COMPETITION AND COOPERATION IN A TWO-SIDED MATCHING MARKET WITH REPLICATION

Journal of Economic Theory 183C (2019) pp. 1030-1056

by

MARILDA SOTOMAYOR

2020, ICERM AT BROWN UNIVERSITY

EPGE- Brazilian School of Economics and Finance

USP/SP-Department of Economics

E-mail: marildas@usp.br URL: www.marildasotomayor.com

REFERENCES

Aumann, R. (1964): “Markets with a continuum of traders”, *Econometrica*, 32, 39-50.

Debreu, G. and H. Scarf (1963): “A limit theorem on the core of an economy”, *International Economic Review*, 4, 235-46

Crawford, V. and E. M. Knoer (1981): “Job matching with heterogeneous firms and workers”, *Econometrica* 49, 437–450.

Demange, G., Gale, D. (1985): “The strategy structure of two-sided matching markets”, *Econometrica* 53, 88–873.

Kelso, A. and Crawford, V. (1982): “Job matching, coalition formation, and gross substitutes”, Econometrica, 50, 1483-1504.

Shapley, L., Shubik, M. (1972): “The assignment game I: The core”, *Int J Game Theory* 1, 111–130.

Sotomayor, M. (1992): “The multiple partners game”, in: *Equilibrium and dynamics: Essays in honor of David Gale*, Majumdar M. (ed), Macmillan Press Ltd, New York.

Sotomayor, M. (1999): "The lattice structure of the set of stable outcomes of the multiple partners assignment game", *Int J Game Theory* 28, 567–583.

_________(2002): "A labor market with heterogeneous firms and workers", *Int J Game Theory* 31, 269-283.

_________(2007): "Connecting the cooperative and competitive structures of the Multiple-partners game", *J Econ Theory* 134, 155–174.

__________(2013): "The time-sharing assignment models, lending new insights to the theory of two-sided matching markets", *working paper.*

__________(2016):"Modeling cooperative decision situations: The deviation function form and the equilibrium concept", *International Journal of Game Theory* 45, 743-768, ISSN 1432-1270.

Winter, E. and M. Wooders (2008): "On large games with bounded essential coalition sizes", *International Journal of Economic Theory* 4, 191–206.

INTRODUCTION

Cooperative games **with a finite number of types of agents and finitely many agents of each type** have been used in the literature by several authors to represent **perfectly competitive markets.** Such models are obtained by replicating an economy finitely many times. Among these authors we can cite Edgeworth (1881), Debreu and Scarf (1963), Winter and Wooders (2008), among others. The common issue of these papers is the relationship between the cooperative and the competitive equilibria of the resulting markets from the replications of an economy, which can be represented in the characteristic function form and has the core as the cooperative solution.

Debreu and Scarf (1963), for example, shows the relationship between the core and the competitive equilibria as an exchange economy grows large. The outstanding application is embodied in the *core equivalence principle*, according to which, in such markets, *the core shrinks to the set of the competitive equilibrium allocations*. The key result behind such equivalence is that *the core has the equal treatment property (ETP).*

Following these ideas, we replicate a continuous two-sided matching market. In a continuous two-sided matching market the players belong to two finite and disjoint sets and interact pairwisely toward reaching an agreement on the monetary gains they will receive for their participation in the partnerships formed.

These markets have particularly been useful to model economies that can be treated cooperatively and competitively in the same environment. (Shapley and Shubik, 1972; Crawford and Knoer (1981), Kelso and Crawford (1982), Demange and Gale, 1985; Sotomayor, 1992, 1999, 2002, 2007, 2020, among others).

Our setting involves to replicate the “Multiple-partners game”, denoted by *M*, to yield a sequence of *sequential games* with infinitely many terms: $S(M) \equiv (M \equiv M^1, M^2, \ldots, M^h, \ldots M^\infty)$.

The first term *M* is the Multiple-partners game. It is the sequential game with one player of each type, called here *simple game*; the final term M^∞ is the sequential game with infinitely many players of each type; the *h*-th term M^h of the sequence is obtained by replicating *h* times the *simple game* *M*.

