Martingale Neural Networks for High-Dimensional Stochastic Optimal Controls

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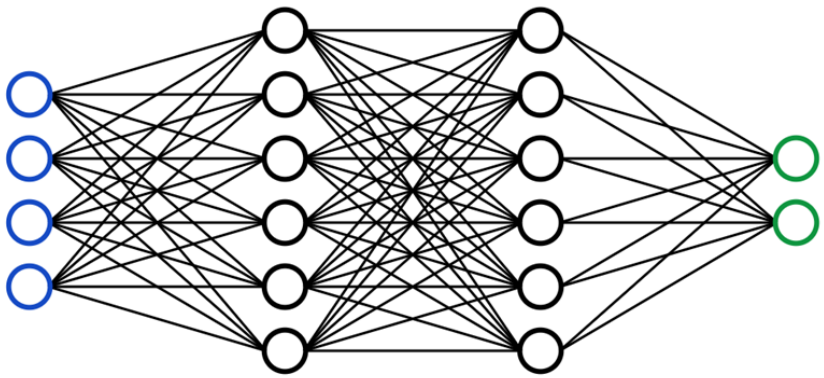
Neural network as a high-dimensional function



Outline

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State equation:

$$X_t^u = x_0 + \int_0^t \bar{\mu}\left(s, X_s^u, u(s, X_s^u)\right) \mathrm{d}s + \int_0^t \bar{\sigma}\left(s, X_s^u, u(s, X_s^u)\right) \mathrm{d}B_s, \quad t \in [0, T], \tag{1}$$

where u is the admissible **feedback control**

Cost functional:

$$J(u) := \mathbb{E}\left[\int_0^T c\left(s, X_s^u, u(s, X_s^u)\right) \mathrm{d}s + g(X_T^u)\right], \quad u \in \mathcal{U}_{\text{ad}} \tag{2}$$

with $c : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}$ and $g : \mathbb{R}^d \to \mathbb{R}$.

Stochastic optimal control problem (SOCP):

$$\text{Find } \; u^* \in \mathcal{U}_{\text{ad}} \; \text{ s.t. } \; J\left(u^*\right) = \inf_{u \in \mathcal{U}_{\text{ad}}} J\left(u\right). \tag{3}$$

Value function:

$$v(t,x) := \inf_{u \in \mathcal{U}_{\text{ad}}} \mathbb{E}\left[\int_t^T c\left(s, X_s^u, u(s, X_s^u)\right) \mathrm{d}s + g(X_T^u) \middle| X_t^u = x\right]$$

for $(t,x) \in [0,T] \times \mathbb{R}^d$.

$$\begin{cases} \partial_t v(t,x) + \inf_{\kappa \in U} H\left(t, x, \kappa, \partial_x v(t,x), \partial_{xx}^2 v(t,x)\right) = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^d, \\ v(T,x) = g(x), \quad x \in \mathbb{R}^d. \end{cases} \tag{4}$$

Classical **verification theorem** [Yong and Zhou, 1999]:
if $v \in C^{1,2}$, then the optimal control u and its trajectory $X^* := X^u$ satisfy

$$\begin{aligned} & H\left(t, X_t^*, u(t, X_t^*), \partial_x v(t, X_t^*), \partial_{xx}^2 v(t, X_t^*)\right) \\ = & \inf_{\kappa \in U} H\left(t, X_t^*, \kappa, \partial_x v(t, X_t^*), \partial_{xx}^2 v(t, X_t^*)\right), \end{aligned} \tag{5}$$

where H is the Hamiltonian given by

$$H(t,x,\kappa,z,p) := \frac{1}{2} \operatorname{Tr}\left(\bar{\sigma}\bar{\sigma}^\top(t,x,\kappa)p\right) + \bar{\mu}^\top(t,x,\kappa)z + c(t,x,\kappa) \tag{6}$$

for $(t,x,\kappa,z,p) \in [0,T] \times \mathbb{R}^d \times U \times \mathbb{R}^d \times \mathbb{R}^{d \times d}$.

The **curse of dimensionality** lies in

- High dimensionality of $x \in \mathbb{R}^d$ and $U \subset \mathbb{R}^m$;
- Optimization problem $\inf_{\kappa \in U} H$ for each (t, x).

$$\begin{cases} \partial_t v(t,x) + \inf\limits_{\kappa \in U} H\left(t, x, \kappa, \partial_x v(t,x), \partial^2_{xx} v(t,x)\right) = 0, & (t,x) \in [0,T) \times \mathbb{R}^d, \\ v(T,x) = g(x), & x \in \mathbb{R}^d. \end{cases} \tag{7}$$

HJB equations and rare event simulation

[Dupuis, Spiliopoulos, and Wang, 2012],[Vanden-Eijnden and Weare, 2012], etc

Previous neural network results on Stochastic Optimal Controls:

- **Direct optimization** using DNN [Han and E, 2016]
- **Dynamic programming**, Hybrid-now method [Huré et al., 2021].
- **SDE based methods**: Actor-critic method [Zhou, Han, and Lu, 2021]. Stochastic maximum principle (SMP) & deep learning [Ji, Peng, Peng, and Zhang, 2022], [Nüsken and Richter, 2021], [Li, Verma, , and Ruthotto, 2024].

Selected other methods for HJB equations (deterministic and stochastic controls, i.e., hyperbolic vs parabolic PDEs)

Darbon and Osher, Resr. math sci, 2016; Darbon, Meng, JCP, 2021; (SDGD) Hu, Shukla, Karniadakis, and K. Kawaguchi, Neural Networks, 2024; Nakamura-Zimmerer, Gong, Kang, SISC, 2021; Dolgov, Kalise,and Kunisch, SISC, 2021, and others

Related work I: Deep learning for the discrete time SOCP:

$$\min_{u\in\mathcal{U}_{\text{ad}}} J(u), \quad J(u) := \mathbb{E}\left[\sum_{n=0}^{N-1} c_n\left(X_n, u_n\right) + g(X_N)\right],$$
$$X_{n+1} = F(X_n, u_n, \xi_{n+1}), \quad n = 0, 1, \cdots, N-1,$$

where the function F is known; $u_n = u_n(X^n)$ is the state feedback control; $\{\xi_0, \cdots, \xi_{N-1}\}$ are i.i.d. random noise.

Direct optimization based on DNN approximations [Han and E, 2016]:

$$u_n(X_n) \approx u_n(X_n; \theta_n), \qquad \min_{(\theta_n)_{n=0}^{N-1}} \mathbb{E}\left[\sum_{n=0}^{N-1} c_n\left(X_n, u_n\left(X_n; \theta_n\right)\right) + g(X_N)\right].$$

Dynamic programming principle: Let

$$J_n(x;u) := \mathbb{E}\left[\sum_{m=n}^{N-1} c_m\left(X_m, u_m\right) + g(X_N)\middle| X_n = x\right], \quad x \in \mathbb{R}^d.$$

For the optimal control $u^* = (u_n^*)_{n=0}^{N-1}$,

$$J_n\left(x;u^*\right) = \mathbb{E}\left[c_n\left(x, u_n^*(x)\right) + J_{n+1}\left(F(x, u_n^*(x), \xi_{n+1}); u^*\right)\middle| X_n = x\right], \tag{8}$$

$$u_n^*(x) = \arg\min_{\kappa \in U} \mathbb{E}\left[c_n\left(x,\kappa\right) + J_{n+1}\left(F(x,\kappa,\xi_{n+1}); u^*\right)\middle| X_n = x\right]. \tag{9}$$

Hybrid-now method [Huré et al., 2021]: For $n = N-1, N-2, \cdots, 0$,

1. Approximate $x \mapsto u_n^*(x)$ and $x \mapsto J_n(x;u^*)$ by DNNs.
2. Train DNNs by enforcing (8) and (9).

Related work II: Actor-critic method [Zhou, Han, and Lu, 2021].

