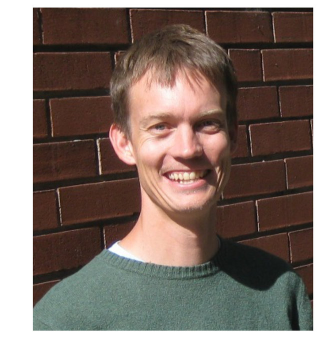
Fast and Accurate Algorithms for Cosmic Microwave Background Radiation Data on HEALPix Points

Kathryn P. Drake
Boise State University

Grady B. Wright
Boise State University

BOISE STATE UNIVERSITY
PhD Program in Computing

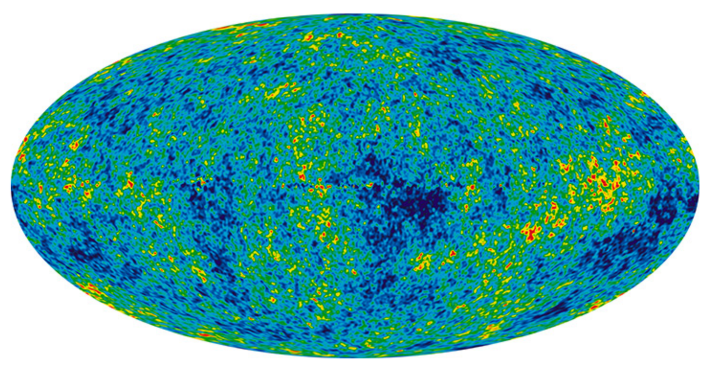
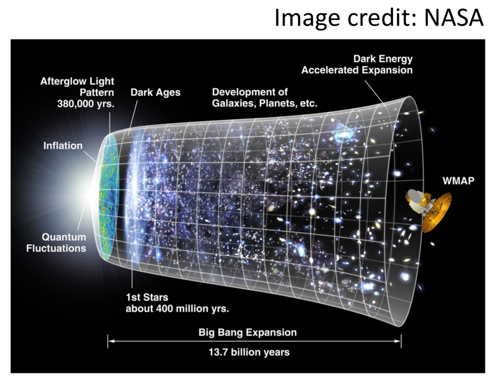
Supported by NASA Idaho Space Grant Consortium

SMART
SCIENCE, MATHEMATICS, AND RESEARCH FOR TRANSFORMATION
PART OF THE NATIONAL DEFENSE EDUCATION PROGRAM

Motivation

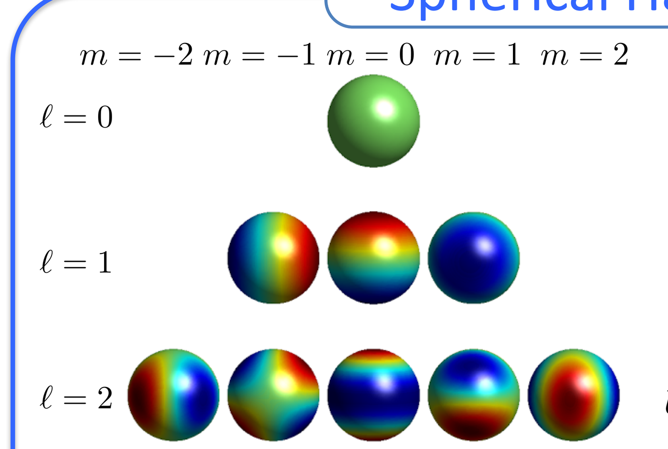
Faintly glowing at the edge of the observable universe, the Cosmic Microwave Background (CMB) is "relic radiation" that represents the first light to travel during the early stages of the universe's development and gives the strongest evidence for the Big Bang theory. Since the discovery of the CMBR in 1964, scientists have worked to measure it in full detail using a Hierarchical Equal Area isoLatitude Pixelization scheme on the sphere. While these "HEALPix" points allow for a quasiuniform discretization of the sphere, they are not well suited for the fast algorithms necessary for mining the massive CMBR data sets. For this work, we apply the Double Fourier Sphere method and fast Fourier transforms for uniform and non-uniform point distributions in order to to efficiently compute the bivariate Fourier series coefficients. We then utilize a state-of-the-art method which computes spherical harmonic coefficients from these bivariate Fourier coefficients. This algorithm enables us to quickly and more accurately compute the angular power spectrum of CMBR data sampled at the HEALPix points.

