Aperiodic tilings (tutorial)

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February 12, 2015, ICERM

Plan of the talk

1. Introduction.
2. Tiling definitions, tiling spaces, tiling dynamical systems.
3. Spectral theory.

I. Aperiodic tilings: some references

- C. Radin, *Miles of Tiles*, AMS Student Math. Library, vol. I, 1999.
- L. Sadun, *Topology of Tiling Spaces*, AMS University Lecture Series, vol. 46, 2008.
- M. Baake and U. Grimm, *Aperiodic Order. Vol. 1: A Mathematical Invitation*, Cambridge, 2013.
- E. Harriss and D. Frettlöh, *Tilings Encyclopedia*, http://tilings.math.uni-bielefeld.de

I. Aperiodic tilings

A **tiling** (or tesselation) of $\mathbb{R}^d$ is a collection of sets, called tiles, which have nonempty disjoint interiors and whose union is the entire $\mathbb{R}^d$.

Aperiodic set of tiles can tile the space, but only non-periodically.

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Origins in Logic: **Hao Wang (1960's)** asked if it is decidable whether a given set of tiles (square tiles with marked edges) can tile the plane?

R. Berger (1966) proved undecidability, and in the process constructed an aperiodic set of 20,426 Wang tiles.

R. Robinson (1971) found an aperiodic set of 6 tiles (up to isometries).

I. Penrose tilings

One of the most interesting aperiodic sets is the set of **Penrose tiles**, discovered by **Roger Penrose (1974)**. Penrose tilings play a central role in the theory because they can be generated by any of the three main methods:

1. local matching rules ("jigsaw puzzle");
2. tiling substitutions;
3. projection method (projecting a "slab" of a periodic structure in a higher-dimensional space to the plane).

I. Penrose and his tiles

Figure: Sir Roger Penrose

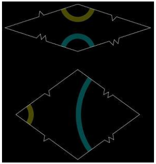


Figure: Penrose rhombi

I. Penrose tiling

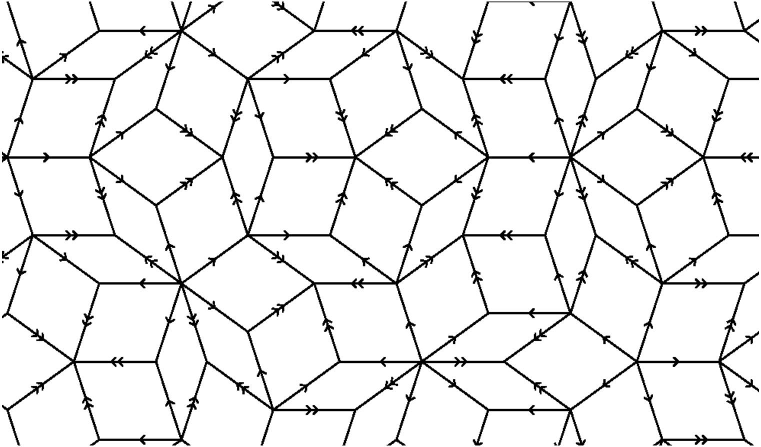
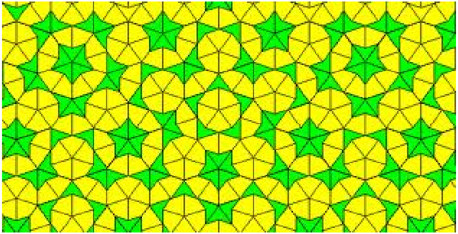


Figure: A patch of the Penrose tiling

I. Penrose tiling (kites and darts)

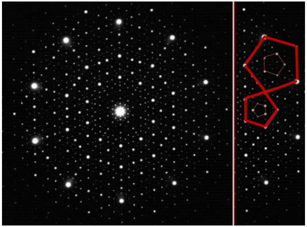


I. Penrose tiling: basic properties

- Non-periodic: no translational symmetries.
- Hierarchical structure, “self-similarity,” or “composition”; can be obtained by a simple “inflate-and-subdivide” process. This is how one can show that the tiling of the entire plane exists.
- “Repetitivity” and uniform pattern frequency: every pattern that appears somewhere in the tiling appears throughout the plane, in a relatively dense set of locations, even with uniform frequency.
- 5-fold (even 10-fold) rotational symmetry: every pattern that appears somewhere in the tiling also appears rotated by 36 degrees, and with the same frequency (impossible for a periodic tiling).