As it is expected, the replications of the market M create new coalitional interactions, which yield cooperative and competitive game structures, distinct from those game stuctures of M. However, in contrast to the traditional setting, the multiple partners game **cannot be fully represented in the characteristic function form and does not have the core concept as its natural solution concept**. The main implication of these distinctions is that there are situations in our model, in which, as the number of replications increases, **the core does not shrink to the set of competitive equilibria and the corewise-stable allocations does not have the ETP** *(ETP: every agent gets the same payoffs).*

The novelty is that, instead of regarding a replicated market as a static game, we also take into account the multi-stage game structure with which these games are endowed. This new way to play the game leads to new cooperative and competitive solution concepts.

MULTIPLE PARTNERS ASSIGNMENT GAME: $M=(B,Q,v,r,s)$

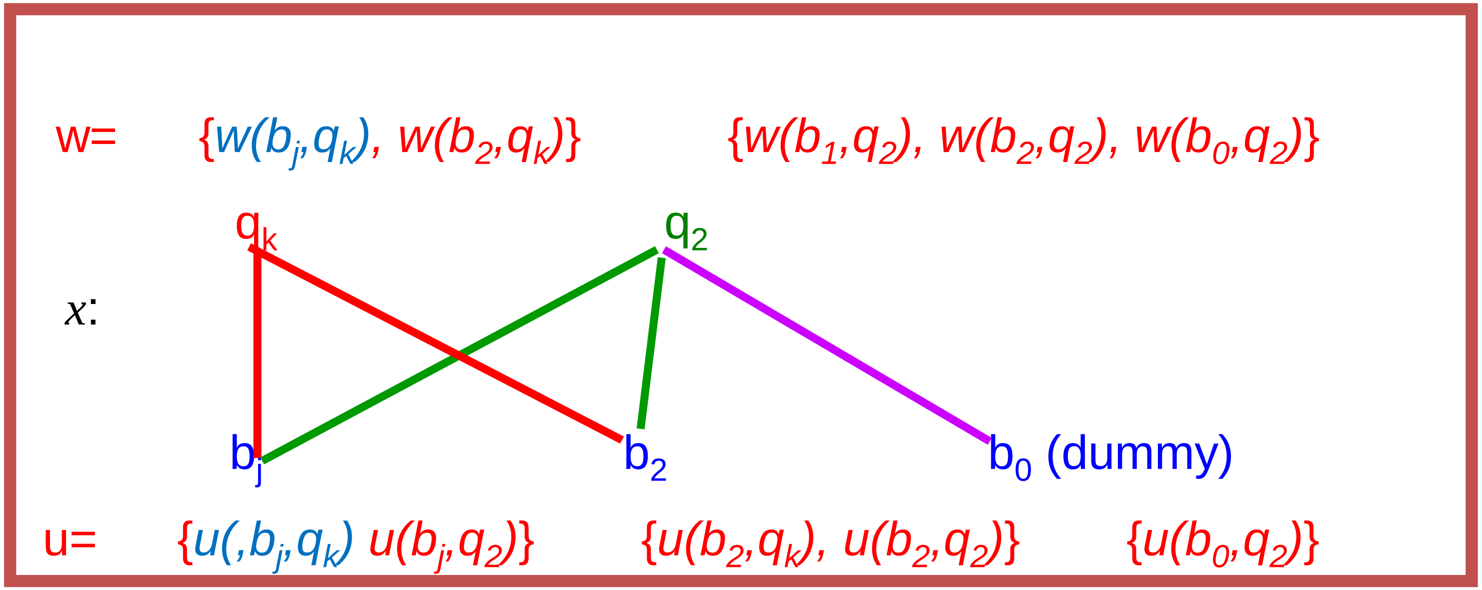
B, ***Q*** – finite and disjoint sets of players (buyers and sellers- dummies are included)

A player of a set may form more than one partnership with different players of the other set.

Quota –$r(b_j)$, $s(q_k)$ - maximum number of partners.

If b_j and q_k become partners, they fulfil an activity together that produces a gain v_{jk}, which is split between them the way both agree. The agents identify utility with money and have a reservation individual payoff of zero. **The main feature of this game is that the trades among the agents are always pairwise and independent.** This means that the trade between two agents who are partners is not affected by any other trade in which these agents may be involved. The Assignment Game of Shapley and Shubik (1972) is the special case, where the quotas are of one.