Zhou, Han, and Lu [2021] focuses the **elliptic problem**

$$\begin{cases} \gamma v(x) - \inf\limits_{\kappa \in U} \left\{ \frac{1}{2} \operatorname{Tr}\left(\bar{\sigma}\bar{\sigma}^\top(x,\kappa)\partial^2_{xx} \right) v(x) + \bar{\mu}^\top(x,\kappa)\partial_x v(x) + f(x,\kappa) \right\} = 0, \quad x \in D, \\ v(x) = g(x), \quad x \in \partial D, \end{cases}$$

where $D \subset \mathbb{R}^d$ and $\gamma > 0$. The dynamic programming reveals that

$$v(x) = \inf_{u \in \mathcal{U}_{\text{ad}}} J(x;u), \quad J(x;u) = \mathbb{E}\left[\int_0^\tau f(X_s^u, u(X_s^u)) e^{-\gamma s} \, \mathrm{d}s + e^{-\gamma\tau} g(X_\tau^u) \,\middle|\, X_0^u = x \right]$$

with $\tau := \inf\{t > 0 : X_t \notin D\}$ and

$$X_t^u = X_0^u + \int_0^t \bar{\mu}(X_s^u, u(s, X_s^u)) \, \mathrm{d}s + \int_0^t \bar{\sigma}(X_s^u, u(s, X_s^u)) \, \mathrm{d}B_s, \quad t \in [0, \infty).$$

Critic: For given $u \in \mathcal{U}_{\text{ad}}$, if $v(x) = J(x;u)$, then Itô's formula implies

$$e^{-\gamma(T\wedge\tau)}v(X^u_{T\wedge\tau}) = v(X^u_0) - \int_0^{T\wedge\tau} f(X^u_s, u(X^u_s))e^{-\gamma s}\,\mathrm{d}s + \int_0^{T\wedge\tau} \partial_x v\sigma(X^u_s, u(X^u_s))e^{-\gamma s}\,\mathrm{d}B_s,$$

which leads to the loss function for v and $z := \partial_x v$ as

$$L_1(v,z;u) := \mathbb{E}\left[\left|\int_0^{T\wedge\tau} f(X^u_s, u(X^u_s))e^{-\gamma s}\,\mathrm{d}s - \int_0^{T\wedge\tau} z\sigma(X^u_s, u(X^u_s))e^{-\gamma s}\,\mathrm{d}B_s + e^{-\gamma(T\wedge\tau)}v(X^u_{T\wedge\tau}) - v(X^u_0)\right|^2\right], \quad X^u_0 \sim \mu.$$

Actor: Given v, the optimal control u can be found by minimizing

$$\tilde{J}(u;v) := \mathbb{E}\left[\int_0^{T\wedge\tau} f(X^u_s, u(X^u_s))e^{-\gamma s}\,\mathrm{d}s + e^{-\gamma\tau}v(X^u_{T\wedge\tau})\right]. \tag{10}$$

Key point: If $v(x) = J(x;u)$ for $x \in \mathbb{R}^d$, then

$$\tilde{J}(u;v) = J(X^u_0;u) \text{ for any } T > 0.$$

Actor-critic method:

1. Introduce neural networks $(u_\alpha, v_\theta, z_\eta) \approx (u, v, \partial_x v)$.
2. **Critics:** Sample the trajectories of X^{u_α} and update

$$(\theta, \eta) \leftarrow (\theta, \eta) - \delta \nabla_{(\theta,\eta)} L_1(v_\theta, z_\eta; u_\alpha)$$

3. **Actor:** Sample the trajectories of X^{u_α} and update

$$\alpha \leftarrow \alpha - \delta \nabla_\alpha \tilde{J}(u_\alpha; v_\theta).$$

4. Repeat the steps 2 and 3 until the training converges.

Related work III: Stochastic maximum principle (SMP) & deep learning [Ji, Peng, Peng, and Zhang, 2022].

SMP: If the tuple (u^*, X^*) is the optimal control and trajectory, then the quadruple (u^*, X^*, p^*, q^*) is the solution to

$$\begin{cases} \mathrm{d}X_t^* = \partial_p H(t, X_t^*, u_t^*, -p_t^*, -q_t^*)\,\mathrm{d}t + \partial_q H(t, X_t^*, u_t^*, -p_t^*, -q_t^*)\,\mathrm{d}B_t, \\ \mathrm{d}p_t^* = \partial_x H(t, X_t^*, u_t^*, -p_t^*, -q_t^*)\mathrm{d}t + q_t^*\ \mathrm{d}B_t, \\ X_0^* = x_0, \quad p_T^* = -\partial_x g(X_T^*), \\ H(t, X_t^*, u_t^*, -p_t^*, -q_t^*) = \min\limits_{\kappa \in U} H(t, X_t^*, \kappa, -p_t^*, -q_t^*) \end{cases} \tag{11}$$

for $t \in [0, T]$, where H is the SMP Hamiltonian.

The **main idea** of Ji, Peng, Peng, and Zhang [2022]:

Algorithm 1: Apply the **DeepBSDE** [E et al., 2017, Han et al., 2018] to the FBSDEs in (11a)-(11c), where

- q_t^* is approximated by a neural network $q_\eta\,(t, X_t^*)$;
- u_t^* is solved by minimizing $H(t, X_t^*, \kappa, -p_t^*, -q_t^*)$ over $\kappa \in U$.

Algorithm 2:

- The condition (11d) is fulfilled by appending a penalty $G(\alpha)$ to the loss function:

$$G(\alpha) := \lambda \int_0^T \mathbb{E}\left[\left|\partial_u H\left(t, X_t^*, u_\alpha(t, X_t^*), -p_t^*, -q_t^*\right)\right|^2\right] \mathrm{d}t, \quad \lambda > 0.$$

  Typical requirement: u is unconstrained and $u \mapsto H$ is convex.

The trajectories of (X^*, p^*) depend on (u_α, q_η), which need to be updated at each training step.

Outline

Derivative-Free Martingale Network (SOC-MartNet)

SOC-MartNet - martingale networks for SOC problem: Using Varadhan martingale problem to convert PDEs problem to a martingale problem of SDEs; sampling SDEs paths to enforce martingale conditions.

Main Aims:

- Addressing cost of updating paths of system SDE under controls
- Avoid computing derivatives of value function $v(t,x)$, i.e. Hessian, in HJB and Optimization of Hamiltonian

SOC-MartNet: Martingale networks for SOC:

$$\boxed{\text{Parabolic PDE}} \qquad \boxed{\inf_{\kappa\in U} H\left(t,x,\kappa,\partial_x v(t,x),\partial^2_{xx} v(t,x)\right)}$$

$$\Downarrow \text{Itô's formula} \qquad \Downarrow \text{Find feedback control}$$

$$\text{Martingale problem} \qquad u(t,x) := \arg\min_{\kappa\in U} H\left(t,x,\kappa,\partial_x v(t,x),\partial^2_{xx} v(t,x)\right)$$

$$\Downarrow \mathbb{E}\left[\cdot|X_t\right] \text{ is a projection} \qquad \Downarrow \text{Neural network } u_\alpha \approx u$$

$$\boxed{\textbf{Galerkin} \text{ Adversarial learning} \;+\; \min_{\alpha} \int_0^T \mathbb{E}\left[\bar{H}(t,X_t,u_\alpha(t,X_t))\right]\,\mathrm{d}t}$$

Using the structure of H, the HJB equation can be rewritten into

$$\partial_t v(t,x) + \inf_{\kappa \in U} \{\mathcal{L}^\kappa v(t,x) + c(t,x,\kappa)\} = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^d \tag{12}$$

with

$$\mathcal{L}^\kappa := \bar{\mu}(t,x,\kappa)\partial_x + \frac{1}{2}\,\mathrm{Tr}\left\{\bar{\sigma}\bar{\sigma}^\top(t,x,\kappa)\,\partial^2_{xx}\right\}.$$

By introducing the optimal feedback control u,

$$u(t,x) := \arg\min_{\kappa \in U} \{\mathcal{L}^\kappa v(t,x) + c\,(t,x,\kappa)\}\,, \quad (t,x) \in [0,T) \times \mathbb{R}^d, \tag{13}$$

the HJB equation (12) becomes a **linear** PDE as

$$\partial_t v(t,x) + \mathcal{L}^u v(t,x) + c\,(t,x,u(t,x)) = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^d. \tag{14}$$

The linear operator $\partial_t + \mathcal{L}^u$ is involved in the Itô formula.

