Enumeration of algebraic and tropical singular hypersurfaces

Uriel Sinichkin (Master thesis under the supervision of prof. Eugenii Shustin)

School of Mathematical Sciences, Tel Aviv University

Questions

How many of the $(n+1)(d-1)^n$ degree d singular hypersurfaces in $\mathbb{C}P^n$ are real and have real singularity?
How many of the $\frac{1}{\delta!}((n+1)(d-1)^n)^\delta + O(d^{n(\delta-1)})$ degree d hypersurfaces with δ isolated singularities in $\mathbb{C}P^n$ are real with all singularities also real?

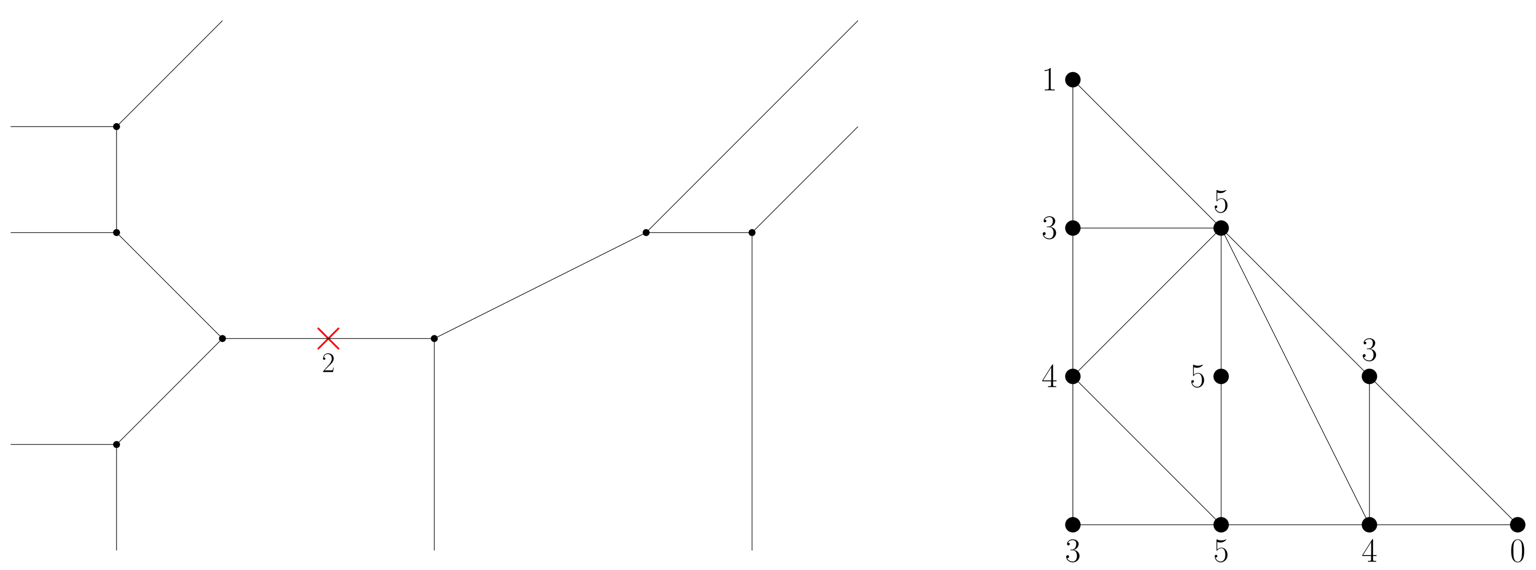

Figure: An example of singular tropical curve (the red cross denotes the singularity) and its dual subdivision.

Theorem: Enumeration of real singular hypersurfaces

For all $n \geq 2$ and $d \gg 1$ there exist a generic real pencil of hypersurfaces of degree d in $\mathbb{P}^n$ that contains at least $\alpha_n d^n + O(d^{n-1})$ singular hypersurfaces, where $(\alpha_n)_{n=1}^{\infty}$ and $(\beta_r))r = 1^{\infty}$ satisfy the recurrence relations

$$\alpha_n = \frac{1}{n}\left(\alpha_{n-1} + \sum_{r=0}^{n-2}(1+\beta_{n-1-r})\alpha_r\right), \ n \geq 2, \quad \alpha_0 = 1, \ \alpha_1 = 2, \tag{1}$$

$$\beta_r = \frac{1}{r}\left(\beta_{r-1} + \sum_{r_0=1}^{r-2}\beta_{r_0}\beta_{r-r_0-1}\right), \ r \geq 1, \quad \beta_1 = 1. \tag{2}$$