I. Quasicrystals

Figure: Dani Schechtman (2011 Chemistry Nobel Prize); quasicrystal diffraction pattern (below)



I. Quasicrystals (aperiodic crystals)

Quasicrystals were discovered by **D. Schechtman (1982)**. A quasicrystal is a solid (usually, metallic alloy) which, like a crystal, has a sharp X-ray diffraction pattern, but unlike a crystal, has an aperiodic atomic structure. Aperiodicity was inferred from a "forbidden" 10-fold symmetry of the diffraction picture.

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Other types of quasicrystals have been discovered by **T. Ishimasa, H. U. Nissen, Y. Fukano (1985)** and others (Al-Mn alloy, 10-fold symmetry, Ni-Cr alloy, 12-fold symmetry; V-Ni-Si alloy, 8-fold symmetry)

I. Other quasicrystal diffraction patterns

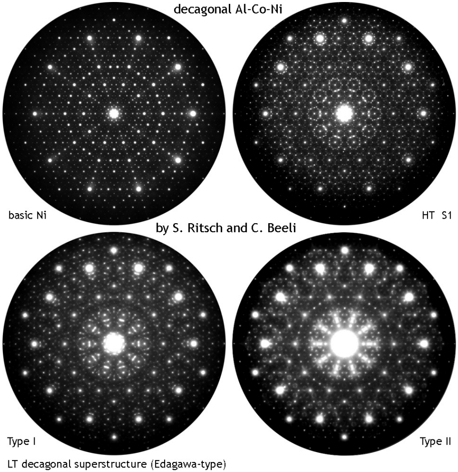

Figure: From the web site of Uwe Grimm (http://mcs.open.ac.uk/ugg2/quasi.shtml)

I. Substitutions

Symbolic substitutions have been studied in Dynamics (coding of geodesics), Number Theory, Automata Theory, and Combinatorics of Words for a long time.

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Symbolic substitution is a map ζ from a finite "alphabet" $\{0, \ldots, m-1\}$ into the set of "words" in this alphabet.

- Thue-Morse: $\zeta(0) = 01,\ \zeta(1) = 10.$ Iterate (by concatenation) :

$$0 \to 01 \to 0110 \to 01101001 \to \ldots$$

$$u = u_0 u_1 u_2 \ldots = \lim_{n\to\infty} \zeta^n(0) \in \{0,1\}^{\mathbb{N}},\ u = \zeta(u)$$

- Fibonacci: $\zeta(0) = 01,\ \zeta(1) = 0.$ Iterate (by concatenation) :

$$0 \to 01 \to 010 \to 01001 \to \ldots u = 0100101001001010010100100\ldots$$

I. Tile-substitutions in $\mathbb{R}^d$

Symbolic substitutions have been generalized to higher dimensions.

One can just consider higher-dimensional symbolic arrays, e.g.

$$\boxed{0} \to \begin{array}{|c|c|}\hline 0 & 0 \\ \hline 0 & 1 \\ \hline\end{array}, \quad \boxed{1} \to \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 1 & 0 \\ \hline\end{array}$$

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More interestingly, one can consider "geometric" substitutions, with the symbols replaced by tiles. Penrose tilings can be obtained in such a way.