Image credit: NASA

CMBR temperature anisotropy map (left) CMBR and the Big Bang (right)

Spherical Harmonics



Spherical harmonics, Y_ℓ^m, degree ℓ and order m

Spherical analogue of Fourier series

Example:

Let $f(\lambda,\theta)$ be a smooth function on the surface of the sphere

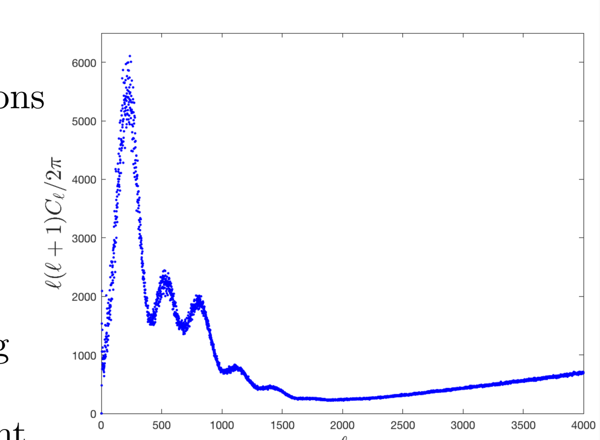
$$f(\lambda,\theta)=\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} b_\ell^m Y_\ell^m(\lambda,\theta),$$

$$b_\ell^m=\int_0^{2\pi}\int_0^{\pi} f(\lambda,\theta)\overline{Y}_\ell^m(\lambda,\theta)\sin\theta d\theta d\lambda$$

The angular power spectrum measures the amplitude of temperature fluctuations of the CMB and is calculated with the spherical harmonic coefficients, b_ℓ^m:

$$C_\ell=\frac{1}{2\ell+1}\sum_m |b_\ell^m|^2$$

CMBR is strongest evidence for the Big Bang theory. Refined analysis of the angular power spectrum promises insight into dark matter (95% of the universe).



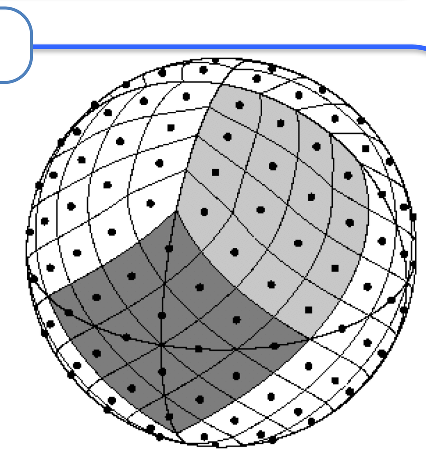
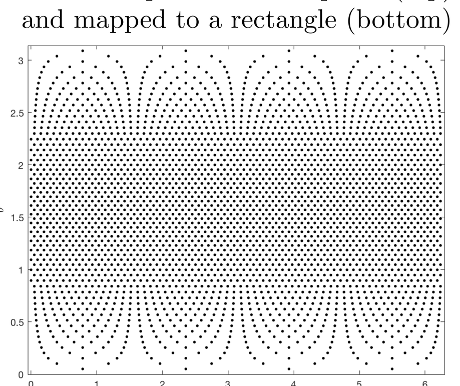
Angular power spectrum of CMBR map

HEALPix Points

Advantages

Hierarchically tessellated: the pixel structure lends itself to simple and direct movements to more refined levels of grid resolution

isoLatitude: Pixels are distributed on lines of constant latitude, which aids univariate Fourier analysis and computations

HEALPix points on the sphere (top) and mapped to a rectangle (bottom)