Theorem: Enumeration of multinodal hypersurfaces

For all $n \geq 2$, $\delta > 1$ and $d \gg 1$ there exist a configuration $\overline{\boldsymbol{p}}$ of $\binom{n+d}{n} - 1 - \delta$ real points in $\mathbb{P}^n$ s.t. there are at least $\frac{1}{\delta!}(\gamma_n d^n)^\delta + O(d^{n\delta-1})$ real hypersurfaces passing through $\overline{\boldsymbol{p}}$ and having δ real nodes, where γ_n satisfies the recurrence relation

$$\gamma_n = \frac{1}{n}\left(\gamma_{n-1} + \sum_{r=0}^{n-2}(1+\beta_{n-1-r})\gamma_r\right), \ n \geq 2, \quad \gamma_0 = 1, \ \gamma_1 = 1. \tag{4}$$

n	2	3	4	∞
α_n	2	$5/2$	$11/4$	≈ 4.228
γ_n	$3/2$	$5/3$	$23/12$	≈ 2.915

Table: Table with some vales of α_n and γ_n. The last column is the limit as $n \to \infty$.

Tropicalization

We can translate the problem to a combinatorial one by *tropicalizing* it. Algebraic hypersurfaces that satisfy point conditions become tropical hypersurfaces, again satisfying point conditions.

Lemma: tropical singularity

Let $S = V(\bigoplus_{\boldsymbol{v}} a_{\boldsymbol{v}} \odot X^{\boldsymbol{v}})$ be a tropical hypersurface. For $c \in \mathbb{R}$ let $F_c := \{\boldsymbol{v} \in \mathbb{Z}^n_{\geq 0} \mid a_{\boldsymbol{v}} = c\}$. Then, there exists an algebraic hypersurface with singularity at $\boldsymbol{1} := (1, \ldots, 1)$ tropicalizing to S iff the points with highest value of $a_{\boldsymbol{v}}$ are affinely dependent and for every $c \in \mathbb{R}$ there exists an affine combination of F_c with nonzero coefficients that lie in the affine span of $\bigcup_{c'>c} F_{c'}$.

