Classical codes in the simplex and their use in quantum

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joint works with Arda Aydin, Dor Elimelech, Victor Albert

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Overview

Classical codes are often useful for the construction of quantum codes

- ▷ Linear self-orthogonal codes over fields
- ▷ Expander-based codes
- ▷ Products of classical codes
- ▷ Codes in the discrete simplex
- ▷ `https://errorcorrectionzoo.org`

This line of work:

▷ we study codes in discrete simplices and their use for exact and approximate QEC for Fock (number) state codes (works with ARDA AYDIN and VICTOR ALBERT; DOR ELIMELECH and ARDA AYDIN)

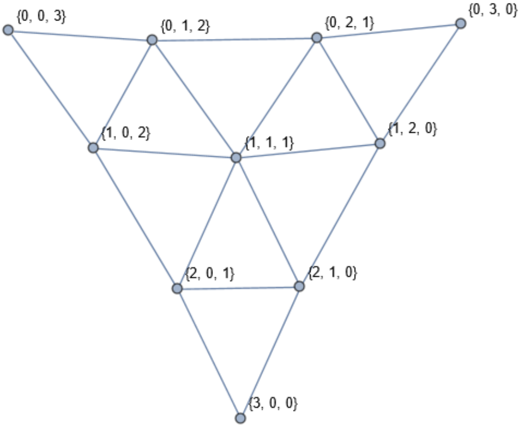
▷ we analyze AQEC in general (ongoing work with DOR ELIMELECH and VICTOR ALBERT).

Plan of the talk

- Construction of Fock state codes from ℓ_1 codes
- Constructions of ℓ_1 codes: algebraic and randomized
- Asymptotically good Fock state codes
- AQEC with Fock state codes
- Asymptotically good qudit PI codes

Simplices

For q, N consider q-tuples $\underline{n} = (n_0, n_1, \ldots, n_{q-1}) \in (\mathbb{Z}_0)^q$ with $\|\underline{n}\|_1 = N$



$\mathcal{S}_{3,3}$

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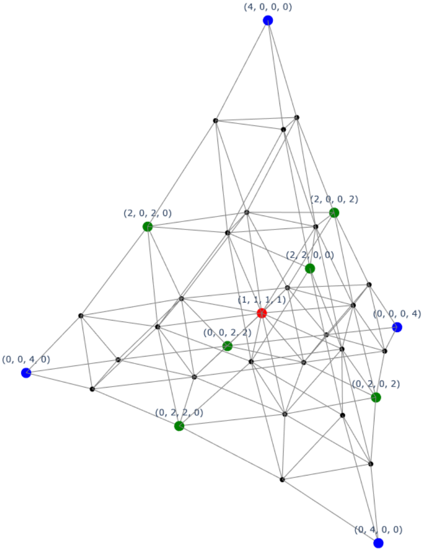


$\mathcal{S}_{4,4}$

Fock state codes

- Let a be the excitation (photon) annihilation operator.

$$a\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle, \quad a^{\dagger}\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle$$

- The single-mode Fock (number) states $\{\left|n\right\rangle\}_{n\geqslant 0}$ are defined by the counting (number) operator of which they are eigenstates:

$$(a^{\dagger}a)\left|n\right\rangle = n\left|n\right\rangle$$

- Let $\underline{n} \in \mathcal{S}_{q,N}$ and define a basis of the q-mode Fock states

$$\left|\underline{n}\right\rangle_b := \left|n_0\right\rangle_b \otimes \left|n_1\right\rangle_b \otimes \ldots \otimes \left|n_{q-1}\right\rangle_b ,$$

where each $\left|n_i\right\rangle_b$ is a single-mode Fock state.

From codes in $\mathcal{S}_{q,N}$ to Fock state codes

We consider a bosonic quantum system with q modes, each associated with the Hilbert space $\mathcal{H}_{\text{Fock}} = \ell^2(\mathbb{N})$, spanned by the number states $\{|n\rangle\}_{n\in\mathbb{Z}_0}$. Joint space of the q-mode system:

$$\mathcal{H}_q \triangleq \bigotimes_{i\in[q]} \mathcal{H}_{\text{Fock}} = \text{Span}\Big\{ |\underline{n}\rangle = |n_0\rangle \otimes \cdots \otimes |n_{q-1}\rangle \ : \ \underline{n} \in \mathbb{Z}_0^q \Big\}.$$