Disadvantages

Sparsity at poles: functions are grossly undersampled near the poles of the sphere

Unequal spacing: Pixels are misaligned and unequally spaced in longitude, making fast bivariate Fourier analysis impossible without modification

Current Method: HEALPix

Step 1: "Zeroth" Order Approximation

HEALPix software estimates the angular power spectrum of a function f on the sphere by first using a low-order approximation to calculate the spherical harmonic coefficients:

$$b_\ell^m \approx \frac{4\pi}{N_{pix}}\sum_{i=1}^{N_{pix}}\overline{Y}_\ell^m(\lambda_i,\theta_i)f(\lambda_i,\theta_i),\ 0\le\ell\le\ell_{max},\ -\ell_{max}\le m\le\ell_{max}$$

Due to the isolatitude nature of the pixels, this computation is done with $\mathcal{O}(N_{pix}^{3/2})$

Step 2: Iterative Refinement

In order to improve the estimate of the b_ℓ^ms, HEALPix uses an iterative process.

Analysis (function to coefficients): $b_\ell^m=\sum_{\ell=0}^{\ell_{max}}\sum_{m=0}^{\ell}\overline{Y}_\ell^m(\lambda,\theta)f(\lambda,\theta)\iff \mathbf{b}=\mathbf{Af}$

Synthesis (coefficients to function): $\widetilde{f}(\lambda,\theta)=\sum_{\ell=0}^{\ell_{max}}\sum_{m=-\ell}^{\ell}Y_\ell^m(\lambda,\theta)b_\ell^m\iff\widetilde{\mathbf{f}}=\mathbf{Sb}$

These operators are used in the iterative refinement scheme, which converges to the least squares solution to the coefficients: $\mathbf{b}^{(0)}=\mathbf{Af},\quad \mathbf{b}^{(k)}=\mathbf{b}^{(k-1)}+\mathbf{A}\left(\mathbf{f}-\mathbf{Sb}^{(k-1)}\right)$

Quadrature Weights

To improve equal weight quadrature above, HEALPix offers quadrature weights:

$$b_\ell^m\approx\sum_{i=1}^{N_{pix}}w_i\overline{Y}_\ell^m(\lambda_i,\theta_i)f(\lambda_i,\theta_i),\ 0\le\ell\le\ell_{max},\ -\ell_{max}\le m\le\ell_{max}$$

"Pixel weights" for *each HEALPix point*, computed using the algorithm from [5]

"Ring weights" for *each ring*, similar algorithm to pixel weights, but wtih a smaller system

New Method: HP2SPH

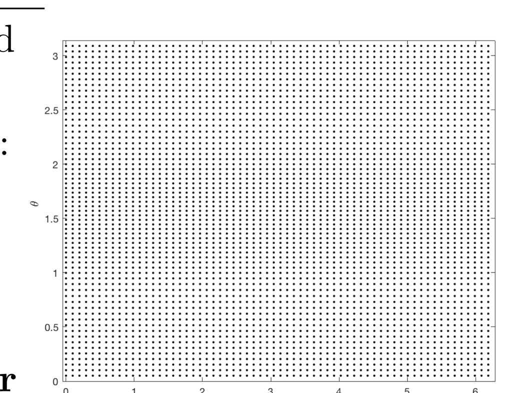
Step 1: Upsample in latitude

In order to use fast Fourier analysis in longitude, we need to upsample to an equal number of points on each ring

