Maximally Complete Maps on Orientable Surfaces

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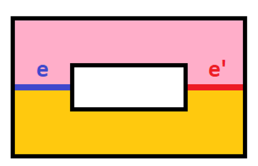
Introduction

- A **map** is an embedding of a connected graph X into a surface[a] S such that each component of $S \setminus X$, called a **face**, is homeomorphic to a disk.
- A face F and a vertex V (edge E) are **incident** if the boundary of F contains V (contains E), and two faces F_1 and F_2 are **incident** if there is some edge that is incident with both F_1 and F_2. Incident faces are also said to be **adjacent**.
- A map is **redundant** if it contains distinct edges incident with the same two faces.

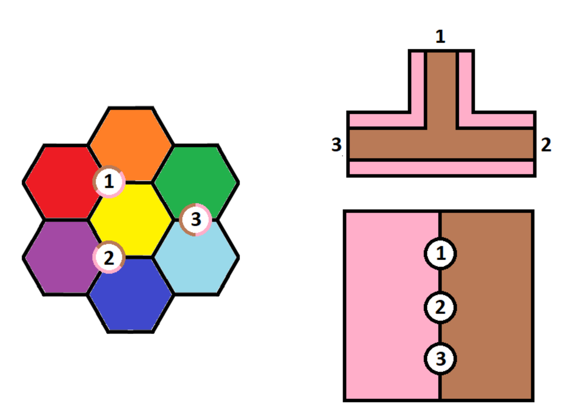


Maximally Complete Maps

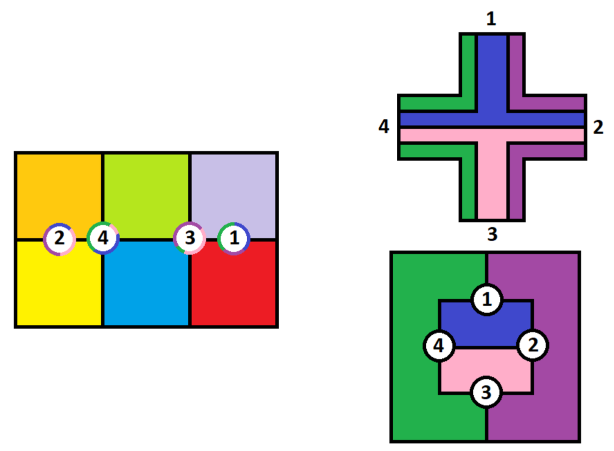
- A map is **complete** if any two faces are adjacent and is **maximally complete** if it is complete and is maximal with respect to the number of faces.
- The number of faces in a maximally complete map on a surface S of genus g is given by the **Heawood number**

$$H(S) = \left\lfloor \frac{7 + \sqrt{1 + 48g}}{2} \right\rfloor$$

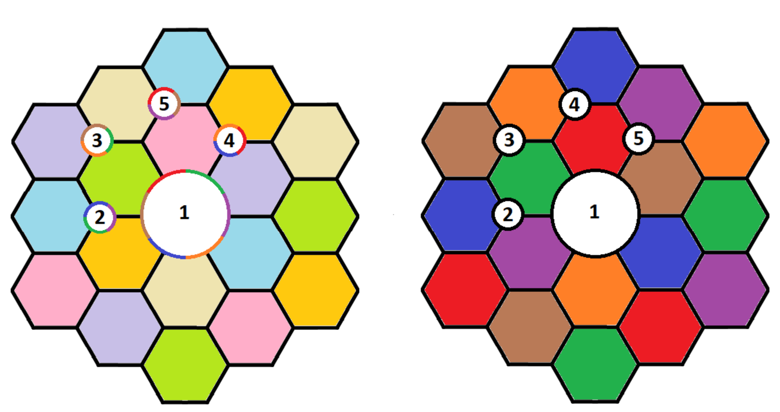
Maximally Complete Maps on Surfaces of Genus 3 through 7