For the controlled state process X^u given in (1), the Itô formula leads to

$$v(t, X_t^u) = v(0, X_0) + \int_0^t (\partial_t + \mathcal{L}^u) v(s, X_s^u)\, \mathrm{d}s + \int_0^t \nabla v(s, X_s^u) \sigma\, \mathrm{d}B_s \tag{15}$$

$$= v(0, X_0) - \int_0^t c\,(s, X_s^u, u_s)\ \mathrm{d}s + \int_0^t \nabla v(s, X_s^u) \sigma\, \mathrm{d}B_s \tag{16}$$

which implies a martingale

$$\mathcal{M}_t^{u,v} :=\ v(t, X_t^u) + \int_0^t c\,(s, X_s^u, u_s)\ \mathrm{d}s \tag{17}$$

with $u_s := u(s, X_s^u)$.

Martingale formulation for (14)**:** under some usual conditions,

$$\mathcal{M}^{u,v} \text{ is a } \mathbb{F}^B\text{-martingale} \ \Leftrightarrow\ (\partial_t + \mathcal{L}^u)\, v(t, X_t^u) + c\,(t, X_t^u, u(t, X_t^u)) = 0, \ \text{ a.s..} \tag{18}$$

Increment of the Martingale and PDE residual

Using the Itô formula, we have

$$\begin{aligned}\mathcal{M}_{t+\Delta t}^{u,v}-\mathcal{M}_t^{u,v} =& v(t+h,X_{t+h}^u)-v(t,X_t^u)+\int_t^{t+h} c\left(s,X_s^u,u_s\right)\,\mathrm{d}s\\ =&\int_t^{t+h}(\partial_t+\mathcal{L}^u)v(s,X_s^u)+c\left(s,X_s^u,u_s\right)\mathrm{d}s+\int_t^{t+h}\nabla v(s,X_s^u)\sigma\,\mathrm{d}B_s\\ =&\int_t^{t+h} R(v(s,X_s^u))\,\mathrm{d}s+\int_t^{t+h}\nabla v(s,X_s^u)\sigma\,\mathrm{d}B_s,\end{aligned}$$

For sufficiently small $h>0$, we have $\Delta\mathcal{M}_h^{u,v}(t,x)$ as the Euler-scheme approximation to $\mathcal{M}_{t+h}^{u,v}-\mathcal{M}_t^{u,v}$ with $X_t^u=x$, i.e.,

$$\Delta\mathcal{M}_h^{u,v}(t,x):=v(t+h,x+\xi_h^{t,x,u})-v(t,x)+hc\left(t,x,u(t,x)\right), \tag{19}$$

$$\xi_h^{t,x,u}:=\bar{\mu}\left(t,x,u(t,x)\right)h+\bar{\sigma}\left(t,x,u(t,x)\right)\left(B_{t+h}-B_t\right). \tag{20}$$

Comment: Increment of the martingale is an random difference (material derivative along random direction) approximation of the PDE.

[1] Cai, et al, Deep random difference method for high-dim. quasilinear parabolic PDEs. J. of Comput. Phys, 2026; [2] Xu, Zhang, A deep shotgun method for solving high-dim parabolic PDEs, J. of Sci. Comput. 2025..

How to fulfill the martingale property in parallel in time?

Let X be any uncontrolled **pilot** process s.t. $\Gamma(X_t) = \Gamma(X_t^*) := \text{Support}(X_t^*)$.

The martingale property of $\mathcal{M}^{u,v}$ can be ensured by

$$\mathbb{E}\left[\left(\mathcal{M}_{t+h}^{u,v} - \mathcal{M}_t^{u,v}\right) | X_t^u = x\right] = 0, \quad x \in \Gamma(X_{t_n}^u), \quad t \in [0, T-h]. \tag{21}$$

$$\Uparrow \text{If } \Gamma(X_t^u) \subset \Gamma(X_t),$$

$$\mathbb{E}\left[\left(\mathcal{M}_{t+h}^{u,v} - \mathcal{M}_t^{u,v}\right) | X_t^u = x\right] = 0, \quad x \in \Gamma(X_t), \quad t \in [0, T-h]. \tag{22}$$

$$\Updownarrow \text{Galerkin method}$$

$$\sup_{\rho \in \mathcal{T}} \int_0^{T-h} \mathbb{E}\left[\rho(t, X_t) \Delta \mathcal{M}_h^{u,v}(t, X_t)\right] \, \mathrm{d}t = 0, \quad t \in [0, T-h], \tag{23}$$

- **Path density pdf weighted Weak form of PINN**

How to avoid computing $\mathbb{E}\left[\mathcal{M}_T^{u,v}|\mathcal{F}_t\right]$ for $t \in [0,T]$?

Using the idea of Weak Adversarial Network (WAN) [Zang, Bao, Ye, and Zhou, 2020],

$$\mathbb{E}\left[(\mathcal{M}_{t+\Delta t}^{u,v} - \mathcal{M}_t^{u,v})\middle|X_t\right] = 0, \quad (t,\omega) \in [0, T-\Delta t] \times \Omega, \ \text{a.e.}, \tag{24}$$

$$\iff \int_0^{T-\Delta t} \mathbb{E}\Big[\rho(t,X_t)\mathbb{E}\left[(\mathcal{M}_{t+\Delta t}^{u,v} - \mathcal{M}_t^{u,v})\middle|X_t\right]\Big]\,\mathrm{d}t = 0, \quad \forall \rho \in \mathcal{T} \tag{25}$$

with $\mathcal{T}$ the set of **test functions** defined by

$$\mathcal{T} := \left\{\rho : [0,T] \times \mathbb{R}^d \to \mathbb{R}\middle|\ \rho \text{ is smooth and bounded}\right\}. \tag{26}$$

Since $\rho(t, X_t)$ is $\mathcal{F}^{X_t}$-measurable,

$$(25) \iff \int_0^{T-\Delta t} \mathbb{E}\left[\mathbb{E}\left[\rho(t,X_t)(\mathcal{M}_{t+\Delta t}^{u,v} - \mathcal{M}_t^{u,v})\middle|X_t\right]\right]\,\mathrm{d}t = 0, \quad \forall \rho \in \mathcal{T}, \tag{27}$$

$$\iff \int_0^{T-\Delta t} \mathbb{E}\left[\rho(t,X_t)(\mathcal{M}_{t+\Delta t}^{u,v} - \mathcal{M}_t^{u,v})\right]\,\mathrm{d}t = 0, \quad \forall \rho \in \mathcal{T}, \tag{28}$$

$$\iff \sup_{\rho\in\mathcal{T}}\left|\int_0^{T-\Delta t} \mathbb{E}\left[\rho(t,X_t)\left(\mathcal{M}_{t+\Delta t}^{u,v} - \mathcal{M}_t^{u,v}\right)\right]\,\mathrm{d}t\right|^2 = 0. \tag{29}$$

How to efficiently compute $\inf_{\kappa\in U} H\left(t,x,\kappa,\partial_x v(t,x),\partial^2_{xx} v(t,x)\right)$**?**

It is sufficient to pursue

$$u(t,X_t) = \arg\min_{\kappa\in U} H\left(t,X_t,\kappa,\partial_x v(t,X_t),\partial^2_{xx} v(t,X_t)\right), \quad (t,\omega)\in[0,T]\times\Omega \ \text{ a.e.}, \tag{30}$$

$$\Updownarrow$$

$$u = \arg\min_{\bar{u}:[0,T]\times\mathbb{R}^d\to U} \int_0^T \mathbb{E}\left[H\left(t,X_t,\bar{u}(t,X_t),\partial_x v(t,X_t),\partial^2_{xx} v(t,X_t)\right)\right] \mathrm{d}t,$$

for $\Gamma(X_t)\supset\Gamma(X_t^*) := \text{Support}(X_t^*)$.