I. Example: chair tiling

Examples are taken from "Tiling Encyclopedia", see http://tilings.math.uni-bielefeld.de/

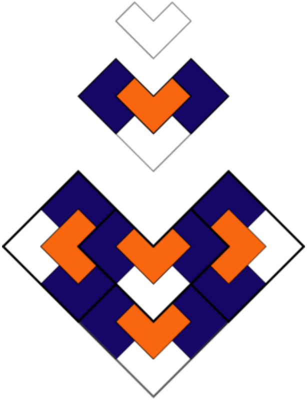

Figure: tile-substitution, real expansion constant $\lambda = 2$

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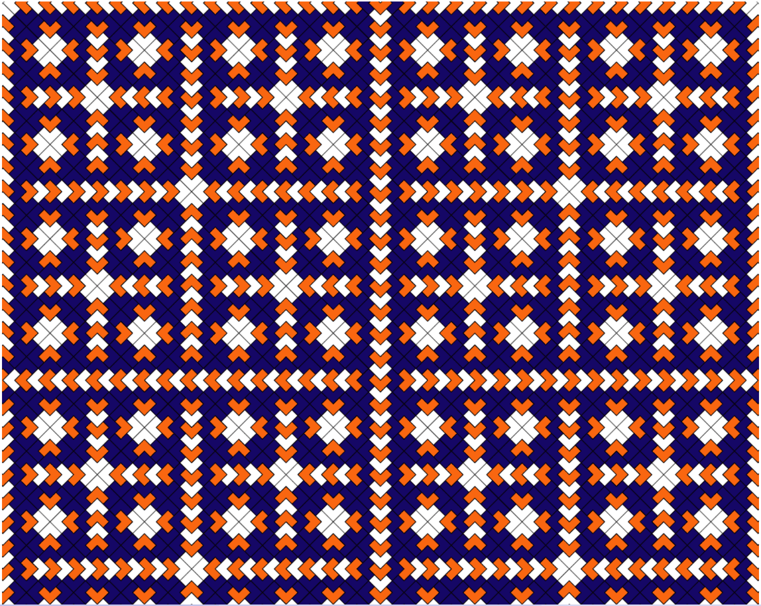

I. Example: Ammann-Beenker rhomb-triangle tiling

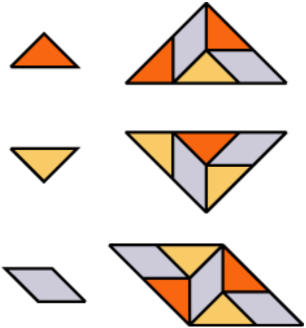

Figure: tile-substitution, real expansion constant $\lambda = 1 + \sqrt{2}$

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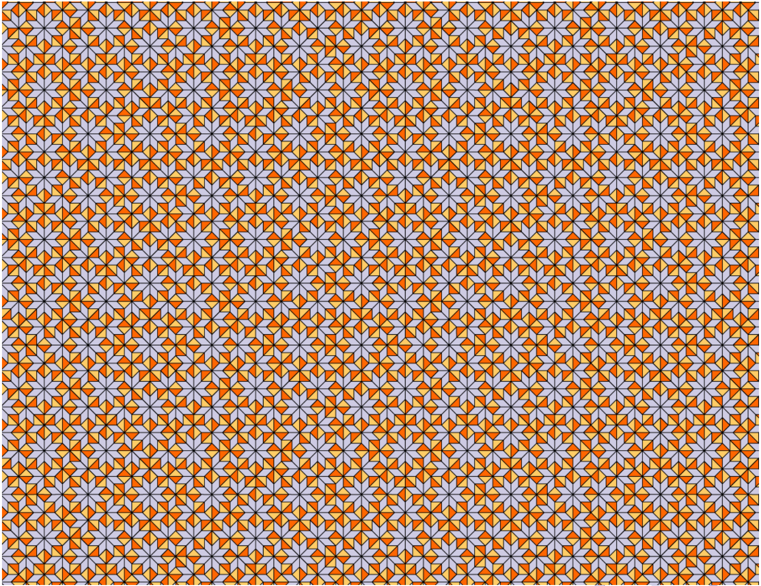