An **allocation** ***(u,w;x)*** specifies a **payoff allocation** ***(u,w)*** (an array of individual payoffs for each player, resulting from the division, among the partners, of the income yielded within each partnership formed: if b_j and q_k become partners at *x* then $u(b_j,q_k)+w(b_j,q_k)=v_{jk}$, $u(b_j,q_k)\geq 0$, $w(b_j,q_k)\geq 0)$ and a **matching** ***x*** (a set of partnerships that respects the quotas of the players).



Facing two arrays of individual payoffs an agent prefers the one which gives to him the highest total payoff.

The cooperative equilibrium concept is captured by the notion of **stability** (Sotomayor, 2016), which, in this model, is identified with the **pairwise-stability concept** (Sotomayor 1992).

PAIRWISE-STABILITY (Sotomayor 1992): *The allocation* $x=(u,w;x)$ *is pairwise stable if it is feasible and, for all pairs* (b_j, q_k), *the sum of any individual payoff of* b_j *with any individual payoff of* q_k *is not less than* v_{jk}.

The interpretation of this condition is the natural one. If it is not satisfied, that is, if $u(b_j,q_t) + w(b_s,q_k) < v_{jk}$, for some b_j and q_k, then these agents, by becoming partners and by breaking their current partnerships with q_t and b_s, respectively, and possibly keeping other partnerships, can obtain, for each one, a higher total payoff than the one given by $(u,w;x)$.

Theorem (Sotomayor 1992)**.** *The corewise-stability concept is not equivalent to the stability concept.*

The competitive treatment, proposed in Sotomayor (2007), is given through an economic structure where the concept of competitive equilibrium allocation was introduced. The prices are not negotiated, but taken as given by the buyers who demand their favorite allowable sets of objects.

COMPETITIVE EQUILIBRIUM (Sotomayor 2007**):** ***The allocation*** **(*u,w;x*)** is a *competitive equilibrium allocation* if it is ***feasible*** *and* each buyer is assigned to a bundle of objects of his demand set at prices ***w;*** (i.e. an allowable set of items that, at the given prices, maximizes his total payoff).

Theorem (Sotomayor, 2007): *An allocation is a competitive equilibrium if and only if it is stable and non-discriminatory for the buyers (each seller sells all his items for the same price)*

THE EXTENDED MARKETS

The *extended markets* are the terms of the sequence $S(M) \equiv (M \equiv M^1, M^2, \ldots, M^\infty)$, yielded by the replications of the simple market M infinitely many times.

M^h - *finite extended market* - is obtained by replicating h times the *simple market* M, so it has h copies of each agent.

M^∞ - *asymptotically extended market*- is the infinite union of all terms of the sequence, so it has infinitely many copies of each agent.

Each copy maintains the same characteristics of the original agent.

$M(t)$ – t-th replica of M.

$b_j(t)$ - buyer of type b_j in $M(t)$;

$q_k(t)$ - seller of type q_k in $M(t)$;

quota of $b_j(t)$: $r(b_j(t)) \equiv r_j$; quota of $q_k(t)$: $s(q_k(t)) \equiv s_k$.

Any sequential game M^h is also a *subgame* of the sequential game M^k, for all $k \geq h$; any finite sequential game is a *subgame* of the *sequential game* M^∞, with *infinitely many stages* (M^∞ is a subgame of itself and M^h is a subgame of itself, for any positive integer h).

The idea behind the *multi-stage cooperative equilibrium* for a sequential game is analogous to that behind a subgame perfect equilibrium of a non-cooperative dynamic game: *a multi-stage cooperative equilibrium for a sequential game is a cooperative equilibrium for the whole game, whose restriction to each of its subgames is also a cooperative equilibrium.*

Therefore, in an environment where the players interact under the assumption of optimal cooperative behavior, the equilibrium for a given sequential game M^* occurs when the players of any previous subgame of M^*, are also playing optimally, not only with respect to the choice of their partners, as well with respect to the division of the income yielded within each partnership formed. Therefore, at the equilibrium, the agreements reached in the previous subgames are preserved in the subsequent subgames. This idea is captured by the concept of *sequential stability*.

A payoff allocation is ***sequentially stable*** *for a finite term* M^h *of the sequence* $S(M)$ *if it is stable and its restriction to any previous term (if any) is stable. It is* ***sequentially stable*** *for the asymptotically extended market* M^∞ *if its restriction to any extended market is stable.*

$$S(M) = M^1, M^2, \ldots, M^{h-1}, M^h, \ldots, M^\infty$$

Cooperative games, which are also endowed with a competitive market game structure, generate sequential games with a structure of competitive market game with multiples stages. For this structure, the idea of the “perfect” competitive equilibrium allocation emerges as the natural solution concept.