Theorem 1

An optimal feedback control u and the value function v satisfying the HJB equation for $t \in [0, T]$ and $x \in \Gamma(X_t)$, can be found by the following two conditions,

$$\boxed{\mathcal{M}_t^{u,v} = \mathbb{E}\left[\mathcal{M}_T^{u,v} | \mathcal{F}_t\right], \quad t \in [0, T],} \tag{31}$$

$$\boxed{\int_0^T \mathbb{E}\left[H_t^{u,v}\right] \mathrm{d}t = \inf_{\bar{u} \in \mathcal{U}_{\mathrm{ad}}} \int_0^T \mathbb{E}\left[H_t^{\bar{u},v}\right] \mathrm{d}t,} \tag{32}$$

How to find the optimal feedback control without computing derivatives in Hamiltonian?

$$\textbf{(optimal control)} \quad \min_{u\in\mathcal{U}_{\text{ad}}} \int_0^T \mathbb{E}\left[\mathcal{L}^u v(t,X_t) + c\left(t,X_t,u(t,X_t)\right)\right] \mathrm{d}t. \tag{33}$$

However,

$$\mathbb{E}\left[\Delta\mathcal{M}_h^{u,v}\left(t,X_t\right)\right] = h\left\{(\partial_t + \mathcal{L}^u)v(t,X_t) + c(t,X_t,u(t,X_t))\right\} + O(h^2)$$

Thus

$$\min_{u\in\mathcal{U}_{\text{ad}}} \int_0^{T-h} \mathbb{E}\left[h^{-1}\Delta\mathcal{M}_h^{u,v}\left(t,X_t\right) - \partial_t v(t,X_t)\right] \mathrm{d}t.$$

i.e. the optimal control can be found from

$$\min_{u\in\mathcal{U}_{\text{ad}}} \mathbb{G}(u,v,1) \tag{34}$$

where

$$\mathbb{G}(u,v,\rho) := \int_0^{T-h} \mathbb{E}\left[\rho(t,X_t)\Delta\mathcal{M}_h^{u,v}\left(t,X_t\right)\right] \mathrm{d}t \tag{35}$$

Derivative-Free SOC-MartNet:

$$(u, v) = \lim_{\lambda \to +\infty} \underset{(\bar{u}, \bar{v}) \in \mathcal{U}_{\text{ad}} \times \mathcal{V}}{\arg\min} \left\{ \sup_{\rho \in \mathcal{T}} \mathbb{L}(\bar{u}, \bar{v}, \rho, \lambda) \right\}, \tag{36}$$

where

$$\begin{aligned} \mathcal{V} &:= \left\{ v : [0, T] \times \mathbb{R}^d \to \mathbb{R} \middle| \; v \in C^{1,2}, \; v(T, x) = g(x), \; \forall x \in \mathbb{R}^d \right\}, \\ \mathbb{L}(u, v, \rho, \lambda) &:= |\mathbb{G}(u, v, 1)|^2 + \lambda |\mathbb{G}(u, v, \rho)|^2, \\ \mathbb{G}(u, v, \rho) &:= \int_0^{T-h} \mathbb{E}\left[\rho(t, X_t) \Delta \mathcal{M}_h^{u,v}(t, X_t) \right] \mathrm{d}t \end{aligned} \tag{37}$$

with $\Delta \mathcal{M}_h^{u,v}(t, x)$ the Euler approximation to $\mathcal{M}_{t+h}^{u,v} - \mathcal{M}_t^{u,v}$ with $X_t^u = x$, i.e.,

$$\Delta \mathcal{M}_h^{u,v}(t, x) := v(t + h, x + \xi_h^{t,x,u}) - v(t, x) + hc(t, x, u(t, x)) \tag{38}$$

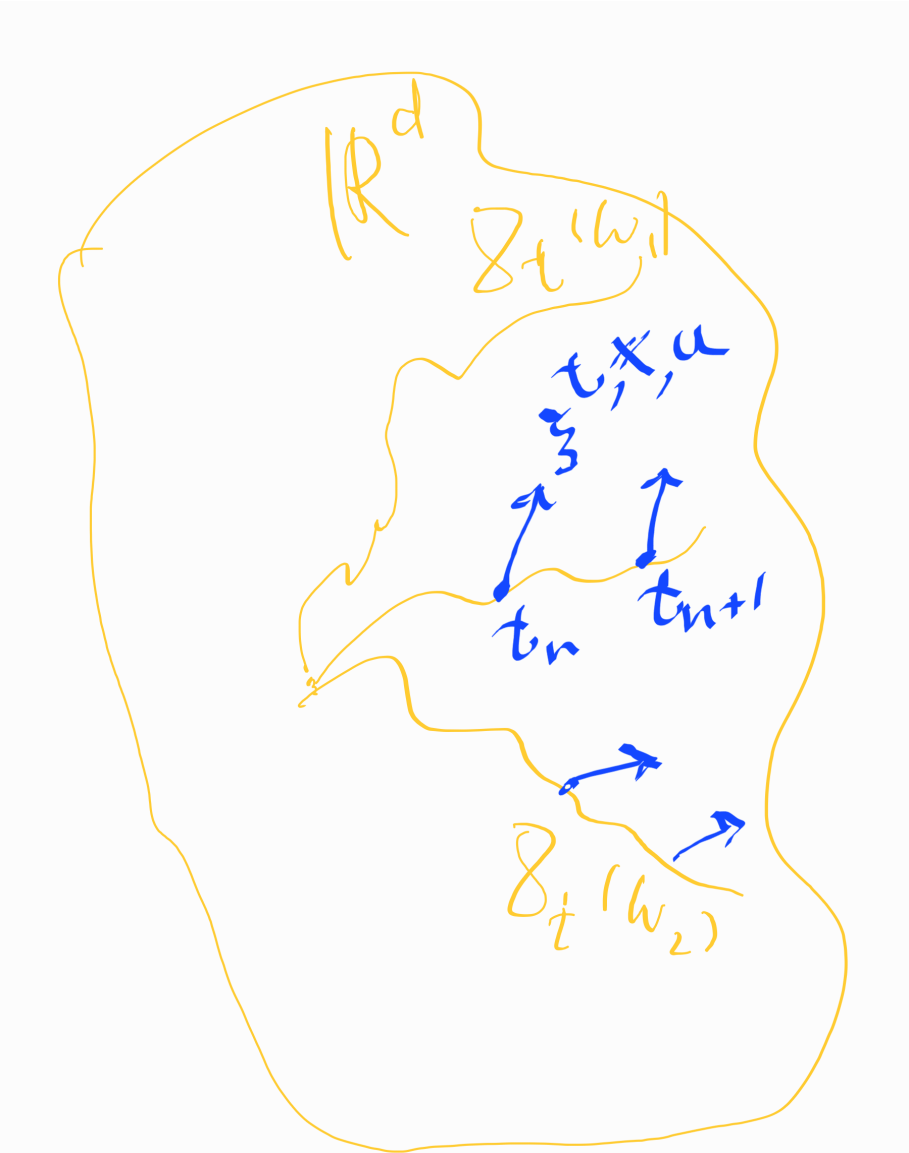
$$\xi_h^{t,x,u} := \bar{\mu}(t, x, u(t, x)) h + \bar{\sigma}(t, x, u(t, x)) (B_{t+h} - B_t). \tag{39}$$

The functions (u, v, ρ) are approximated by neural networks, i.e,

$$(u, v, \rho) \approx (u_\alpha, v_\theta, \rho_\eta) \tag{40}$$

with α, θ and η the parameters.