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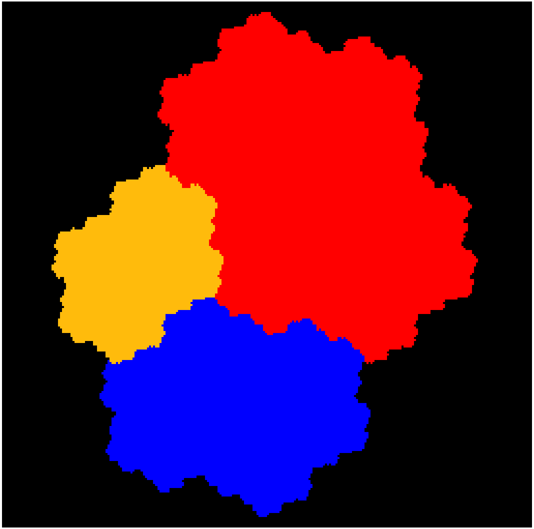
- Markov partitions for hyperbolic toral automorphisms in dimensions 3 and larger
- Numeration systems with a complex base.

Rauzy tiling is a famous example.
let λ be the complex root of $1 - z - z^2 - z^3 = 0$ with positive imaginary part, $z \approx -0.771845 + 1.11514i$. Then (Lebesgue) almost every $\zeta \in \mathbb{C}$ has a unique representation

$$\zeta = \sum_{n=-N}^{\infty} a_n \lambda^{-n},$$

where $a_n \in \{0,1\}$, $a_n a_{n+1} a_{n+2} = 0$ for all n.

Rauzy tiles



Gerard Rauzy

Figure: Gerard Rauzy (1938-2010)

II. Tiling definitions

- **Prototile set:** $\mathcal{A} = \{A_1, \ldots, A_N\}$, compact sets in $\mathbb{R}^d$, which are closures of its interior; interior is connected. (May have "colors" or "labels".)

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- A **tiling** of $\mathbb{R}^d$ with the prototile set $\mathcal{A}$: collection of tiles whose union is $\mathbb{R}^d$ and interiors are disjoint. All tiles are isometric copies of the prototiles.

- A **patch** is a finite set of tiles. $\mathcal{A}^+$ denotes the set of patches with tiles from $\mathcal{A}$.

Finite local complexity

- Several options: identify tiles (patches) up to (a) **translation**; (b) orientation-preserving isometry (**Euclidean motion**); (c) isometry. Let G be the relevant group of transformations.

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Definition. A tiling $\mathcal{T}$ is said to have **finite local complexity** (FLC) with respect to the group G if for any $R > 0$ there are finitely many $\mathcal{T}$-patches of diameter $\leq R$, up to the action of G.

II. Tile-substitutions in $\mathbb{R}^d$

Let $\phi : \mathbb{R}^d \to \mathbb{R}^d$ be an expanding linear map, that is, all its eigenvalues are greater than 1 in modulus. (Often ϕ is assumed to be a similitude or even a pure dilation $\phi(x) = \lambda x$.)

Definition. Let $\{A_1, \ldots, A_m\}$ be a finite prototile set. A **tile-substitution** with expansion ϕ is a map $\omega : \mathcal{A} \to \mathcal{A}^+$, where each $\omega(A_i)$ is a patch $\{g(A_j)\}_{g \in D_{ij}}$, where D_{ij} is a finite subset of G, such that

$$\mathrm{supp}(\omega(A_i)) = \phi(A_i), \ \ i \le m.$$

The substitution is extended to patches and tilings in a natural way.

Substitution matrix counts the number of tiles of each type in the substitution of prototiles: $M = (M_{ij})_{i,j \le m}$, where $M_{ij} = \# D_{ij} = \#$ tiles of type i in $\omega(A_j)$.