- ▷ The total excitation of a number state $|\underline{n}\rangle \in \mathcal{H}_q$ is given by $N = \sum_{i=0}^{q-1} n_i$.
- ▷ q-mode number states $|\underline{n}\rangle$ with constant excitation N are in one-to-one correspondence with vertices of the simplex $\mathcal{S}_{q,N}$

A K-dimensional q-mode Fock state code Q_F with total excitation N is a $\mathbb{C}$-linear space with the basis

$$|\boldsymbol{c}_i\rangle = \sum_{\underline{n}\in\mathcal{S}_{q,N}} \alpha_{\underline{n}}^{(i)} \, |\underline{n}\rangle_b \,, \quad i = 0, 1, \ldots, K-1.$$

Examples

▷ A 2-mode code with the basis states

$$|\boldsymbol{c}_0\rangle = \sqrt{\frac{3}{10}}\,|0,7\rangle_b + \sqrt{\frac{7}{10}}\,|5,2\rangle_b$$
$$|\boldsymbol{c}_1\rangle = \sqrt{\frac{7}{10}}\,|2,5\rangle_b - \sqrt{\frac{3}{10}}\,|7,0\rangle_b$$

with total excitation $N = 7$ and bosonic distance $d_b = 3$.

▷ A 3-dimensional Fock state code with the basis states

$$|\boldsymbol{c}_0\rangle = \frac{1}{3}\,|18,0\rangle_b + \frac{\sqrt{7}}{3}\,|9,9\rangle_b + \frac{1}{3}\,|0,18\rangle_b$$
$$|\boldsymbol{c}_1\rangle = \frac{\sqrt{3}}{3}\,|15,3\rangle_b + \frac{\sqrt{6}}{3}\,|6,12\rangle_b$$
$$|\boldsymbol{c}_2\rangle = \frac{\sqrt{6}}{3}\,|12,6\rangle_b + \frac{\sqrt{3}}{3}\,|3,15\rangle_b\ ,$$

with total excitation $N = 18$ and bosonic distance $d_b = 3$.

From Simplex codes to Fock state codes

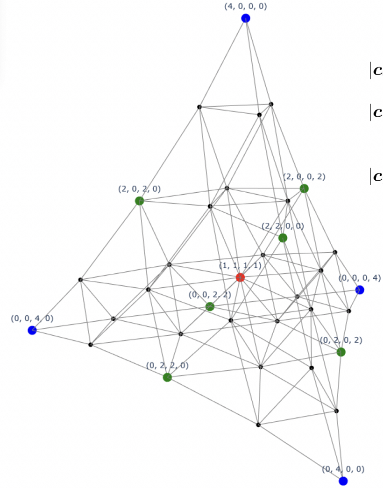
Construction (AAB'26)

Let $C_N \subseteq \mathcal{S}_{q,N}$ *be a classical code. Given a partition of* C_N *into disjoint subsets* $I_0, I_1, \ldots, I_{K-1}$*, consider the Fock state code* Q_N *with the basis* $\{|c_0\rangle, \ldots, |c_{K-1}\rangle\} \subset \mathcal{H}_{q,N}$ *where*

$$|c_i\rangle = \sum_{\underline{n} \in I_i} \alpha_{\underline{n}}^{(i)} |\underline{n}\rangle , \tag{1}$$

where the coefficients $\{ \alpha_{\underline{n}}^{(i)} \}_{\underline{n} \in I_i}$ *satisfy* $\sum_{\underline{n} \in I_i} |\alpha_{\underline{n}}^{(i)}|^2 = 1$*.*

Code example



$$
\begin{aligned}
|\boldsymbol{c}_0\rangle &= \frac{1}{2}(|4,0,0,0\rangle_b + |0,4,0,0\rangle_b + |0,0,4,0\rangle_b + |0,0,0,4\rangle_b \\
|\boldsymbol{c}_1\rangle &= \frac{1}{\sqrt{6}}(|2,2,0,0\rangle_b + |2,0,2,0\rangle_b + |2,0,0,2\rangle_b + |0,2,2,0\rangle_b \\
&\quad + |0,2,0,2\rangle_b + |0,0,2,2\rangle_b) \\
|\boldsymbol{c}_2\rangle &= |1,1,1,1\rangle_b \,.
\end{aligned}
$$

Error correction conditions

Fock state codes are used for information protection against the amplitude damping noise.