A competitive equilibrium allocation for a sequential game is ***a perfect competitive equilibrium allocation*** *for this game if its restriction to each of its subgames is a competitive equilibrium allocation.*

NOTATION:

H= maximum quota of the agents. **$L(M)=(M^{H+1}$, ..., $M^{\infty})$.**

M^* is a **sufficiently large extended market** if $M^* \in$ **$L(M;$**

$M^*=M^h$ is a small extended market if $h \in [2, H]$.

The positive integer number h is a **sufficiently large number if $h>H$**.

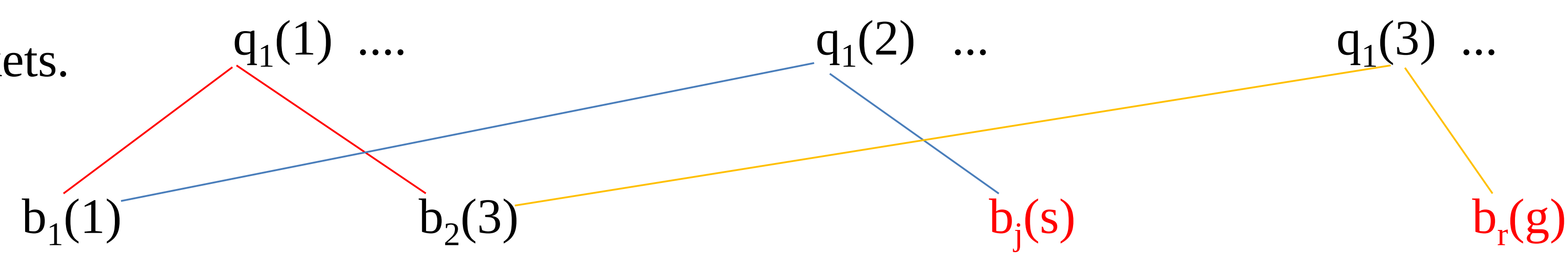
Matching markets have been replicated and only studied under the static approach. We analyze how the replications of the simple market M, a sufficiently large number of times, affect the cooperative and competitive game structures of the resulting extended markets, under the static and the sequential approaches. Such game structures are of economic interest, given their implications in the way the market operates.

Basically, we establish, under both approaches, the relationships between the four solution sets considered: the sets of stable allocations, competitive equilibrium allocations, sequentially stable allocations and perfect competitive equilibrium allocations, and show that not all stable allocations of a sufficiently large extended market succeed in being extensions of stable allocations of the simple market M, and not all stable allocations of the simple market M succeed in extending to stable allocations of a sufficiently large extended market. Then, we characterize the successful allocations for these markets.

The main point is that the replications of the market M create new coalitional interactions, which yield cooperative and competitive game structures, distinct from those of M.

While in the market M an agent cannot trade more than once with the same partner, in an extended market any agent is allowed to form partnerships, up to his/her quota, with more than one copy of the same agent. Thus, any buyer, for example, can acquire more than one object of the same type, as if she was trading more than once with the same seller. These two ways to play the game yield distinct types of deviations.

On the other hand, when the extended market belongs to the sub-sequence *L(M)*, a buyer, for example, **cannot form partnerships with all copies of any given seller, because her quota is less than the number of copies of any seller.** Hence, he becomes available to form a destabilizing pair with some of the copies of any seller. This is not true for the other extended markets.



$b_1(1)$ cannot form partnerships with all copies of q_1 in M^3 ($H=2$).

The new kinds of coalitional interactions become available to the agents and increase the possibilities of cooperation among them. The deviations caused by these coalitional interactions make unstable every discriminatory allocation: **every stable allocation is non-discriminatory**.