II. Substitution tiling space

Definition. Given a tiling substitution ω on the prototile set $\mathcal{A}$, the **substitution tiling space** X_ω is the set of all tilings whose every patch appears as a "subpatch" of $\omega^k(A)$, for some $A \in \mathcal{A}$ and $k \in \mathbb{N}$.

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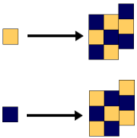
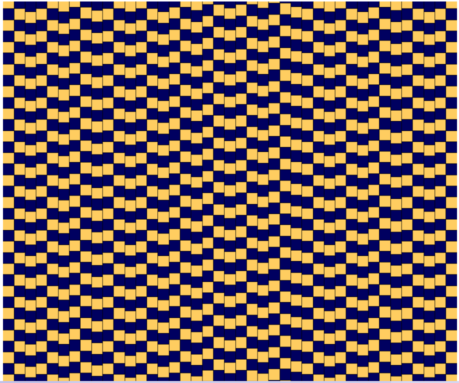
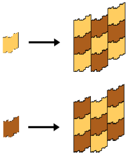
- Tile-substitution is **primitive** if the substitution matrix is primitive, that is, some power of M has only positive entries (equivalently, $\exists\, k \in \mathbb{N}$, $\forall\, i \le m$, the patch $\omega^k(A_i)$ contains tiles of all types). For a primitive tile-substitution, $X_\omega \ne \emptyset$ (in fact, there exists an ω-periodic tiling in X_ω; that is, $\omega^\ell(\mathcal{T}) = \mathcal{T}$ for some ℓ).
- The tile-substitution ω and the space X_ω are said to have **finite local complexity** (FLC) with respect to the group G if for any $R > 0$ there are finitely many patches of diameter $\le R$ in tilings of X_ω, up to the action of G.

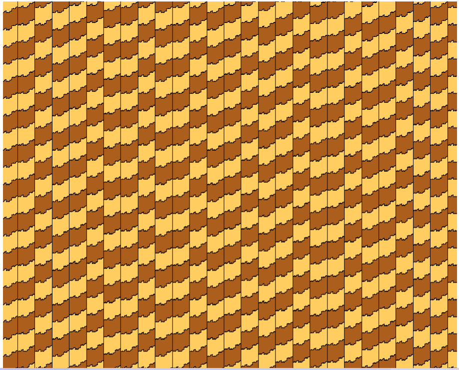
II. Example: Kenyon's non-FLC tiling space



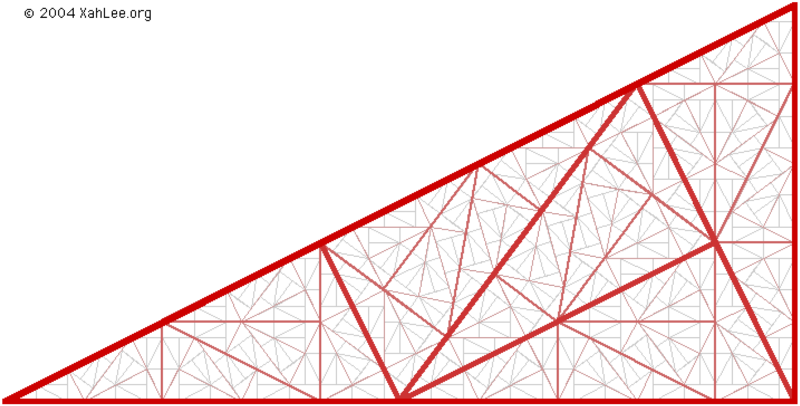


II. Example: Kenyon's non-FLC substitution tiling space





Conway-Radin pinwheel tiling



The prototiles appear in infinitely many orientations; the tiling is FLC with $G =$ Euclidean group.

Work on non-FLC tilings

- Non-FLC substitution tilings with a translationally-finite prototile set have been studied by L. Danzer, R. Kenyon, N. P. Frank-E. A. Robinson, Jr., N. P. Frank-L. Sadun, and others.