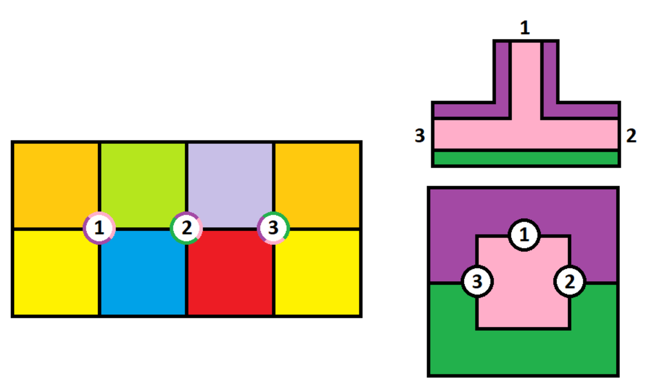
9 Color Map on Genus 3 Surface



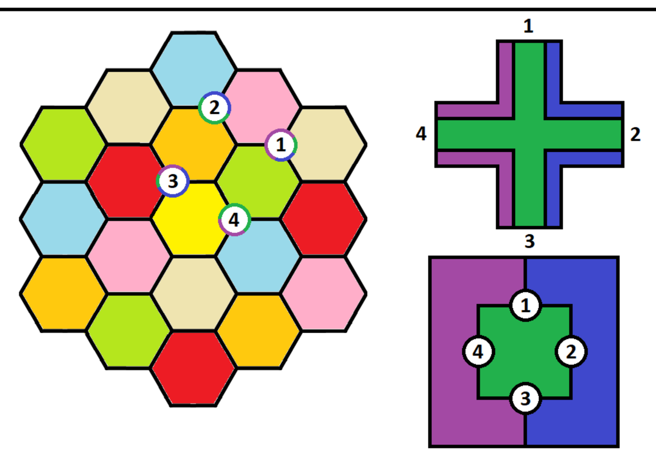
10 Color Map on Genus 4 Surface



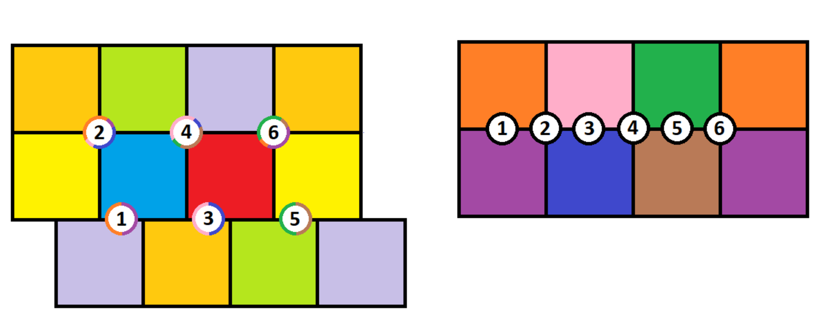
12 Color Map on Genus 6 Surface



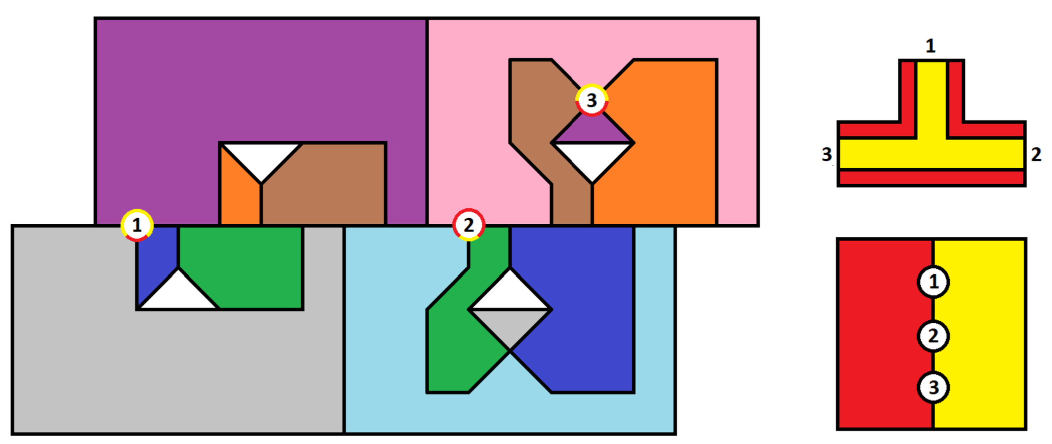
9 Color Map on Genus 3 Surface



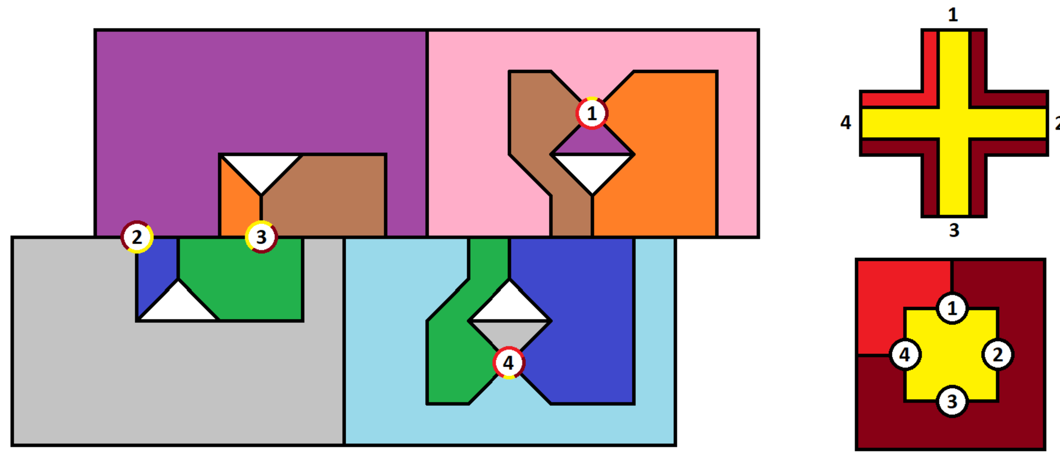
10 Color Map on Genus 4 Surface



12 Color Map on Genus 7 Surface



10 Color Map on Genus 4 Surface



11 Color Map on Genus 5 Surface

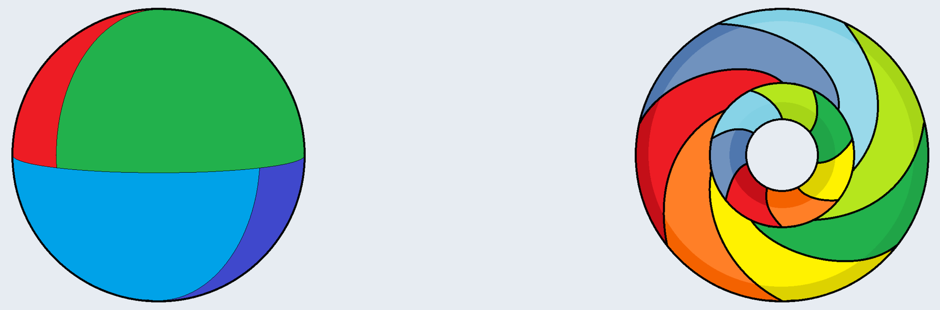
- Each of the maps consists of two complete base maps with numbered boundary circles.
- There are exactly two circles of each number, one black and the other divided into colored arcs.
- Each two circles of the same number are identified so that the arc of the black circle that bounds the face of a given color is identified with the arc of the same color of the colored circle.

Regular Maps

- An **automorphism** of a map is an incidence-preserving permutation of each of the sets of vertices, edges and faces.
- A map is **regular** if every incident vertex-edge-face triple can be transformed into any other by an automorphism of the map.

Main Theorem

The only regular, maximally complete maps are on the sphere and torus.

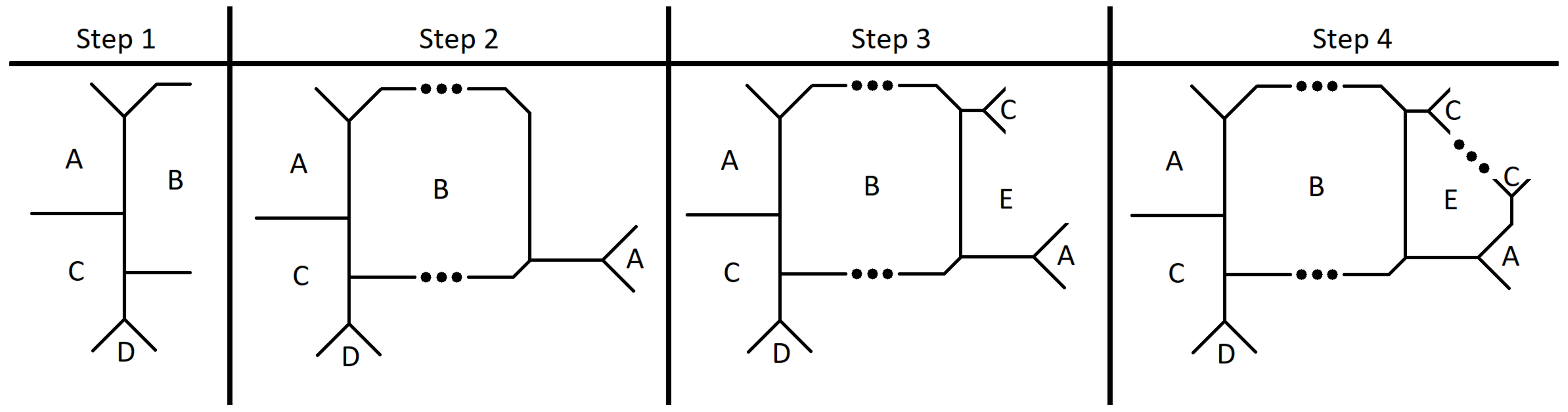


Proof of Main Theorem

We first prove the following proposition:

- **Proposition:** Regular, maximally complete maps are non-redundant and trivalent.

We then show that any regular map with more than 7 faces is redundant and conclude that a maximally complete, regular map, being non-redundant, has at most 7 faces.



The only surfaces S such that $H(S) \leq 7$ are the sphere and the torus; hence these are the only surfaces that admit a regular, maximally complete map.

Further Research

- The **type-n connected sum** of two surfaces S and T is the surface obtained by removing n disjoint open disks from each of S and T and identifying resulting boundary circles in pairs.
- Each maximally complete map we give is the **type-n connected sum** of two complete maps on surfaces of smaller genus.
- **Open Question:** When is it possible to construct a maximally complete map via a **type-n connected sum** of two given complete base maps?

References

[1] Yanbing Gu, Connor Stewart, and Ajmain Yamin. Maximally complete and regular maps.

[2] Marston Conder. Symmetries of maps and abstract polytopes: Lecture 2.

Acknowledgements

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[a] All surfaces are assumed to be closed, connected, and orientable.