Theorem 1. (First Equivalence Theorem): *For any* $M^* \in L(M)$ *the sets of stable allocations, of competitive equilibrium allocations and of non-discriminatory stable allocations coincide.*

This equivalence holds for M^{H+1} and it is preserved in all the subsequent terms. Thus, **as the number of terms in** ***S(M)*** **increases, the set of stable allocations and the set of competitive equilibrium allocations (these allocations always exist) of the replicated markets shrink to the set of non-discriminatory stable allocations.**

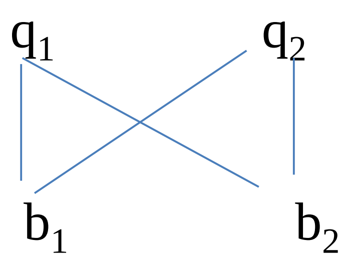
The characterization of the set of competitive equilibrium allocations for M^{∞}, provided by Theorem 1, is distinct from that suggested by the well-known result of Aumann (1964) for pure exchange economies with a continuum of agents. For such economies, this author proves that the set of competitive equilibrium allocations always coincides with the core. However, Example 1 shows that this conclusion does not carry over to our setting.

Example 1. $B=\{b_1, b_2\}$, $Q=\{q_1,q_2\}$, every agent has a quota of *2* and

w= (1,0) (1, 2)

$$v=\begin{array}{c|cc|} & q_1 & q_2 \\ b_1 & 2 & 3 \\ b_2 & 1 & 2 \end{array}$$

σ:



u= (1, 2) (1, 0)

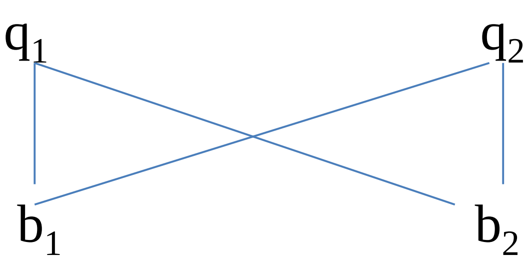
$\sigma=(u,w;x)$ is stable, so it is in the core.

σ is discriminatory.

Let $M^* \in L(M)$ and let σ^* be the extension of σ to M^*. Then, σ^* is also discriminatory, so, by Theorem 1, **σ^* is not stable for M^*.** We claim that σ^* **is in the core of M^***. In fact, consider the ND allocation $\sigma'=u',w';x)$ for M:

w'= (0.5, 0.5) (1.5, 1.5)

σ':



u'= (1.5, 1.5) (0.5, 0.5)

$U_1=U'_1=3$; $U_2=U'_2=1$; $W_1=W'_1=1$; $W_2=W'_2=3$.

Let σ'^* be the extension of σ' to M^*. Then,

$U^*(b_1(t))=U'^*(b_1(t))$; $U^*(b_2(t))=U'^*(b_2(t))$; $W^*(q_1(t))=W'^*(q_1(t))$; $W^*(q_2(t))=W'^*(q_2(t))$.

Since the definition of a corewise-stable allocation only involves the total payoffs of the agents, σ^* is in the core of M^* if and only if σ'^* is in the core of M^*. It is a matter of verification that σ'^* is stable for M^*. Then, σ'^* is in the core of M^*, and so $\boldsymbol{\sigma^*}$ **is in the core of** $\boldsymbol{M^*}$**.** Since σ^* is unstable, we have that $\boldsymbol{\sigma^*}$ **is not a competitive equilibrium allocation**. Hence, **the core and the set of competitive equilibrium allocations of** $\boldsymbol{M^*}$ **are distinct.** ■

In Example 1, **the replication σ^h of σ to any M^h is in the core of M^h, but σ is not a competitive equilibrium allocation for M.** The situation presented in this example never occurs when the market M is the exchange economy of Debreu and Scarf (1963): for any pure exchange economy E, satisfying certain conditions on the preferences of the agents, if σ is some allocation in E and **the extension of σ to any economy E^h** resulting from the replication of E, h times, **is in the core of that economy, then σ is a competitive equilibrium allocation of E.**

Every perfect competitive equilibrium allocation is competitive and every sequentially-stable allocation is stable, by definition. Example 2 shows that not all competitive equilibrium allocation is perfect and not all stable allocation is sequentially-stable.