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- Non-FLC substitution tilings with a translationally-finite prototile set have been studied by L. Danzer, R. Kenyon, N. P. Frank-E. A. Robinson, Jr., N. P. Frank-L. Sadun, and others.
- Pinwheel-like tilings have been studied by C. Radin, L. Sadun, N. Ormes, and others.

II. Tiling spaces (not just substitutions)

Tiling metric in the G-finite setting: two tilings are close if after a transformation by small $g \in G$ they agree on a large ball around the origin.

More precisely (in the translationally-finite setting):

$$\widetilde{\varrho}(\mathcal{T}_1, \mathcal{T}_2) \;:=\; \inf\{r \in (0, 2^{-1/2}) : \; \exists g \in B_r(0) : \mathcal{T}_1 - g \text{ and } \mathcal{T}_2 \text{ agree on } B_{1/r}(0)\}.$$

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Tiling space: a set of tilings which is (i) closed under the translation
ction and
(ii) complete in the tiling metric. The **hull** of $\mathcal{T}$, denoted by $X_{\mathcal{T}}$, is the closure of the $\mathbb{R}^d$-orbit $\{\mathcal{T} - x : \; x \in \mathbb{R}^d\}$ in the tiling metric.

II. Local topology of the tiling space

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There is a lot of interesting work on the **algebraic** topology of tiling spaces: e.g.
[J. Anderson and I. Putnam '98], [F. Gähler '02], [A. Forrest, J. Hunton, J. Kellendonk '02], [J. Kellendonk '03], [L. Sadun '03,'07, AMS Lecture Series '08], [L. Sadun and R. Williams '03], [M. Barge and B. Diamond '08],...

II. Tiling dynamical system

Important properties of a tiling are reflected in the properties of the tiling space and the associated dynamical system:

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Theorem. *$\mathcal{T}$ has finite local complexity (FLC) $\iff$ $X_{\mathcal{T}}$ is compact.*

$\mathbb{R}^d$ acts by translations: $T^{\mathbf{t}}(\mathcal{S}) = \mathcal{S} - \mathbf{t}$. Topological dynamical system (action of $\mathbb{R}^d$ by homeomorphisms):

$$(X_{\mathcal{T}}, T^{\mathbf{t}})_{\mathbf{t}\in\mathbb{R}^d} = (X_{\mathcal{T}}, \mathbb{R}^d)$$

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Definition. A topological dynamical system is *minimal* if every orbit is dense (equivalently, if it has no nontrivial closed invariant subsets).

Theorem. *$\mathcal{T}$ is repetitive $\Longleftrightarrow$ $(X_{\mathcal{T}}, \mathbb{R}^d)$ is minimal.*

II. Substitution action and non-periodicity

Recall that the substitution ω extends to **tilings**, so we get a map

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Theorem [B. Mossé '92], [B. Sol. '98] *The map* $\omega : X_{\mathcal{T}} \to X_{\mathcal{T}}$ *is invertible iff* $\mathcal{T}$ *is non-periodic.*

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Theorem [B. Mossé '92], [B. Sol. '98] *The map $\omega : X_{\mathcal{T}} \to X_{\mathcal{T}}$ is invertible iff $\mathcal{T}$ is non-periodic.*

Useful analogy:

substitution $\mathbb{Z}$-action by ω $\sim$ geodesic flow

translation $\mathbb{R}^d$-action $\sim$ horocycle flow

II. Uniform patch frequencies

For a patch $P \subset \mathcal{T}$ consider

$$N_P(\mathcal{T}, A) := \#\{\mathbf{t} \in \mathbb{R}^d : \ -\mathbf{t} + P \text{ is a patch of } \mathcal{T} \text{ contained in } A\},$$

the number of $\mathcal{T}$-patches equivalent to P that are contained in A.