Achieved using trigonometric interpolation and the DFT:

$$f(x)=\sum_{k=0}^{N-1}c_k e^{-2\pi ikx},\qquad x\in[0,1]$$

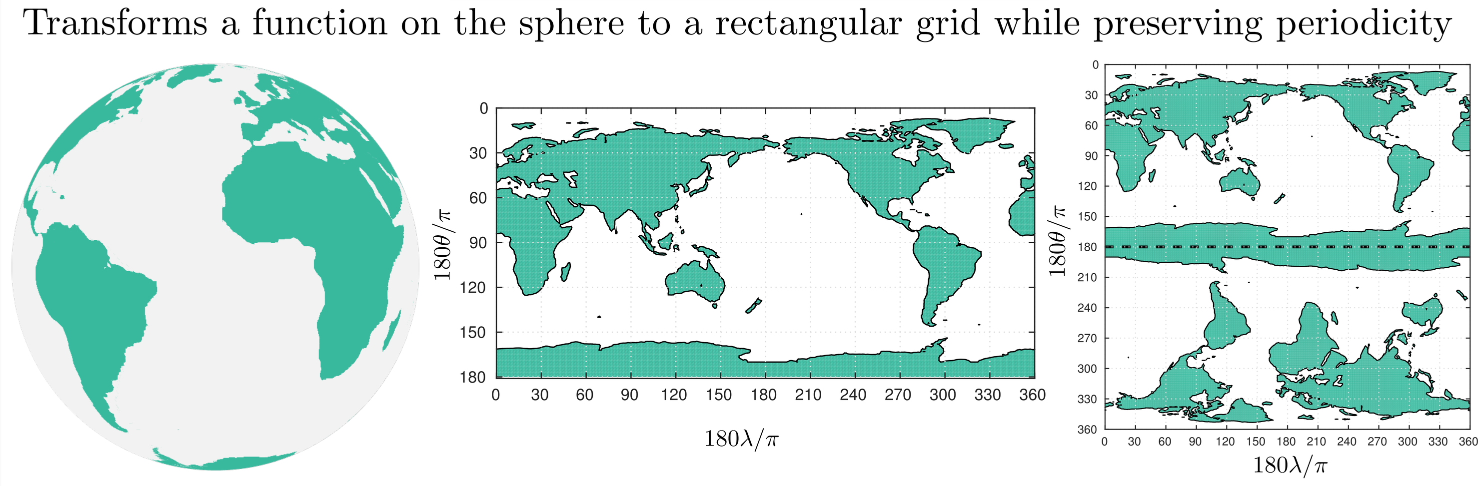
Since the points are equispaced in latitude, we can solve for the upsampled function values using the **fast Fourier transform** with a complexity of $\mathcal{O}(N\log N)$



HEALPix points upsampled in longitude

Step 2: Double Fourier Sphere

Transforms a function on the sphere to a rectangular grid while preserving periodicity



(a) The surface of the Earth (b) projected onto a lat-lon grid (c) "doubled up" with DFS

The Double Fourier Sphere technique allows for fast Fourier analysis in longitude

Step 3: Nonuniform FFT

HEALPix points are non-equispaced in longitude, so we use the **Nonuniform FFT**

Let $\mathbf{x}$ be non-equispaced and $0\le j,k\le N-1$:

$$f_j=\sum_{k=0}^{N-1}c_k e^{-2\pi i x_j k}\quad\iff\quad \mathbf{f}=\widetilde{F}\mathbf{c},\ (\widetilde{F})_{jk}=e^{-2\pi i x_j k}$$

By relating $\widetilde{F}$ to the standard DFT matrix $(F_{jk}=e^{-2\pi ijk/N})$ with a low rank approximation (rank K), the NUFFT simplifies to K FFT computations [4]