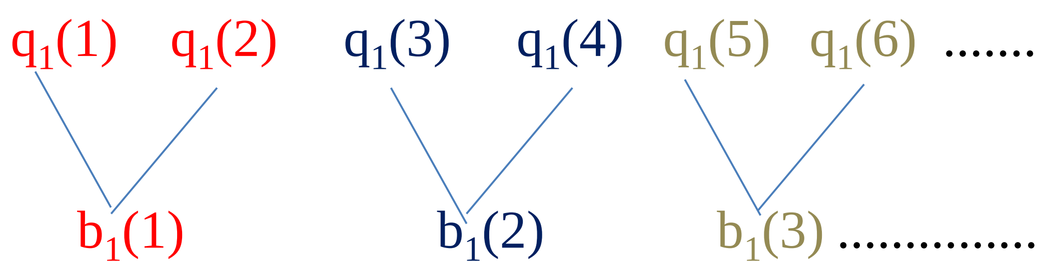
Example 2. $B=\{b_1,b_0\}$, $Q=\{q_1,q_2,q_0\}$, $r(b_1)=2$, $s(q_1)=s(q_2)=1$;

$$v= \begin{array}{c|cc|} & q_1 & q_2 \\ b_1 & 3 & 2 \end{array}$$

Consider the allocation $(\sigma^{\infty}=(u^{\infty},w^{\infty}); x^{\infty})$ which matches every copy of b_1 with two copies of q_1 and leaves all copies of q_2 unmatched.

$w^{\infty} =$ 0 0 0 0 0 0 0 0

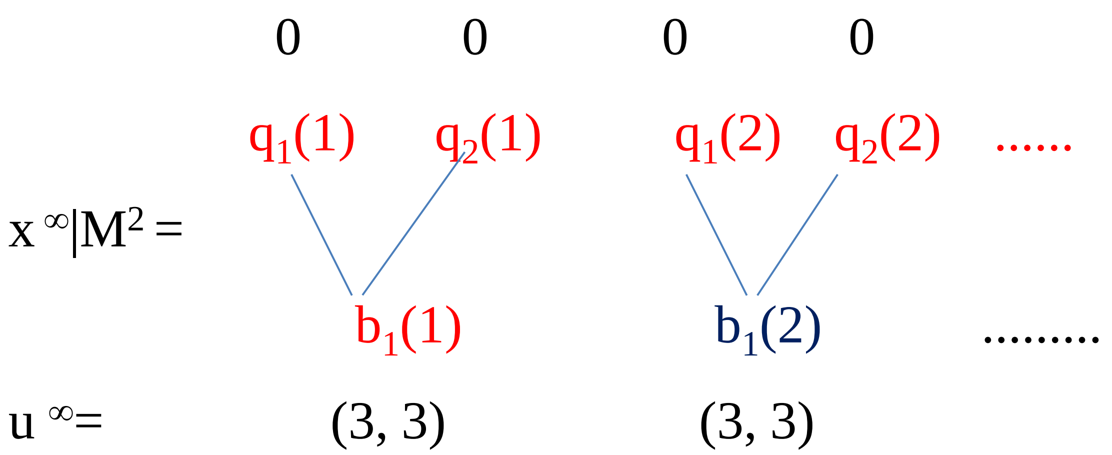
$x^{\infty} =$



$u^{\infty}=$ (3, 3) (3, 3) (3, 3) 0 0

The payoff allocation $\sigma^{\infty}=(u^{\infty},w^{\infty})$ is **stable** and **competitive for** M^{∞}.

But for any h>0, $\sigma^{\infty} \mid M^h$ is not feasible, so it is not stable for M^h, and so it is not competitive for that market. Then, $\boldsymbol{\sigma}^{\infty}$ **is not sequentially stable** and **it is not perfect competitive equilibrium.**



Furthermore, $\sigma^{\infty}=(u^{\infty},w^{\infty})$ is the extension of $\sigma=(u,w)$, given by: $u(b_1)=(3,3)$, $w(q_1)=w(q_2)=0$, which is not compatible with any feasible matching for M, so it is not feasible and so it is unstable. Then, **it is not a competitive allocation** in the market M.

What are the stable payoffs that are sequentially stable? What are the competitive payoffs that are perfect?

Theorem 2. *Let $M^* \in L(M)$. Let x^* be the extension of an optimal matching of M to M^*. Let σ^* be a stable payoff for M^*. Then σ^* is a sequentially stable payoff if and only if σ^* is compatible with x^*.*

Theorem 3. *Let $M^* \in S(M)$. Then, the set of perfect competitive equilibrium allocations is the intersection of the set of competitive equilibrium allocations with the set of sequentially-stable allocations.*

Thus, if an allocation in a market in $M^* \in L(M)$ is sequentially stable, then it is stable, so it is competitive by Theorem 1 and so it is a perfect competitive equilibrium, by Theorem 3.