Single-mode Kraus operators

$$A_k = \sum_{n=k}^{\infty} \sqrt{\binom{n}{k}} \sqrt{(1-\gamma)^{n-k}\gamma^k} |n-k\rangle\langle n|, \quad k = 0, 1, 2, \ldots .$$

The AD channel on a q-mode bosonic space, $\mathscr{N} : \mathcal{H}_q \to \mathcal{H}_q$, is defined by the set of Kraus operators $\{A_{\underline{r}}\}_{\underline{r}\in\mathbb{Z}_0^q}$:

$$A_{\underline{r}} = A_{r_0} \otimes \cdots \otimes A_{r_{q-1}},$$

Asymptotically good codes

A sequence of codes $\mathcal{Q} = (Q_N)_N, N \geqslant 1$, $Q_N \subseteq \mathcal{H}_{q_N,N}$ is called *asymptotically good* if the distance $d(Q_N)$ grows linearly with N and the rate stays positive as N increases:

$$\min\left(R(\mathcal{Q}), \liminf_{N\to\infty} \frac{d(Q_N)}{N}\right) > 0, \tag{2}$$

where $R(\mathcal{Q}) := \liminf_{N\to\infty} R(Q_N)$.

We show that for a sequence of Fock state codes $(Q_N)_N$ to support noise mitigation on the AD channel, they necessarily should have nonvanishing distance.

Constructing ℓ_1 codes: Sidon sets

- Let G be an Abelian group, written additively. A subset $B \subseteq G$ is a *t-Sidon set* if the sums

  $$b_{i_1} + b_{i_2} + \ldots + b_{i_t}$$

  are all distinct for $0 \leqslant i_1 \leqslant i_2 \leqslant \ldots \leqslant i_t \leqslant |G|$.
- Let G be an Abelian group that contains a t-Sidon set of cardinality $q \geqslant 2$. Then for every $N > t \geqslant 1$, there exists an ℓ_1 code $C \subset \mathcal{S}_{q,N}$ with distance $\geqslant t+1$ such that

  $$|C| \geqslant \frac{|\mathcal{S}_{q,N}|}{|G|}.$$

Why Sidon sets guarantee ℓ_1 separation:

$$C = \{\underline{x} \in \mathcal{S}_{q,N} : \sum_{i=0}^{q-1} x_i g_i = g\} \quad \text{for some fixed } (g_0, \ldots, g_q);\, g \in G$$

Error correction (KL) conditions

KL conditions: orthogonality and non-deformation:

$$\langle \boldsymbol{c}_i | A_a^\dagger A_b | \boldsymbol{c}_j \rangle = 0 \text{ and } \langle \boldsymbol{c}_i | A_a^\dagger A_b | \boldsymbol{c}_i \rangle = g_{ab} \text{ for all } i \neq j \text{ and all } a, b,$$

The following conditions are sufficient for correcting t errors with PI and Fock state codes:

$$\sum_{\underline{n} \in \mathcal{S}_{q,N}} (\alpha_{\underline{n}}^{(i)})^* \alpha_{\underline{n}-\underline{e}+\underline{f}}^{(j)} \frac{\binom{N-t}{\underline{n}-\underline{e}}}{\sqrt{\binom{N}{\underline{n}}\binom{N}{\underline{n}-\underline{e}+\underline{f}}}} = 0$$

$$\sum_{\underline{n} \in \mathcal{S}_{q,N}} \left((\alpha_{\underline{n}}^{(i)})^* \alpha_{\underline{n}-\underline{e}+\underline{f}}^{(i)} - (\alpha_{\underline{n}}^{(j)})^* \alpha_{\underline{n}-\underline{e}+\underline{f}}^{(j)} \right) \frac{\binom{N-t}{\underline{n}-\underline{e}}}{\sqrt{\binom{N}{\underline{n}}\binom{N}{\underline{n}-\underline{e}+\underline{f}}}} = 0$$

Here $\underline{e}, \underline{f} \in \mathcal{S}_{q,t}$ where t is the number of errors that a qudit PI code can detect, and $i, j \in \{0, 1, \ldots, K-1\}, i \neq j$.

This is shown by

1) considering the action of deletions on PI codes

2) showing an equivalence between PI and Fock state codes.

