Kanev surfaces as hypersurfaces in toric 3-folds

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General setting:
M: 3-dim. lattice of characters, N: dual lattice of one-parameter subgroups, $T = Hom(M, \mathbb{C}) \cong (\mathbb{C}^*)^3$ the torus,
$\Delta \subset M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ a 3-dimensional lattice polytope given by

$$\Delta = \{x \in M_{\mathbb{R}} | \langle x, \nu_i \rangle \geq r_i, \, i = 1, ..., s\}$$

with $\nu_1, ..., \nu_s \in \Sigma_\Delta[1]$ the inner facet normals, $r_1, ..., r_s \in \mathbb{Z}$ (Σ_Δ= normal fan of Δ).
$l(\Delta) := |\Delta \cap M|$ the number of lattice points in Δ and $l^*(\Delta) := |Int(\Delta) \cap M|$ the number of interior lattice points.
$\mathbb{P}_\Delta$ denotes the projective toric variety to the polytope Δ given by the construction with the normal fan.
$L(\Delta)_{reg}$: The nondegenerate Laurent polynomials with Newton polytope Δ, $f \in L(\Delta)_{reg}$. $Z_f := \{f = 0\} \subset T$, the associated hypersurface in the torus, Z_Δ the closure of Z_f in the toric variety $\mathbb{P}_\Delta$.
For a birational toric (rational) map $\mathbb{P}_{\tilde{\Delta}} \to \mathbb{P}_\Delta$ we write $Z_{\tilde{\Delta}}$ for the proper transform of the hypersurface Z_Δ in $\mathbb{P}_{\tilde{\Delta}}$.

The Fine interior and canonical closure (compare [Bat20]):
For $\nu \in N$ let

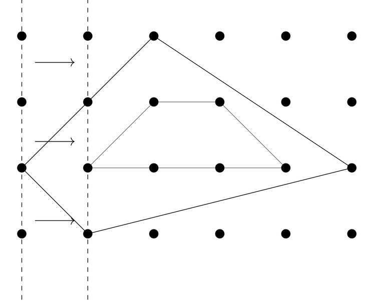
$$ord_\Delta(\nu) := \min_{m \in \Delta \cap M} \langle m, \nu \rangle.$$

Then define the Fine interior

$$\Delta^{FI} := \{x \in M_{\mathbb{R}} | \langle x, \nu \rangle \geq ord_\Delta(\nu) + 1, \, \nu \in N \setminus \{0\}\}.$$

See the picture for a 2-dim. example. In dimension 2 the Fine interior always equals the convex span of the interior lattice points. The support $S_{\Delta^{FI}}(\Delta)$ of Δ^{FI} is given by the lattice points $\nu \in N \setminus \{0\}$ with

$$ord_{\Delta^{FI}}(\nu) = ord_\Delta(\nu) + 1.$$



Picture Δ and Δ^{FI}

The polytope

$$C(\Delta) := \{x \in M_{\mathbb{R}} \,|\, \langle x, \nu \rangle \geq ord_\Delta(\nu) \;\; \forall \, \nu \in S_{\Delta^{FI}}(\Delta)\}$$

is called the canonical closure of Δ. We obtain $\Delta^{FI} \subset \Delta \subset C(\Delta)$.
Let $\tilde{\Delta} := C(\Delta) + \Delta^{FI}$ be the Minkoswki sum. Then there is a diagram of toric morphisms

$$\mathbb{P}_{C(\Delta)} \xleftarrow{\rho_1} \mathbb{P}_{\tilde{\Delta}} \xrightarrow{\rho_2} \mathbb{P}_{\Delta^{FI}} \qquad (1)$$

due to the Minkowski sum $\tilde{\Delta} = C(\Delta) + \Delta^{FI}$. Further there exists a crepant morphism $\mathbb{P}_{\Delta^{min}} \to \mathbb{P}_{\tilde{\Delta}}$ such that $\mathbb{P}_{\Delta^{min}}$ has at most terminal singularities, i.e. $Z_{\Delta^{min}}$ gets a smooth minimal model.