Therefore we obtain the second equivalence theorem, which establishes the equivalence between the concepts of cooperative equilibrium and competitive equilibrium for the terms of the subsequence $L(M)$, viewed as sequential games.

Theorem 4 (Second Equivalence Theorem). *Let* $M^* \in L(M)$. *Then, the set of perfect competitive equilibrium allocations and the set of sequentially stable allocations coincide.*

Theorem 4 implies that the cooperative equilibrium allocation yielded **in an extended market of** $L(M)$**, when the agents adopt a cooperative optimal behavior in every subgame, can also occur in a different scenery, with reference to the prices, where the agents behave as price takers in every subgame, and vice-versa.** This is not necessarily true for the other extended markets.

What are the stable allocations of the markets in *L(M),* which are extensions of stable allocations of *M*?

What are the stable allocations of *M*, which succeed in extending to stable allocations in the markets of *L(M)*?

Theorem 5 (Characterization of the successful feasible allocations of *L(M)*). *Let $M^* \in$* ***L(M).*** *Let σ^* be a stable payoff for M^*. Then, σ^* is the extension of a feasible payoff of M if and only if σ^* is compatible with the extension of an optimal matching.*

Theorem 6 (Characterization of the successful feasible allocations of *M*). *Let $M^* \in S(M)$. Let (σ,x) be a feasible allocation of M. Then, the extension of (σ,x) to M^* is stable if and only if (σ,x) is stable and non-discriminatory.*

Then, the competitive equilibrium allocations, which extend to stable allocations of *L(M)* are those that do not discriminate the sellers.

Corollary 1: *Let* $M^* \in$ ***L(M)*****.** *Let* σ^* *be a stable payoff for* M^**. Then* σ^* *is the extension of a feasible payoff of M if and only if it is sequentially stable.*

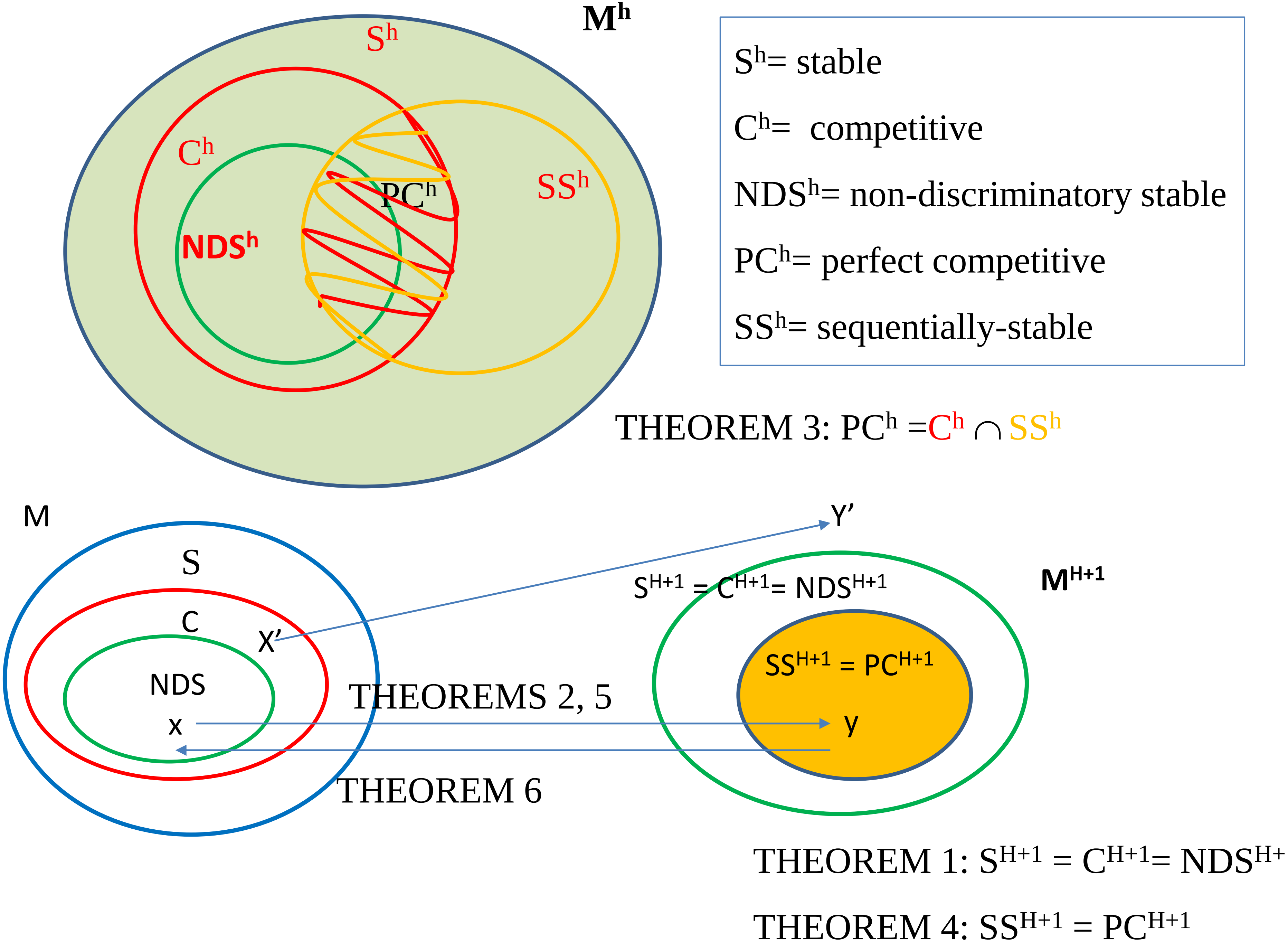
However, the set of non-discriminatory stable allocations of *M* may be empty. In these cases, no stable allocation of the markets in *L(M)* is extension of a feasible allocation of *M*. Consequently, **the set of sequentially stable allocations (or the set of perfect competitive equilibrium payoffs) is empty when the set of non-discriminatory stable allocations of *M* is empty.**