Schematics of DF SOC-Martnet - Random jumps under control along Pilot Paths



Error bounds for linear PDEs

We consider

$$\begin{cases} \mathcal{D}v(t,x) = f\big(t,x,v(t,x)\big), & (t,x) \in [0,T) \times \mathbb{R}^d, \\ v(T,x) = g(x), & x \in \mathbb{R}^d. \end{cases}$$

Assumptions:

1. Lipschitz continuity of f: $\exists\, C_0 > 0$ s.t.

$$|f(t,x,y_1) - f(t,x,y_2)| \le C_0 \, |y_1 - y_2|, \quad \forall y_1, y_2 \in \mathbb{R}, \quad (t,x) \in [0,T] \times \mathbb{R}^d.$$

2. Linearity of $\mathcal{D}$:

$$\mathcal{D} = \partial_t + \mu^\top \partial_x + \frac{1}{2} \operatorname{Tr}\left\{\sigma\sigma^\top \partial^2_{xx}\right\}, \tag{41}$$

where $\mu = \mu\big(t,x\big)$ and $\sigma = \sigma\big(t,x\big)$ are **independent of** $v(t,x)$.

Error metric: At each time step t_n, define the mean squared error by

$$\mathbb{M}_n^2 \left[v - v_\theta\right] := \int_{\mathbb{R}^d} \left|v(t_n,x) - v_\theta(t_n,x)\right|^2 \mathcal{P}_{X_n}(x) \, \mathrm{d}x, \tag{42}$$

where $x \mapsto \mathcal{P}_{X_n}(x)$ is the probability density function of X_n generated by $\mathcal{D}$, i.e.,

$$X_{n+1} = X_n + \mu(t_n, X_n)\, h + \sigma(t_n, X_n)\, \sqrt{h}\, \xi_{n+1}, \quad \xi_{n+1} \sim \mathrm{N}(0, I_q). \tag{43}$$

Theorem 2 (Cai-Fang-Zhou 2026)

Suppose Assumptions 1-2 hold, $v \in C_b^{2,4}$, *and* $0 < h < \min\{1, (2\sqrt{2}C_0)^{-1}\}$. *Then*

$$\max_{0\le n\le N-1} \mathbb{M}_n^2 \left[v - v_\theta\right] \le C\left(\mathbb{M}_N^2 \left[v - v_\theta\right] + L_{\mathrm{rdm},\pi}(v_\theta) + h^2\right) \tag{44}$$

for some constant $C > 0$ *independent of the time step size* h, *where*

- $\mathbb{M}_N^2 \left[v - v_\theta\right]$ *is due to the residual of the terminal condition;*
- $L_{\mathrm{rdm},\pi}(v_\theta)$ *is induced by the residual of the Random difference method, i.e.,*

$$L_{\mathrm{rdm},\pi}(v_\theta) := \frac{T}{N} \sum_{n=0}^{N-1} \int_{\mathbb{R}^d} \left|\xi R(t_n, x, \xi; v_\theta)\right|^2 \mathcal{P}_{X_n}(x)\, \mathrm{d}x; \tag{45}$$

- h^2 *is due to the Random difference method approximation* $\mathcal{D}_h v(t,x) = \mathcal{D}v(t,x) + \mathcal{O}(h)$.

• **Cai W, Fang S, Zhou T. (2026)** Deep random difference method for high dimensional quasilinear parabolic PDEs. J. of Comput. Phys, Vol. 555, 114767.

The derivative free SOC-MartNet (36) enjoys the following features:

- Its loss function (37) is free of computing $\partial_x v$ and $\partial^2_{xx} v$.
- The pilot diffusion process X for exploration is not necessarily dependent on $\mathcal{L}v(t,x)$, providing more flexibility to design the sample distribution.

The price of the derivative-free formulation:

- The random jumps $\xi_h^{t,x,u}$ in (38) need to be updated along with the optimal control, which can not be pre-calculated before the training as in the original SOC-MartNet [Cai et al, SISC, 2025].

Outline

Time partition:

$$\pi_N := \{t_0, t_1, \cdots, t_N\} \text{ s.t. } 0 = t_0 < t_1 < t_2 < \cdots < t_n < t_{n+1} < \cdots < t_N = T. \quad (46)$$

$$\Delta t_n := t_{n+1} - t_n, \quad \Delta B_{n+1} := B_{n+1} - B_n.$$

Numerical approximations:

1. Apply the **Euler** approximation to pilot paths $(X_{t_n})_{n=0}^N$:

$$X_{n+1} = X_n + \mu(t_n, X_n)\Delta t_n + \sigma(t_n, X_n)\Delta B_{n+1}, \quad n = 0, 1, 2, \cdots, N-1. \quad (47)$$

2. Apply the **trapezoid formula** to the time integral $\int_{t_n}^{t_{n+1}} H_s^{u,v}\, \mathrm{d}s$:

$$\mathcal{M}_{t_{n+1}}^{u_\alpha, v_\theta} - \mathcal{M}_{t_n}^{u_\alpha, v_\theta} \approx \Delta\mathcal{M}_{n+1}^{\alpha,\theta}, \quad n = 0, 1, \cdots, N-1,$$

$$\Delta\mathcal{M}_{n+1}^{\alpha,\theta} := v_\theta(t_{n+1}, \mathbf{X}_{\mathbf{n+1}}^{\mathbf{u}_\alpha}) - v_\theta(t_n, X_n) + \frac{1}{2}\left(c_n^{\alpha,\theta} + c_{n+1}^{\alpha,\theta}\right)\Delta t_n, \quad (48)$$

$$c_n^{\alpha,\theta} := c\left(t_n, X_n, u_\alpha(t_n, X_n)\right), \quad c_{n+1}^{\alpha,\theta} := c\left(t_{n+1}, \mathbf{X}_{\mathbf{n+1}}^{\mathbf{u}_\alpha}, u_\alpha(t_{n+1}, \mathbf{X}_{\mathbf{n+1}}^{\mathbf{u}_\alpha})\right) \quad (49)$$

3. Approximate $\mathbb{E}[\cdot]$ by the **Monte-Carlo** method based on the i.i.d. samples

$$\{(X_n^{(m)}, c_n^{\alpha,\theta,(m)}, c_{n+1}^{\alpha,\theta,(m)}, \Delta\mathcal{M}_n^{\alpha,\theta,(m)})\}_{n=0}^N, \quad m = 1, 2, \cdots, M. \quad (50)$$

Mini-batch loss function:

$$L(\alpha, \theta, \eta, \lambda; A_1, A_2) := |G(\alpha, \theta, 1; A_1)|^2 + \lambda |G(\alpha, \theta, \eta; A_2)|^2 , \tag{51}$$

$$G(\alpha, \theta, \eta; A) := \frac{1}{|A|} \sum_{(n,m)\in A} \rho_\eta(t_n, X_n^{(m)}) \Delta \mathcal{M}_{n+1}^{\alpha,\theta,(m)} \Delta t_n, \tag{52}$$

where $A \subset \{0, 1, \cdots, N\} \times \{1, 2, \cdots, M\}$ is updated at each optimization step.