Step 4: Fast Spherical Harmonic Transform

Finally take the FFT in latitude to get the bivariate Fourier coefficients

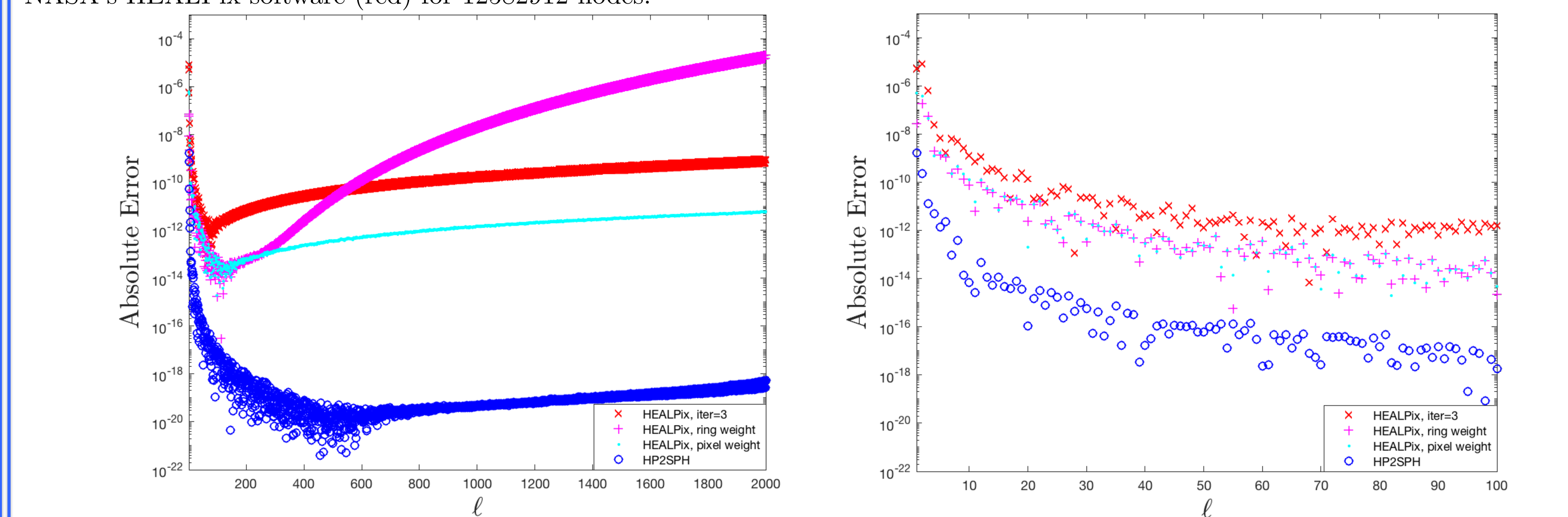
Bivariate Fourier Series represent smooth functions on the sphere with exponentials:

Let $f(\lambda,\theta)$ be bi-periodic on $[-\pi,\pi]^2$: $f(\lambda,\theta)=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}a_{n,m}e^{in\theta}e^{im\lambda}$

The Fast Spherical Harmonic Transform [1] quickly computes spherical harmonic coefficients from the bivariate Fourier coefficients with a complexity of $\mathcal{O}(N\log N)$ (minus some precomputational cost).

Numerical Test #1

Absolute error for the angular power spectrum of f for $0\le\ell\le2000$ (left) and $0\le\ell\le100$ (right) calculated by HP2SPH (blue) and NASA's HEALPix software (red) for 12582912 nodes.