The key lemma for the proofs of Theorems 5 and 6 is the following:

LEMMA 1. *Let M^* be any extended market. Let $\sigma^*=(u^*,w^*)$ be a non-discriminatory stable payoff of M^*. Then σ^* has the ETP* - all agents of the same type receive the same payoff vector.

In opposition to the result of Debreu and Scarf (1963), **the ETP is not shared with all corewise-stable allocations**.

Example 3. Consider the allocations σ and σ' presented in Example 3.1 to define the allocation θ for $M\#$, as follows: the agents in $B(t)$ and in $Q(t)$ are allocated according to σ for all $t=1,3,5,\ldots$ and according to σ' for all $t=2,4,6,\ldots$ Clearly, θ is unstable but it is in the core of $M\#$ (it is feasible and the agents have the same total payoffs as in $\sigma\#$, which is in the core). However, θ does not have the ETP. This setting also illustrates that some agents may be discriminated at a corewise-stable allocation for $M\#$. ■



M^h
S^h
C^h
NDS^h
PC^h
SS^h
S^h= stable
C^h= competitive
NDS^h= non-discriminatory stable
PC^h= perfect competitive
SS^h= sequentially-stable
THEOREM 3: $PC^h = C^h \cap SS^h$
M
S
C
X'
NDS
x
Y'
$S^{H+1} = C^{H+1} = NDS^{H+1}$
M^{H+1}
$SS^{H+1} = PC^{H+1}$
y
THEOREMS 2, 5
THEOREM 6
THEOREM 1: $S^{H+1} = C^{H+1} = NDS^{H+1}$
THEOREM 4: $SS^{H+1} = PC^{H+1}$

Although the set of non-discriminatory stable payoffs of the market M, as well of the small extended markets, may be empty, the non-discriminatory stable payoffs of the sufficiently large extented markets always exist. More precisely,

Theorem 7. *The set of stable allocations of any extended market is always non-empty.*

In any finitely extended market, the core, the set of stable allocations and the set of competitive equilibrium allocations are always non-empty (Sotomayor 1992, 2007). For the asymptotically extended market, the idea of the proof is, first, to consider the market M^{∞} as resulting from the replication infinitely many times of an extended market M^h in $L(M)$. Next, to extend a stable payoff of M^h, which is non-discriminatory, by Theorem 1, to M^{∞}. This allocation is stable, by Theorem 6.