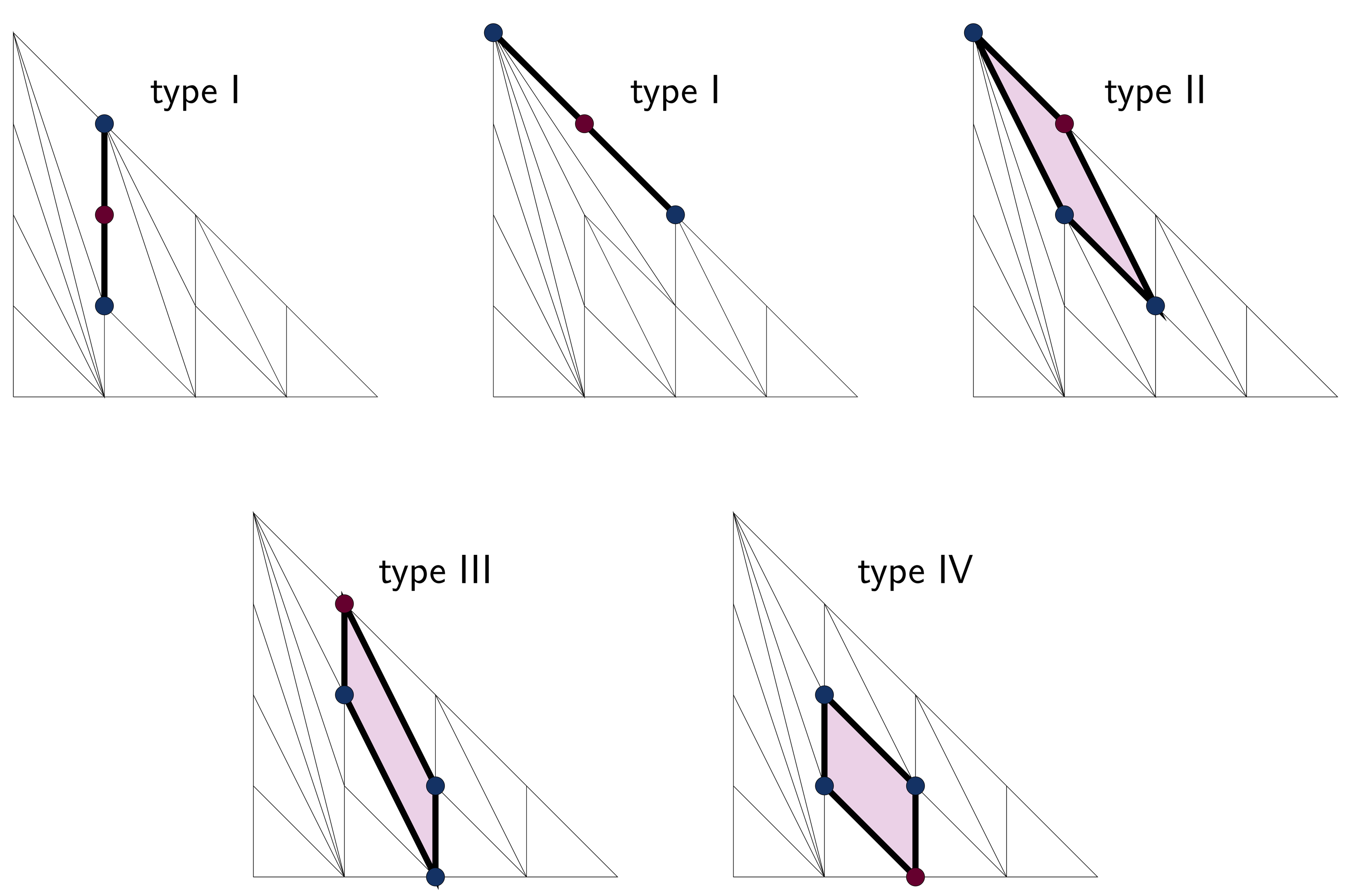

Figure: Some elementary Mikhalkin circuits in dimension $n = 2$ and degree $d = 4$.

Main ideas of the proof

1. We begin by putting the point conditions in the special *Mikhalkin position*.
2. This allow us to classify the possible circuits that can occur in our situation (as the exponents with highest tropical coefficients), see Figure 1.
3. We then show how those circuits can be *glued* (see Figure 3) to form graded circuits which correspond to singular points on a tropical hypersurface passing through the prescribed points.
4. Using Viro's *patchworking* technique we are able, for every singular point $\boldsymbol{\psi} \in S$ were S is a tropical hypersurface, to enumerate all the pairs $\boldsymbol{q} \in \mathcal{S}$ such that $\mathcal{S}$ is a complex algebraic hypersurface tropicalizing to S, and $\boldsymbol{q}$ is a singular point of $\mathcal{S}$ with $\text{trop}(\boldsymbol{q}) = \boldsymbol{\psi}$.
5. By carefully picking the signs of the coordinates of the point conditions we are able to ensure that some of the solutions are in fact real.

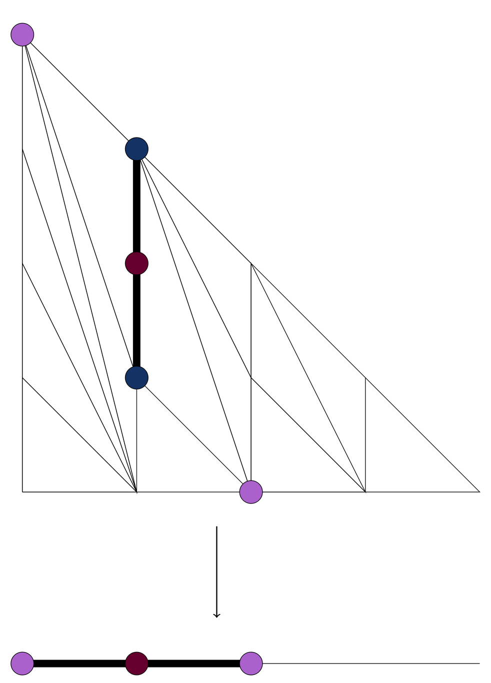

Figure: Gluing of two circuits of type I to a 2 dimensional graded circuit.

Definition: graded circuit

A *Graded circuit* is a sequence of subsets $C_i \subset \mathbb{Z}^n_{\geq 0}$ $(i \in \mathbb{N})$ s.t.

1. C_1 is a circuit.
2. For every $i > 1$, C_i is affinely independent and there exists a **unique** affine combination

$$\sum_{\boldsymbol{x} \in C_i} \alpha_{\boldsymbol{x}} \boldsymbol{x} \in \text{Aff}\Big(\bigcup_{j<i} C_j\Big) \tag{3}$$

with $\alpha_{\boldsymbol{x}} \neq 0$ for all $\boldsymbol{x} \in C_i$.

$n \leq 3$

Graded circuits of dimension $n \leq 3$ that *admit convex function* were classified in [1, 2]. In higher dimensions this is a difficult open problem.

Contact Information

Email: sinichkin@mail.tau.ac.il

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References

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