Results to canonical Fano 3-polytopes (compare [Sch18]):
We restrict ourselve to Δ a canonical Fano 3-polytope with dim $\Delta^{FI} = 3$. There are 49 such polytopes Δ and by their classification, all facets of Δ have distance 1 to the origin (= unique interior lattice point) except from one facet, which we call Δ_{can} which has distance 2 to the origin. Further in 46 examples we have $l^*(\Delta_{can}) = 2$ and in 3 examples $l^*(\Delta_{can}) = 3$.
It follows from these results and ([DK87]) that $p_g(Z_{\tilde{\Delta}}) = l^*(\Delta) = 1$ and $p_g(D_{can|Z_{\tilde{\Delta}}}) \in \{2, 3\}$ with D_{can} the toric divisor to the facet Δ_{can}.

Facts (compare [Bat20]):
$\mathbb{P}_{\tilde{\Delta}}$ is $\mathbb{Q}$-Gorenstein and has at most canonical singularities. The adjoint divisor

$$K_{\mathbb{P}_{\tilde{\Delta}}} + Z_{\tilde{\Delta}}$$

is $\mathbb{Q}$-Cartier and nef. Thus the hypersurface $Z_{\tilde{\Delta}}$ has at most $\mathbb{Q}$-Gorenstein canonical singularities and by the adjunction formula the canonical divisor

$$K_{Z_{\tilde{\Delta}}} = (K_{\mathbb{P}_{\tilde{\Delta}}} + Z_{\tilde{\Delta}})_{|Z_{\tilde{\Delta}}} \qquad (2)$$

is nef. The facet presentation of Δ with Δ_{can} of distance 2 and the others facets of distance 1 to the origin yields

$$Z_\Delta \sim_{lin} \sum_{i=0}^{r} D_i + 2D_{can}$$

with the torus invariant divsiors $D_0, ..., D_r$ to the remaining facets. By abuse of notation we also write $Z_{\tilde{\Delta}} \sim_{lin} \sum_{i=0}^{r} D_i + 2D_{can}$ with $D_0, ..., D_r, D_{can}$ the proper transforms in $\mathbb{P}_{\tilde{\Delta}}$ of the original $D_0, ..., D_r, D_{can}$. We observed that in all 49 examples ρ_1 is an iso. in codimension 1, thus we have $K_{\mathbb{P}_{\tilde{\Delta}}} = -\sum_{i=0}^{r} D_i - D_{can}$ and we get with (2):

$$K_{Z_{\tilde{\Delta}}} \sim_{lin} D_{can|Z_{\tilde{\Delta}}}.$$

Thus $p_g(K_{Z_{\tilde{\Delta}}}) \in \{2, 3\}$ and by the adjunction formula equivalently $K^2_{Z_{\tilde{\Delta}}} = p_g(K_{Z_{\tilde{\Delta}}}) - 1 \in \{1, 2\}$. Everything is very similar to reflexive polytopes, where we have

$$Z_{\tilde{\Delta}} \sim_{lin} \sum_{i=0}^{r} D_i, \quad K_{Z_{\tilde{\Delta}}} = \mathcal{O}_{Z_{\tilde{\Delta}}}$$

and we get a K3-surface.
The singularities of $Z_{\tilde{\Delta}}$ could be resolved by a crepant morphism, therefore $Z_{\tilde{\Delta}}$ lies between a minimal and a canonical model of a surface Y (of general type) with

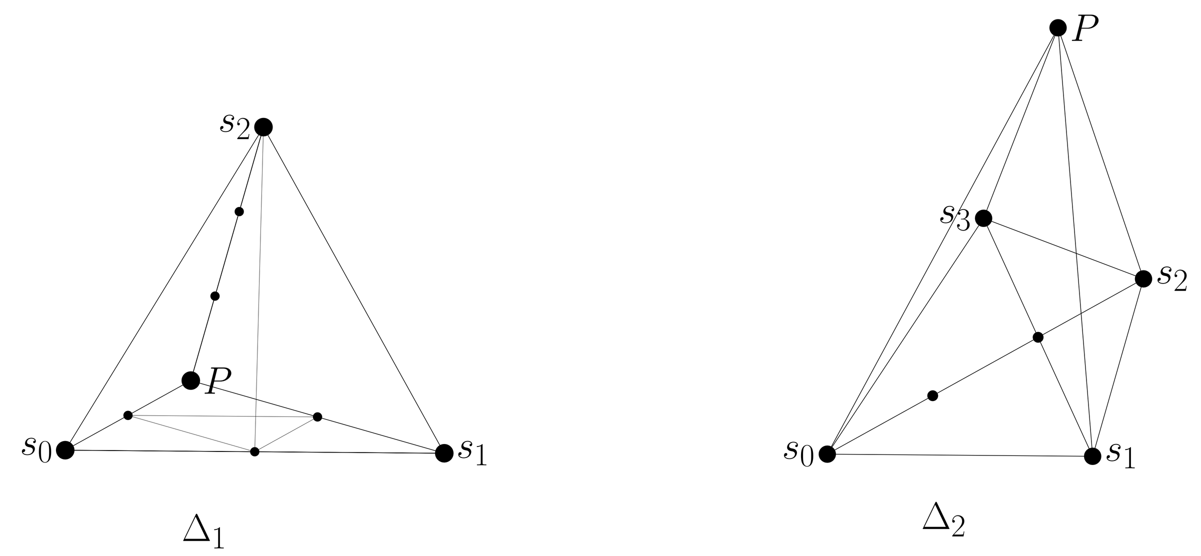
$$p_g(Y) = 1, \quad K_Y^2 \in \{1, 2\}$$

