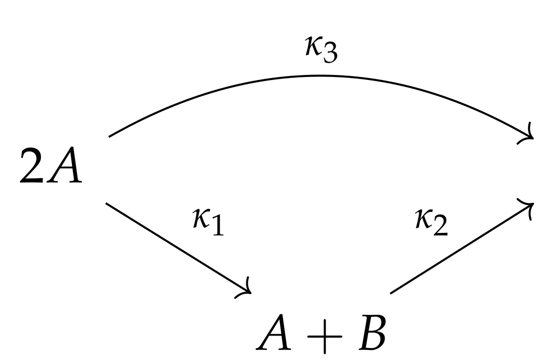
Background and Notation

A **chemical reaction network** (CRN) is a triple $N = (\mathcal{S}, \mathcal{C}, \mathcal{R})$ where $\mathcal{S}$ is a finite set of chemical species, $\mathcal{C} \subset \mathbb{Z}^{\mathcal{S}}_{\geq 0}$ is a finite set of chemical complexes, and $\mathcal{R}$, a set of reactions, is a relation $\rightarrow$ on $\mathcal{C}$. Each complex $y_i = \sum_{s \in \mathcal{S}} \gamma_{si} s \in \mathcal{C}$ is associated to a monomial $x^y = \prod_{s \in \mathcal{S}} x_s^{\gamma_{si}}$ where x_s is the concentration of species s. Under the assumption of mass action kinetics, we obtain a system of differential equations:

$$\dot{x}_s = \frac{dx_s}{dt} = \sum_{y_i \rightarrow y_i' \in \mathcal{R}} \kappa_i (\gamma_{si'} - \gamma_{si}) x^{y_i},$$

where κ_i is the rate constant for $y_i \rightarrow y_i'$. The **steady state ideal** is $\mathcal{I}(N) = \langle \dot{x}_s : s \in \mathcal{S} \rangle$. We mostly restrict to CRNs such that $\gamma_{si} \in \{0, 1\}$ and call such networks, 0,1-networks.

Example: Consider the following CRN with three complexes :



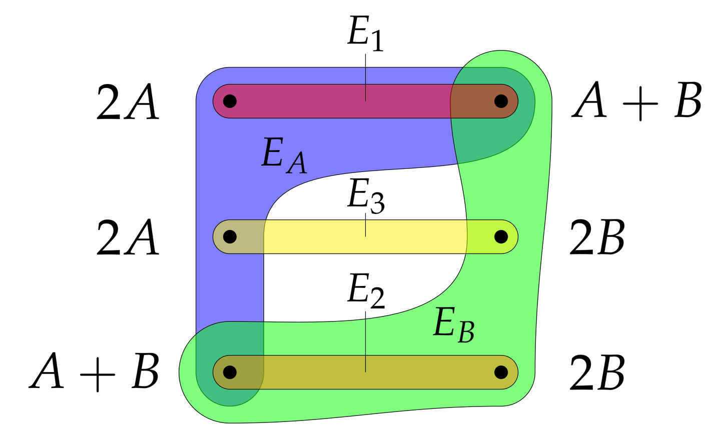
$$\dot{x}_A = -\kappa_1 x_A^2 - \kappa_2 x_A x_B - 2\kappa_3 x_A^2$$

$$\dot{x}_B = \kappa_1 x_A^2 + \kappa_2 x_A x_B + 2\kappa_3 x_A^2$$

Combinatorics of CRNs

Definition: Given a CRN, $N = (\mathcal{S}, \mathcal{C}, \mathcal{R})$, let $u_i = y_i$ and $v_i = y_i'$ for each $y_i \rightarrow y_i' \in \mathcal{R}$. The **associated hypergraph** is $\mathcal{H}(N) = (V, E)$ where $V = \{u_i, v_i : y_i \rightarrow y_i' \in \mathcal{R}\}$ and E is a set containing the hyperedges $E_i = \{u_i, v_i\}$ whenever $y_i \rightarrow y_i' \in \mathcal{R}$ and the hyperedges $E_s = \{u \in V : s \in \text{supp}(u)\}$ for each $s \in \mathcal{S}$.

Example: Associated hypergraph of previous example is below. The hyperedge $E_1 = \{2A, A+B\}$ is red and corresponds to $2A \rightarrow A+B$ etc.



Definition: Given a hypergraph $\mathcal{H}(N) = (V, E)$, a **bi-colored edge set** is a multiset $\mathcal{E}$ with underlying set E where we color the edges two colors and write $\mathcal{E} = \mathcal{E}_r \cup \mathcal{E}_b$. A vertex $v \in V$ is said be **almost balanced** with respect to $\mathcal{E}$ if $\deg_{\mathcal{E}_r}(v) \neq \deg_{\mathcal{E}_b}(v)$ and $\deg_{\mathcal{E}_r}(u) = \deg_{\mathcal{E}_b}(u)$ for all $u \in \mathcal{H}(V) \setminus \{v\}$.