Definition. A tiling $\mathcal{T}$ has *uniform patch frequencies* (UPF) if for any non-empty patch P, the limit

$$\mathrm{freq}(P, \mathcal{T}) := \lim_{R\to\infty} \frac{N_P(\mathcal{T}, \mathbf{t} + Q_R)}{R^d} \geq 0$$

exists uniformly in $\mathbf{t} \in \mathbb{R}^d$. Here $Q_R = [-\frac{R}{2}, \frac{R}{2}]^d$.

II. Unique ergodicity for tiling systems

Theorem. *Let $\mathcal{T}$ be a tiling with FLC. Then the dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d)$ is uniquely ergodic, i.e. has a unique invariant probability measure, if and only if $\mathcal{T}$ has UPF.*

Theorem. *Let $\mathcal{T}$ be a self-affine tiling, for a primitive FLC tile-substitution. Then the dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d)$ is uniquely ergodic.*

Denote by μ the unique invariant measure.

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Note: if X_S is nonempty, but every tiling in X_S is non-periodic, we say that S is an **aperiodic set**.

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S. Mozes (1989): for any primitive aperiodic substitution ω in $\mathbb{R}^2$, with square tiles, there exists a set S and a factor map $\Phi : X_S \to X_\omega$ which is 1-to-1 outside a set of measure zero (for all translation-invariant measures).

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This was extended by C. Radin (1994) to the pinwheel tiling, and by C. Goodman-Strauss (1998) to **all** substitution tilings.

III. Diffraction spectrum of a tiling

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For self-affine tilings, the **autocorrelation measure** is well-defined and equals

$$\gamma = \sum_{z \in \Lambda - \Lambda} \nu(z) \delta_z,$$

where $\nu(z)$ is the frequency of the cluster $\{x, x+z\}$ in Λ.

III. Diffraction spectrum and dynamical spectrum

The autocorrelation γ is positive-definite, and its Fourier transform $\widehat{\gamma}$ is a measure which gives the diffraction pattern, or diffraction spectrum, of the tiling, see **[A. Hof '95]**.

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Dynamical spectrum of the tiling $\mathcal{T}$ is the spectral measure of the tiling dynamical system, or equivalently, of the group of unitary operators $\{U_{\mathbf{t}}\}_{\mathbf{t}\in\mathbb{R}^d}$, where $U_{\mathbf{t}}f(\mathcal{S}) = f(\mathcal{S} - \mathbf{t})$ for $\mathcal{S} \in X_{\mathcal{T}}$ and $f \in L^2(X_{\mathcal{T}}, \mu)$.

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Remarkably, there is a connection! **[S. Dworkin '93]**: the diffraction is (essentially) a "part" of the dynamical spectrum.

Theorem [J.-Y. Lee, R. V. Moody, B. S. '02] *The diffraction is pure point if and only if the dynamical spectrum is pure discrete, i.e. there is a basis for $L^2(X_{\mathcal{T}}, \mu)$ consisting of eigenfunctions.*

Definition. $\alpha \in \mathbb{R}^d$ is an eigenvalue for the measure-preserving $\mathbb{R}^d$-action $(X, T^{\mathbf{t}}, \mu)_{\mathbf{t}\in\mathbb{R}^d}$ if $\exists$ eigenfunction $f_\alpha \in L^2(X,\mu)$, i.e., f_α is not 0 in L^2 and for μ-a.e. $x \in X$

$$f_\alpha(T^{\mathbf{t}}x) = e^{2\pi i\langle \mathbf{t},\alpha\rangle} f_\alpha(x),\ \ \mathbf{t} \in \mathbb{R}^d.$$

Here $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbb{R}^d$.