i.e. a Kanev ($K_Y^2 = 1$) or Todorov ($K_Y^2 = 2$) surface. In the following we restrict ourselve to the 46 polytopes yielding Kanev surfaces.

Result 1. *([Gie21a])*
The birational transform $Z_{\Delta^{FI}}$ of Z_Δ gets a canonical model of a Kanev surface.
Among the 46 examples of 3-dim. canonical Fano Newton polytopes yielding Kanev surfaces, there are two maximal polytopes Δ_1 and Δ_2 and for them the two morphisms ρ_1 and ρ_2 are isomorphisms. Thus we get ($i = 1, 2$)

$$\mathbb{P}_{\Delta_i} \cong \mathbb{P}_{\tilde{\Delta}_i} \cong \mathbb{P}_{\Delta_i^{FI}}, \text{ and } Z_{\Delta_i} \cong Z_{\tilde{\Delta}_i} \cong Z_{\Delta_i^{FI}}.$$

Δ_1 is a simplex, Δ_2 has one more vertex (compare picture):



By considering monomials x^m to all lattice points $m \in \Delta \cap M$ and varying the coefficients a_m, such that the Zariski open condition

$$f = \sum_{m \in \Delta \cap M} a_m x^m \in L(\Delta)_{reg}$$

is satisfied, we get a family $Z_{a,\Delta^{FI}}$ of Kanev hypersurfaces in $\mathbb{P}_{\Delta^{FI}}$. All families are gotten from the two maximal families by setting some coefficients a_i to 0.
The two maximal polytopes both have an axis of symmetry. (In $\Delta_1 : \langle P, s_2, \frac{1}{2}(s_0 + s_1) \rangle$, in $\Delta_2 : \langle s_0, s_2, P \rangle$). If we subdivide the two polytopes at this axis we get polytopes all of them yielding the same toric variety $\mathbb{P}(1, 2, 2, 3)$. In fact we have:

Result 2. *([Gie21a])*
The two families of Kanev hypersurfaces Z_{a,Δ_1} and Z_{a,Δ_2} are related by a common degeneration: Let X be a general member of the degeneration of one of these two families. Then X has two components each of which defines a del Pezzo hypersurface of degree 2 with at most canonical singularities in the toric variety $\mathbb{P}(1, 2, 2, 3)$.

Further results:
We have

$$l(\Delta_1) = 18, \quad l(\Delta_2) = 18.$$

and naturally the question arises what the number of moduli for the two maximal families are:

Result 3. *([Gie21a])*
The number of moduli of $Z_{a,\Delta}$ is 12 for $\Delta = \Delta_1^{min}$ and 14 for $\Delta = \Delta_2^{min}$.

Some Kanev surfaces (the so called special Kanev surfaces) first attracted attention as counterexamples to the infinitesimal Torelli theorem. Usui proved that the mixed infinitesimal Torelli theorem holds for them however ([Us883]).

Result 4. *([Gie21b])*
The family $Z_{a,\Delta}$ meets the special Kanev surfaces in an 6-dimensional subfamily for $\Delta = \Delta_1^{min}$, and in an 8-dim. subfamily for $\Delta = \Delta_2^{min}$.
For both $\Delta = \Delta_1^{min}$ and $\Delta = \Delta_2^{min}$ the family $Z_{a,\Delta}$ fails the infinitesimal Torelli theorem but fulfills the mixed infinitesimal Torelli theorem.

References

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