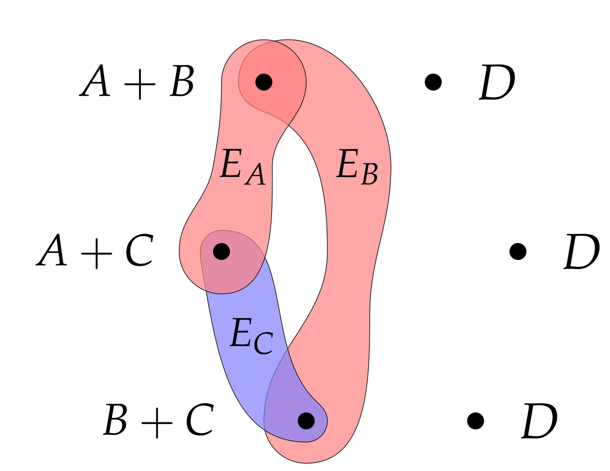
Lemma: Let $N = (\mathcal{S}, \mathcal{C}, \mathcal{R})$ be an 0,1-network. If $y_i \in \mathcal{C}$ is almost balanced with respect to $\mathcal{E} = \mathcal{E}_r \cup \mathcal{E}_b$, then $\kappa_i c^{y_i} = \sum_{E_s \in \mathcal{E}_r} \dot{x}_s - \sum_{E_s \in \mathcal{E}_b} \dot{x}_s$.

Example: Consider the following 0,1-network with its associated hypergraph (only edges in bi-colored edge set are drawn).

$$A + B \xrightarrow{\kappa_1} D$$

$$A + C \xrightarrow{\kappa_2} D$$

$$B + C \xrightarrow{\kappa_3} D$$



We see that $A+B$ is almost balanced with $\mathcal{E}_r = \{E_A, E_B\}$ and $\mathcal{E}_b = \{E_C\}$. Further, observe that:

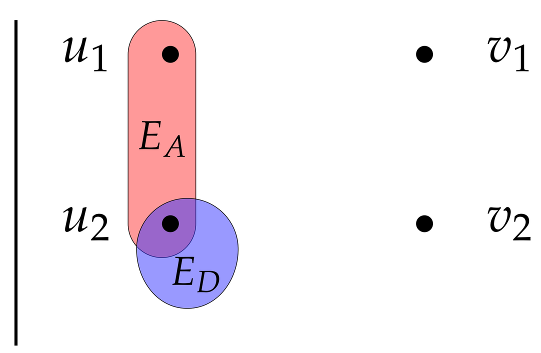
$$\begin{aligned}\sum_{E_s \in \mathcal{E}_b} \dot{x}_s - \sum_{E_s \in \mathcal{E}_r} \dot{x}_s &= \dot{x}_C - \dot{x}_A - \dot{x}_B \\ &= (-\kappa_2 x_A x_C - \kappa_3 x_B x_C) \\ &\quad - (-\kappa_1 x_A x_B - \kappa_2 x_A x_C) \\ &\quad - (-\kappa_1 x_A x_B - \kappa_3 x_B x_C) \\ &= 2\kappa_1 x_A x_B.\end{aligned}$$

Network Operations

Theorem 1: Let N be a 0,1-network and suppose $u_i \in V$ is almost balanced with respect to the two-colored multiset $\mathcal{E} = \mathcal{E}_r \sqcup \mathcal{E}_b$. Suppose s_k is a species such that $s_k \notin \text{supp}(y_i')$, and let $N' = (N \setminus \{y_i \rightarrow y_i'\}) \cup \{y_i \rightarrow y_i' + s_k\}$. If $E_{s_k} \notin \mathcal{E}$, then $\mathcal{I}(N') = \mathcal{I}(N)$.

$$A \xrightarrow{\kappa_1} C$$

$$A + D \xrightarrow{\kappa_2} B$$

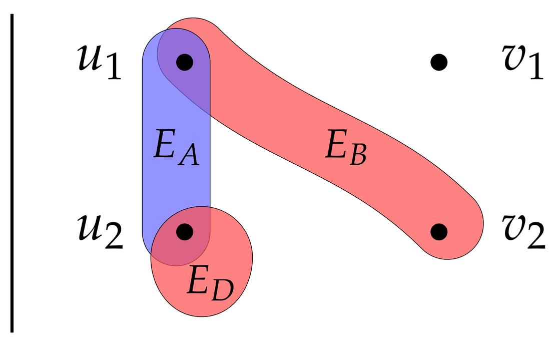


Theorem 2: Let N be a 0,1-network. Suppose there are distinct reactions $y_i \rightarrow y_i'$ and $y_j \rightarrow y_j'$ such that $x^{y_j} \mid x^{y_i}$ and u_j is almost balanced with respect to some two-colored edgeset $\mathcal{E} = \mathcal{E}_r \sqcup \mathcal{E}_b$. Let s_k be a species such that $s_k \notin \text{supp}(y_i)$, and let $N' = (N \setminus \{y_i \rightarrow y_i'\}) \cup \{y_i + s_k \rightarrow y_i'\}$. If $E_{s_k} \notin \mathcal{E}$, then $\mathcal{I}(N') = \mathcal{I}(N)$.

Theorem 3: Let N be a 0,1-network. Suppose there is some vertex $v_i \in V$ such that v_i is almost balanced with respect to the two-colored multiset $\mathcal{E} = \mathcal{E}_r \sqcup \mathcal{E}_b$. Let $N' = N \cup \{y_i \rightarrow \varnothing\}$. If $E_i \notin \mathcal{E}$, then $\mathcal{I}(N') = \mathcal{I}(N)$.

$$A \xrightarrow{\kappa_1} C$$

$$A + D \xrightarrow{\kappa_2} B$$



Question: For monomial ideals, are these ideal preserving operations minimal in the sense that any ideal preserving operation is some combination of the above three?

References

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[3] A. Shiu and B. Sturmfels. *Siphons in chemical reaction networks*. Bulletin of mathematical biology 72 6 (2010), 1448-63.