Warning: eigenvalue is a vector! (like wave vector in physics)

III. Characterization of eigenvalues

Return vectors for the tiling:

$$\mathcal{Z}(\mathcal{T}) := \{z \in \mathbb{R}^d : \ \exists\, T, T' \in \mathcal{T},\ \ T' = T + z\}.$$

Theorem [S. 1997] *Let $\mathcal{T}$ be a non-periodic self-affine tiling with expansion map ϕ. Then the following are equivalent for $\alpha \in \mathbb{R}^d$:*
(i) *α is an eigenvalue for the measure-preserving system $(X_{\mathcal{T}}, \mathbb{R}^d, \mu)$;*
(ii) *α satisfies the condition:*

$$\lim_{n\to\infty} \langle \phi^n z, \alpha\rangle \ (mod\ 1) = 0 \ \ \text{for all } z \in \mathcal{Z}(\mathcal{T}).$$

III. When is there a discrete component of the spectrum?

Theorem [S.'07] *Let $\mathcal{T}$ be a self-similar tiling of $\mathbb{R}^d$ with a pure dilation expansion map $\mathbf{t} \mapsto \lambda\mathbf{t}$. Then the associated tiling dynamical system has non-trivial eigenvalues (equivalently, is not weakly mixing) iff λ is a Pisot number. Moreover, in this case the set of eigenvalues is relatively dense in $\mathbb{R}^d$.*

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[F. Gähler and R. Klitzing '97] have a result similar to the theorem above, in the framework of diffraction spectrum.

III. Necessity of the Pisot condition

The necessity of the Pisot condition for existence of non-trivial eigenvalues, when the expansion is pure dilation by λ, follows from the characterization of eigenvalues and the classical theorem of Pisot:

$$\langle \phi^n z, \alpha \rangle = \lambda^n \langle z, \alpha \rangle \to 0 \pmod 1, \text{ as } n \to \infty,$$

and we can always find a return vector z such that $\langle z, \alpha \rangle \neq 0$ if $\alpha \neq 0$.

III. When is there a large discrete component of the spectrum?

Theorem *Let $\mathcal{T}$ be self-affine with a diagonalizable over $\mathbb{C}$ expansion map ϕ. Suppose that all the eigenvalues of ϕ are algebraic conjugates with the same multiplicity. Then the following are equivalent:*
(i) *the set of eigenvalues of the tiling dynamical system associated with $\mathcal{T}$ is relatively dense in $\mathbb{R}^d$;*
(ii) *the spectrum of ϕ is a Pisot family: for every eigenvalue λ of ϕ and its conjugate γ, either $|\gamma| < 1$, or γ is also an eigenvalue of ϕ.*

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(i) $\Rightarrow$ (ii) was proved by **[E. A. Robinson '04]**, using the criterion for eigenvalues in **[S. '97]**.

(ii) $\Rightarrow$ (i), the more technically difficult part, is proved in **[J.-Y. Lee & S. '12]**.

III. Pure discrete spectrum

Question: when is the spectrum of a tiling (diffraction or dynamical) pure discrete?

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Pisot discrete spectrum conjecture: *A primitive irreducible symbolic substitution $\mathbb{Z}$-action (or $\mathbb{R}$-action) of Pisot type has pure discrete spectrum.*

Settled only in the 2-symbol case: **[M. Barge and B. Diamond '02]** with some contribution by **[M. Hollander, Thesis '96], [M. Hollander and B. Solomyak '03]**.

III. More on diffraction spectrum

This slide was not presented at the workshop, but should have been!

Those interested in the topic should read

[J. Lagarias, Mathematical quasicrystals and the problem of diffraction] in "Directions in Mathematical quasicrystals", CRM monograph series, Volume 13, Amer. Math. Soc., 2000.

This is a comprehensive account of the knowledge up to 2000, with a large bibliography and many open questions. Some of the open questions have been resolved in

[J.-Y. Lee and B. Solomyak, Pure point diffractive Delone sets have the Meyer property], Discrete Comput. Geom. (2008), and

[N. Lev and A. Olevskii, Quasicrystals and Poisson's summation formula], math. arXiv:1312.6884