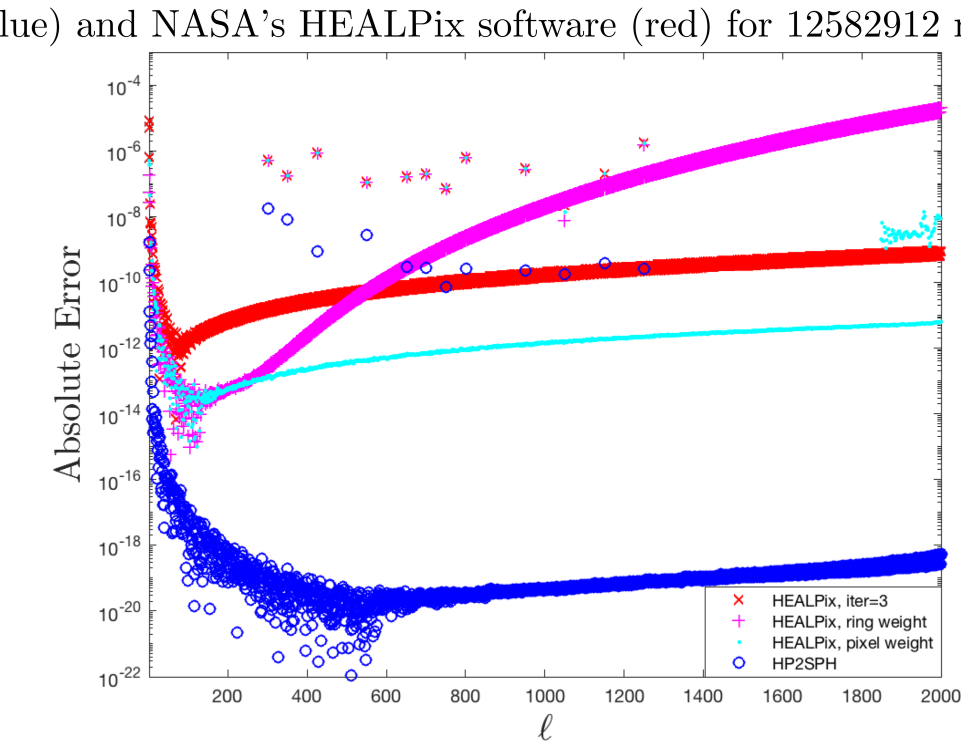


Test function : $f(\lambda,\theta)=5g(\lambda,\theta,l_1,t_1)-3g(\lambda,\theta,l_2,t_2)+8g(\lambda,\theta,l_3,t_3),\ g(\lambda,\theta,l,t)=(2-2(x\cdot x(l,t)+y\cdot y(l,t)+z\cdot z(l,t)))^{\frac{3}{2}}$

$x=\cos(\lambda)\sin(\theta),\ y=\sin(\lambda)\sin(\theta),\ z=\cos(\theta),\ 0<l_1,l_2,l_3<2\pi,\ 0<t_1,t_2,t_3<\pi$

Numerical Test #2

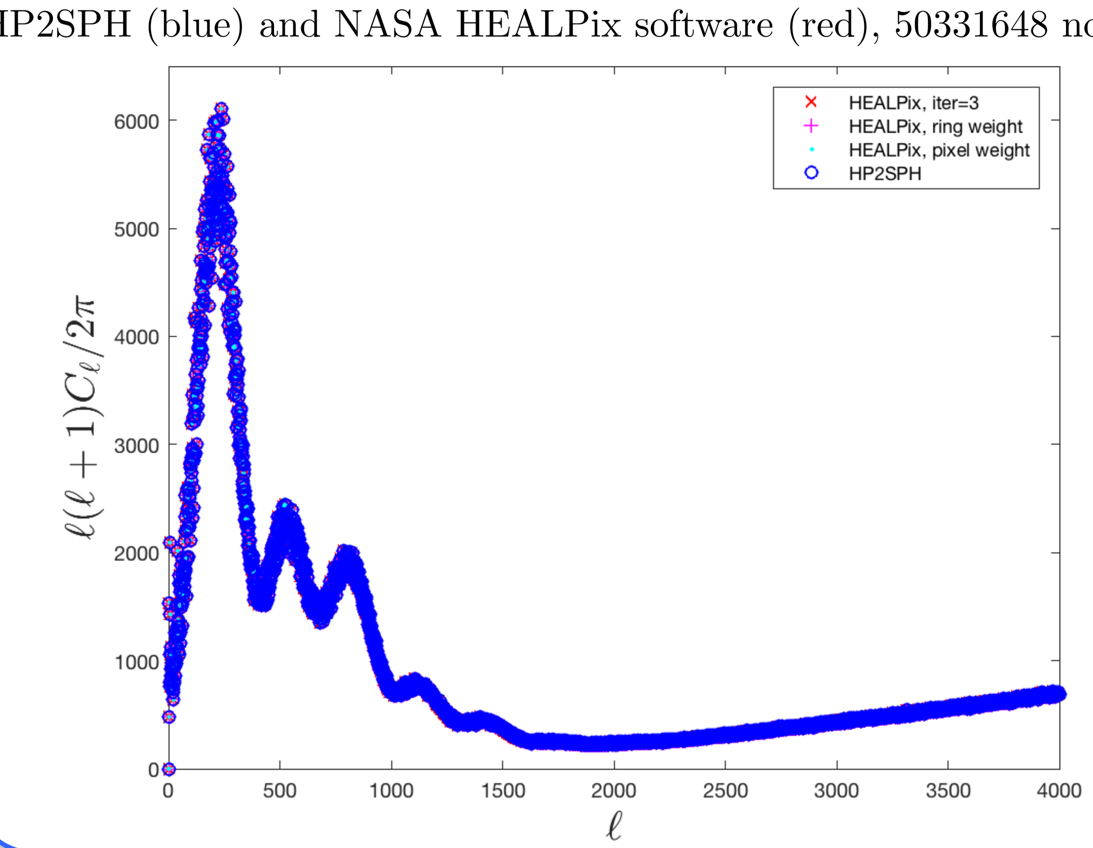
Absolute error for the angular power spectrum of h for HP2SPH (blue) and NASA's HEALPix software (red) for 12582912 nodes.