High efficiency comes from

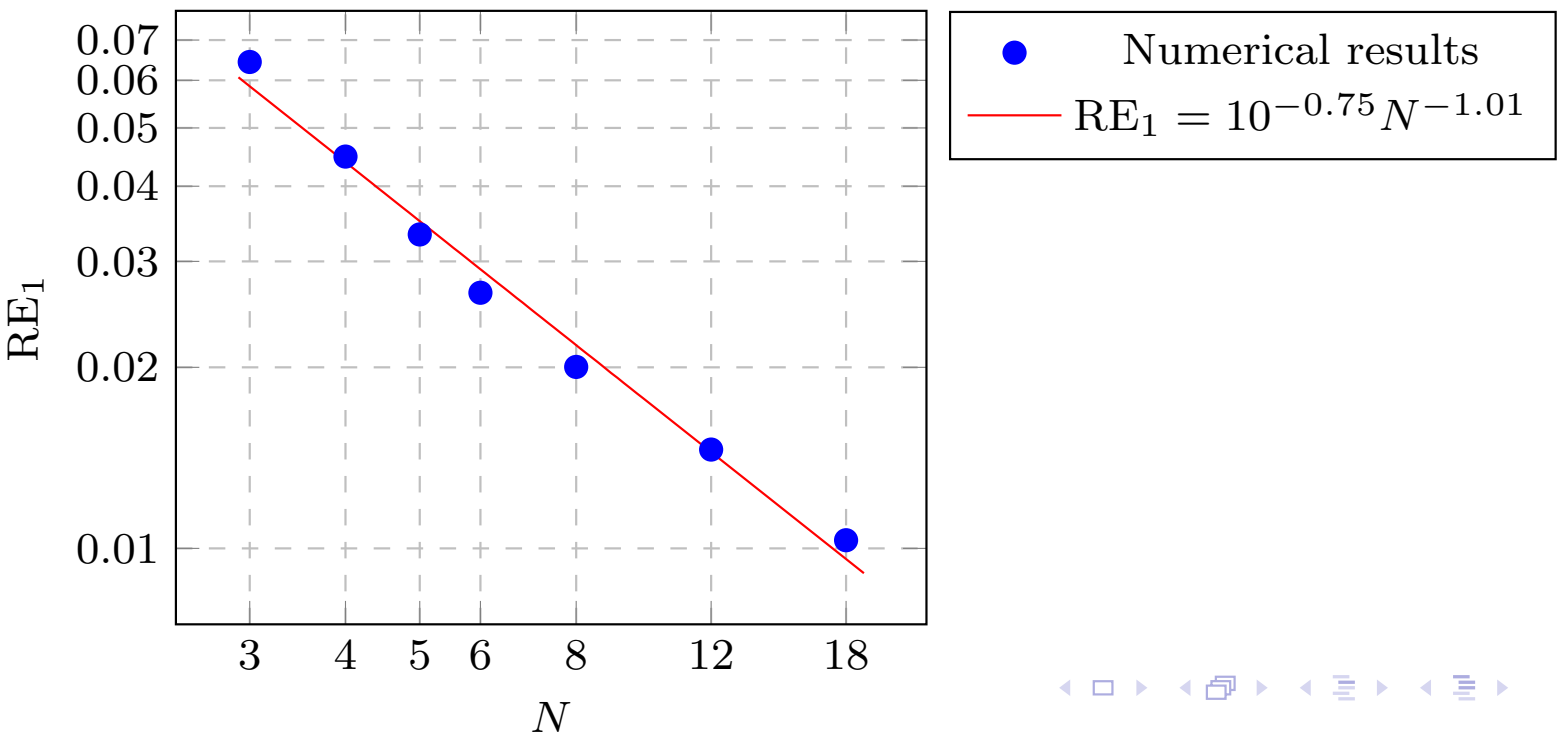
- **Offline** computation for pilot paths $\{X_n^{(m)}\}_{n=0}^N$,
- **Parallel** computation for $\sum_{(n,m)\in A} \cdots$.

We consider an Allen-Cahn-type equation

$$\partial_t v(t,x) + \mathcal{L}v(t,x) + f\left(t, x, v(t,x), \partial_x v(t,x), \partial^2_{xx} v(t,x)\right) = 0 \tag{53}$$

$$\mathcal{L} = \sum_{i=1}^{d} \sin(2x_i)\partial_{x_i} + \frac{1}{2}\sum_{i=1}^{d} (1 + 0.5\sin(5t + x_i))^2 \partial^2_{x_i}, d = 100 \tag{54}$$

$$f(t, x, v, \partial_x v, \partial^2_{xx} v) = v - v^3 + \bar{f}(t, x),$$



HJB equation [Bachouch et al., 2022, Section 3.1]:

$$\begin{cases} (\partial_t + \Delta_x)\, v(t,x) + \inf\limits_{\kappa \in \mathbb{R}^d} \left(2\kappa^\top \partial_x v(t,x) + |\kappa|^2 \right) = 0, \quad (t,x) \in [0,T) \times \mathbb{R}^d, \\ v(T,x) = 1 + g(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{55}$$

The HJB equation (55) is associated with the SOCP:

$$u = \underset{\kappa \in \mathcal{U}_{\text{ad}}}{\arg\min}\, J(\kappa), \quad J(\kappa) := 1 + \mathbb{E}\left[\int_0^T |\kappa_s|^2 \, \mathrm{d}s + g\left(X_T^\kappa\right) \right], \tag{56}$$

$$\mathcal{U}_{\text{ad}} = \left\{ \kappa : [0,T] \times \Omega \to \mathbb{R}^d : \kappa \text{ is } \mathbb{F}^B\text{-adapted} \right\}, \tag{57}$$

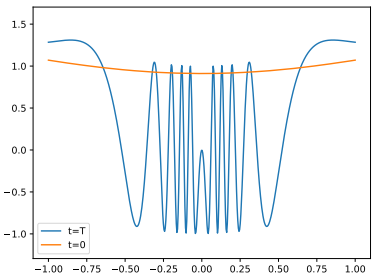
$$X_t^\kappa = X_0 + \int_0^t 2\kappa_s \, \mathrm{d}s + \int_0^t \sqrt{2}\, \mathrm{d}B_s, \quad t \in [0,T]. \tag{58}$$

In (55), $\inf_{\kappa \in \mathbb{R}^d}(\cdots)$ is obtained by enforcing

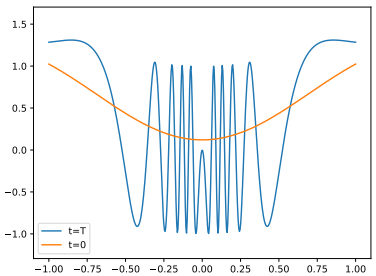
$$\int_0^T \mathbb{E}\left[H_t^{u,v} \right] \mathrm{d}t = \inf_{\bar{u} \in \mathcal{U}_{\text{ad}}} \int_0^T \mathbb{E}\left[H_t^{\bar{u},v} \right] \mathrm{d}t. \tag{59}$$