Test function : $h(\lambda,\theta)=f(\lambda,\theta)+Y_{300}^{75}(\lambda,\theta)+Y_{350}^{100}(\lambda,\theta)+Y_{425}^{75}(\lambda,\theta)+Y_{550}^{25}(\lambda,\theta)+Y_{600}^{50}(\lambda,\theta)+Y_{700}^{75}(\lambda,\theta)+Y_{800}^{25}(\lambda,\theta)+Y_{950}^{25}(\lambda,\theta)+Y_{1050}^{50}(\lambda,\theta)+Y_{1150}^{50}(\lambda,\theta)+Y_{1250}^{50}(\lambda,\theta)$

Numerical Test #3

Angular power spectrum of CMBR temperature anisotropy map for HP2SPH (blue) and NASA HEALPix software (red), 50331648 nodes.



Discussion & Conclusions

The values of the angular power spectrum at higher degrees, ℓ, are crucial to modern cosmology and are more difficult to calculate. The first two numerical tests demonstrate the accuracy of the new HP2SPH method relative to the HEALPix approach for problems involving synthetic data with no noise. The results from the second numerical test specifically show that when the function is deterministic, the HP2SPH method can more accurately recover the angular power spectrum at high degrees when compared to the method used by the HEALPix software. In the case of a real CMB map, which is inherently noisy, the power spectrums computed by the two methods show good agreement, as seen in the third numerical test.

Our new method, HP2SPH, utilizes the FFT, NUFFT, and the FSHT to accurately compute the angular power spectrum of CMBR data maps, even at higher spherical harmonic degrees, with near optimal computational complexity ($\mathcal{O}(N\log N)$, for N total nodes)

Future Work & References

Future Work

- Rigorously test our method against HEALPix's method with more complicated deterministic and noisy data on the sphere.
- Make the implementation of our method even faster by transcribing the code into a lower-level language (C or C++).
- Use our method to compute the angular power spectrum of CMBR data maps and compare to HEALPix software.
- Modify our method to include the ability to perform Fourier synthesis on a CMBR map to return corresponding map values.

References

[1] R. M. Slevinsky. Fast and backward stable transforms between spherical harmonic expansions and bivariate Fourier series, in press at Appl. Comput. Harmon. Anal., 2017.

[2] K. M. Gorski, B. D. Wandelt, E. Hivon, and F. K. Hansen, The HEALPix Primer, (2010).

[3] A. Townsend, H. Wilber, and G. B. Wright, Computing with functions in spherical and polar geometries I. The sphere, SIAM J. Sci. Comp., 38-4 (2016), pp. C403-C425.

[4] D. Ruiz-Antolín and A. Townsend, A Nonuniform Fast Fourier Transform Based on Low Rank Approximation, SIAM J. Sci. Comp., 40-1 (2017), p. 529-547.

[5] J. Keiner and D. Potts, Fast evaluation of quadrature formulae on the sphere, Math. Comp., 77 (2008), pp. 397–419.