Consider the HJB equation (55) with an oscillatory terminal function

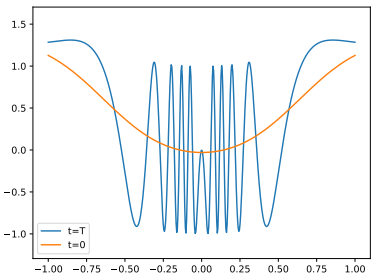
$$g(x) := \frac{1}{d}\sum_{i=1}^{d}\left\{\sin(x_i - \frac{\pi}{2}) + \sin\left(\left(\delta + x_i^2\right)^{-1}\right)\right\}, \quad x \in \mathbb{R}^d, \quad \delta = \pi/10. \tag{60}$$



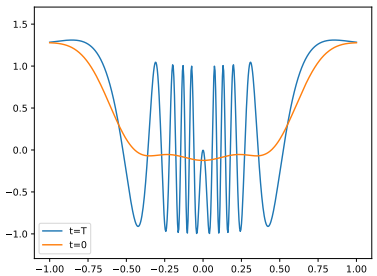
(a) $T = 1$



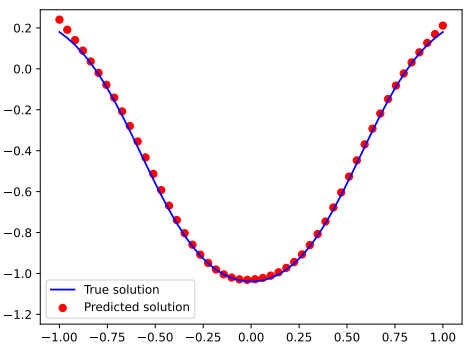
(b) $T = 0.1$



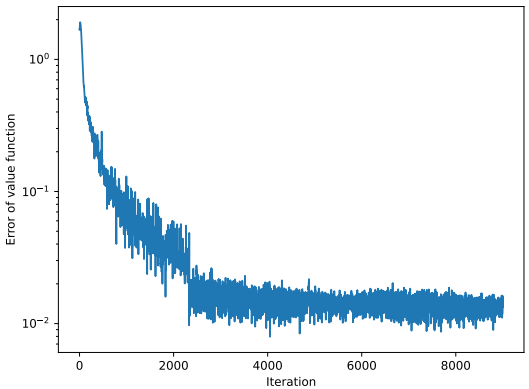
(c) $T = 0.05$



(d) $T = 0.01$



(e)



(f)

Figure 2: Numerical results of derivative-free SOC-MartNet for the $d = 10,000$ HJB equation (55) with oscillatory terminal function (60). The running time is 2814 seconds.

HJB equation without explicit $\inf_u H$

We consider the following HJB equation:

$$\partial_t v + \inf_{\kappa \in \mathbb{R}^d} \left\{ (b+\kappa)^\top \partial_x v + \frac{1}{2}|\kappa|^2 + \epsilon \sin(\mathbf{1}_d^\top \kappa) \right\} + \frac{1}{8}\Delta_x v = 0 \tag{61}$$

When $\epsilon = 0$, the value function $v(t,x)$ admits a closed-form solution given by

$$v(t,x) = -\frac{1}{4} \ln \left(\mathbb{E}\left[\exp\left(-4v(T, X_T^0)\right) \middle| X_t^0 = x \right] \right) \tag{62}$$

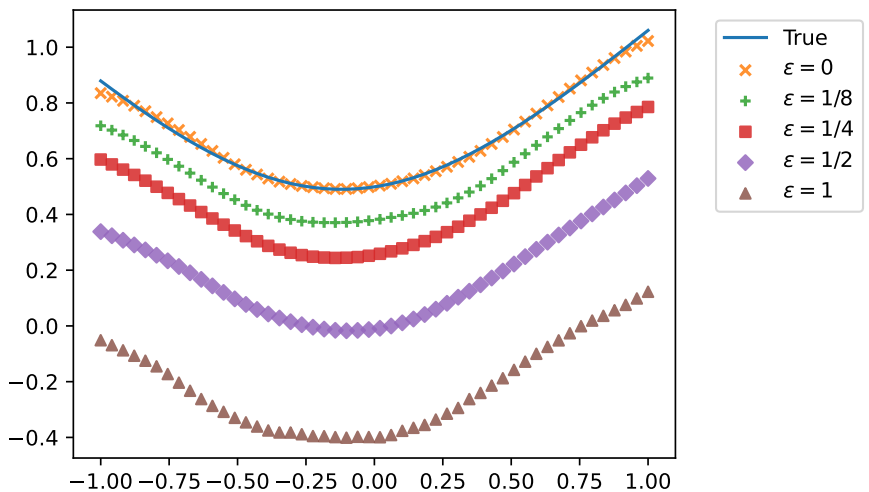


Figure 3: Numerical results of $\mathrm{alg}_a mnet for s \mapsto v(0, s\mathbf{1}_d)$, $s \in [-1,1]$ from (61) with $d = 1,000$ and $\varepsilon = 1, 1/2, 1/4, 1/8, 0$ (from bottom to top). The top blue curve without markers represents the reference solution of (61) with $\varepsilon = 0$.

Convergence in control (LQ Control Problem)

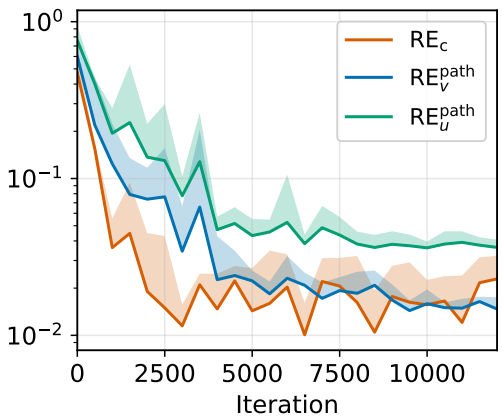
A LQ control problem (i.e.,$c(t,x,\kappa)=\kappa^2$) with the drift and diffusion coefficients

$$\mu_i(t,x,\kappa)=x_{i+1}-x_i+2q_i\kappa,\quad \sigma(t,x,\kappa)=\sqrt{2}QQ^\top,\quad (t,x,\kappa)\in[0,T]\times\mathbb{R}^d\times U,$$
$$x_{d+1}:=x_1,\quad q:=(q_i)_{i=1}^d=0.5\times(\sqrt{2},\sqrt{2},0,0,\cdots,0)^\top\in\mathbb{R}^d,\quad T=1,$$

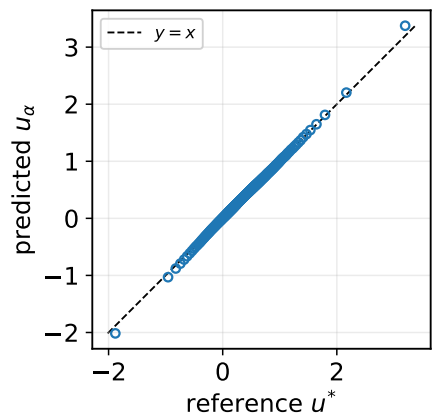
where the optimal feedback control and the value function are given by

$$u^*(t,x)=-q^\top\nabla v(t,x),\quad v(t,x)=\frac{1}{2}\log\left(1+C_{T-t}\right)+\frac{\left(q^\top e^{(T-t)L}x-1\right)^2}{2\left(1+C_{T-t}\right)},\qquad (63)$$

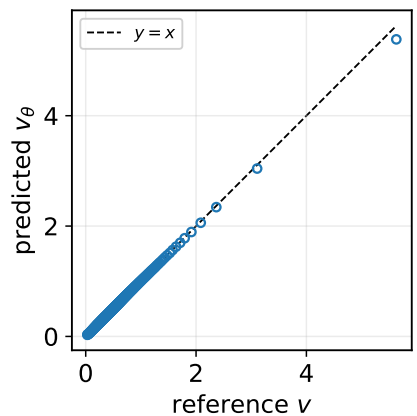
where $C_{T-t}:=2\int_0^{T-t}|q^\top e^{rL}Q|^2\,\mathrm{d}r$



(a) REs vs Iter.



(b) QQ plot of u_α vs u^*



(c) QQ plot of v_θ vs v

Figure 4: Results for the SOCP in $\mathrm{sec}_k 1 pairwith d = 10^4$. In (a), shading denotes the five-run mean plus twice the SD. In (b) and (c), the quantile-quantile (QQ) plots compare u_α and v_θ.

$d = 10^5$ dim. quasilinear PDEs with Allen-Cahn-type source term

$$\mathcal{D}v(t,x) = f(t,x,v(t,x)), \quad (t,x) \in [0,T] \times \mathbb{R}^d \tag{64}$$

$$\mathcal{D} = \partial_t + \frac{v^2}{2}\sum_{i=1}^{d} \partial_{x_i}^2, \quad f(t,x,v) = v - v^3 + Q(t,x); \tag{65}$$

$Q(t,x)$ and the terminal conditions $v(T,x) = g(x)$ are chosen for an exact solution

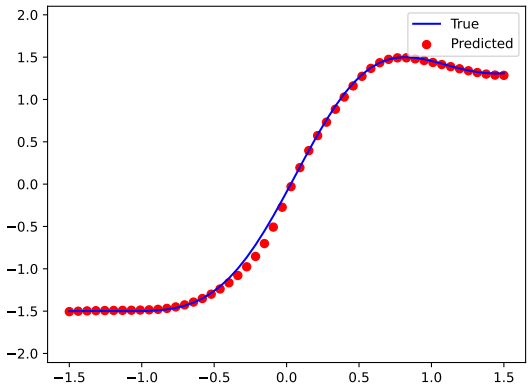
$$v(t,x) = V\big((t-0.5)\mathbf{1}_d + x\big), \quad (t,x) \in [0,T] \times \mathbb{R}^d, \tag{66}$$

where $\mathbf{1}_d := (1,1,\cdots,1)^\top \in \mathbb{R}^d$, and V is given by

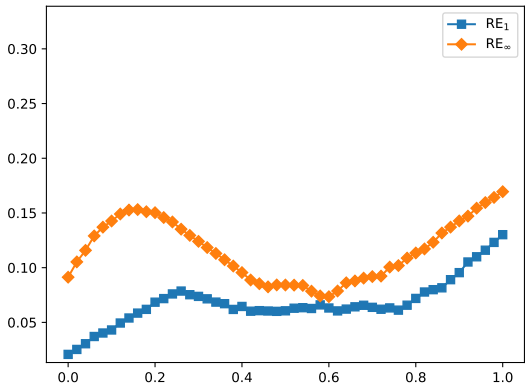
$$V(x) := \sum_{i=1}^{d-1} c_i K(x_i, x_{i+1}) + c_d K(x_d, x_1) \tag{67}$$

with

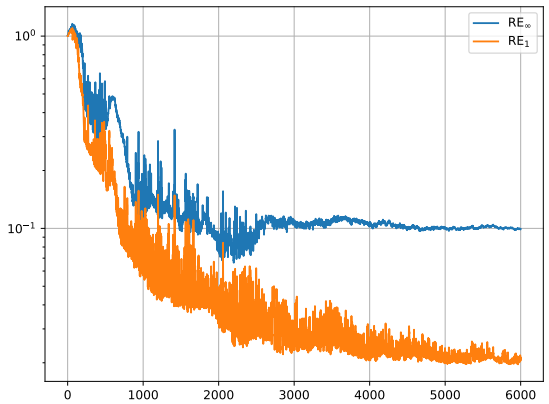
$$c_i := (1.5 - \cos(i\pi/d))/d, \quad K(x_i,x_j) := \sin\big(x_i + \cos(x_j) + x_j\cos(x_i)\big).$$



(a) $s \mapsto v(0, s\mathbf{1}_d)$



(b) $t \mapsto \mathrm{RE}(t)$



(c) RE vs Iter.

Figure 5: Numerical results with $d = 10^5$. The network width for v_θ is set to $W = 10010$, and the number of time steps is $N = 32$. The runtime is 3112 seconds at the 6000-th iteration.

W. Cai, S. Fang, T. Zhou. Deep random difference method for high dimensional quasilinear parabolic partial differential equations. J. of Comput. Phys, Vol. 555, 2026, 114767.

Outline

Table 1: Comparison of PDE and SDE-based and hybrid DNNs for solving high-dimensional quasi-linear parabolic equation.

DNN sol. $u_\theta(t, \mathbf{x})$	**Strategy A: PDE based DNNs**	**Strategy B: SDE based DNNs**	**Strategy C: Random differences**
Math. behind the loss	PDE residuals at $\mathbf{t} = \{t_j\}, \mathbf{X} = \{\mathbf{x}_i\}$, $R(u_\theta, \mathbf{t}, \mathbf{X})$ and BC	Probabilistic form for PDE solution and BC	Itô formula (**??**) applied on diffusion paths and BC
Sampling strategy	Monte Carlo sampling of domain by a given distribution or SDEs	Random walk by stochastic systems with the PDE operator generator	Random walk by stochastic systems with the PDE operator generator
∇, $\nabla\nabla^T$	SDGD, HTE, RS	Itô formula	Itô formula
Training aims	Reducing PDE residuals of DNNs at sampled locations in strong form in l_2 or l_∞ $\min_\theta \|\|R(u_\theta, \mathbf{t}, \mathbf{X})\|\|$	Enforcing the nonlinear Feynman-Kac formula for PDE solution and FBSDEs $u_\theta(t, X_t) = Y_t$, $\nabla u_\theta(t, X_t) = Z_t$	Reducing the PDE residual of neural network in $\pi(X_t)$-weighted Galerkin weak form $\min_\theta \max_\eta$ $(\rho_\eta, R(u_\theta, t, X_t))_{\pi(X_t)}$

Pros	Parallel, any PDEs, error estimate for elliptic PDEs, SDGD-PINN/RS-PINN for derivatives	Hessian matrix-free, system path-importance sampling, parallel between SDE paths	Total derivative-free, path-importance sampling, parallel within each path, first order accuracy in Δt, reduced loss variance.
Cons	Simple PINNs require derivatives, lack sampling distribution	need paths, not parallel within a path, only parabolic PDEs	need paths, test network ρ_η, min-max saddle point, only parabolic PDEs
Rep. works	PINN, DeepRitz, DGM, SDGD, HTE, RS-PINN	DeepBSDE, FBSNN, Actor-Critic, Diffusion, Feynman-Kac, Picard Iter, Diffusion MC	Weak form using test function: DeepMartNet, DRDM and, strong PINN-form: Shotgun, DGM, RS-PINN

• **W.Z. Zhang, Z.Y. Hu, W. Cai, G. Karniadakis, Deep Neural networks for solving high-dimensional parabolic partial differential equations, arXiv:2601.13256, January, 2026**.

Conclusions and future work

Conclusions:

$$\left.\begin{array}{l}\text{Martingale Formulation}\\ \text{Adversarial Learning}\\ \text{Maximum Principle}\end{array}\right\} \Longrightarrow \boxed{\text{SOC-MartNet}} \Longrightarrow \left\{\begin{array}{l}\text{SOCP}\\ \text{HJB equation}\\ \text{Parabolic equation}\end{array}\right.$$

- **Desirable features:** high efficiency and ultra-high dimensional applicability.

Future works:

- Extension to the degenerated HJB equation without classical solution $v \in C^{1,2}$;
- Theoretical analysis for SOCP (random difference operator approach).
- Rare events and committer functions.

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Thank you!