The construction

- ▷ The general form of the nondeformation KL condition becomes

$$\sum_{\underline{h}\in I_0} x_{\underline{h}} \frac{\binom{N-t}{\underline{h}-\underline{e}}}{\binom{N}{\underline{h}}} = \ldots = \sum_{\underline{h}\in I_{K-1}} x_{\underline{h}} \frac{\binom{N-t}{\underline{h}-\underline{e}}}{\binom{N}{\underline{h}}} \qquad \forall \underline{e} \in \mathcal{S}_{q,t}.$$

$$\sum_{\underline{h}\in I_0} x_{\underline{h}} = \sum_{\underline{h}\in I_1} x_{\underline{h}} = \ldots = \sum_{\underline{h}\in I_{K-1}} x_{\underline{h}}$$

  where the variables $x_{\underline{h}} \geqslant 0$ will give the values of the α's.

- ▷ To show that this system has a nonnegative solution, first write it generically as

$$\sum_{i\in I_0} x_i a_{e,i} = \sum_{i\in I_1} x_i a_{e,i} = \cdots = \sum_{i\in I_{K-1}} x_i a_{e,i} \quad \forall e \in \mathcal{S}_{q,t}$$

$$\sum_{i\in I_0} x_i = \sum_{i\in I_1} x_i = \cdots = \sum_{i\in I_{K-1}} x_i$$

The construction, II

- $K = 2$: Rephrasing, let $A = \{A_i\}, A_i = (a_{e,i}, e \in \mathcal{S}_{q,t})^T$ be a set of points (coefficient vectors of our system). If there are $\geqslant m+2$ points, then there are subsets $A^{(1)}, A^{(2)}$ such that

$$\sum_{i \in A^{(1)}} x_i a_{j,i} + \sum_{i \in A^{(2)}} x_i a_{j,i} = 0, \quad j = 1, \dots, m$$

$$\sum_{i \in A^{(1)}} x_i + \sum_{i \in A^{(2)}} x_i = 0$$

  and $A^{(1)} \sqcup A^{(2)} = A$, $x_i \geqslant 0, i \in A^{(1)}; x_i < 0, i \in A^{(2)}$ (Radon's lemma).
- Extending this to general K is covered by Tverberg's theorem (1966): If our set A is formed of $\geqslant (m+1)(K-1)+1$ points, it can be partitioned into K parts whose convex combinations have a point in common.
- The system has a nonnegative solution, i.e., it is possible to satisfy the Knill-Laflamme conditions.
- Rephrasing, if there is an l_1 code $C \subset \mathcal{S}_{q,N}$ with distance $\geqslant t+1$ and $|C| \geqslant (K-1)|\mathcal{S}_{t,q}| + 1$, then there exists a $Q(N, K, q, t+1)$ Fock space code.

Fock state codes from Sidon sets

Theorem (AAB '26)

Let $\mathcal{Q}$ *be a PI code or a Fock state code. For* $N \to \infty$ *and any* K, d *that satisfy*

$$K = o(2^N) \text{ and } d = o\left(\frac{N}{\log N}\right),$$

there exists a sequence of $\mathcal{Q}(N, K, q = N, d)$ *codes.*

- ▷ This improves upon the known constructions, but is not sufficient to guarantee error control on the AD channel with a fixed (constant) loss parameter γ
- ▷ Finding Tverberg partitions is computationally hard
- ▷ No-dimensional Tverberg theorem? (Adiprasito e.a., 2020)

Randomized constructions, I

- Greedy GV-type argument? The problem with this is that $\mathcal{S}_{q,N}$ is not homogeneous in the sense that the volume of the ball depends on the center.
- There is a line of work in coding theory on GV bounds in such (finite metric) spaces (Kolesnik-Krachkovsky '91, Gu-Fuja '94, Tolhuizen '96, Kovacevic-Tan '18, Goyal e.a. '25.)
- This yields asymptotically good codes, although constructing them is difficult. Additionally, the Fock state codes obtained do not satisfy the bounded per-mode occupancy constraint

- Typicality arguments?
  For large N, q, a typical vertex $\underline{n} \in \mathcal{S}_{\alpha N,N}$ satisfies several constraints
  (compare with weight $n/2$ vectors in the binary Hamming space)
- Proposition: With high probability a uniform random vector $\underline{\mathsf{X}} \in \mathcal{S}_{\alpha N,N}$ satisfies

$$\frac{1}{N}\operatorname{supp}(\underline{\mathsf{X}}) \approx \frac{\alpha}{1+\alpha}$$

$$\|\underline{\mathsf{X}}\|_\infty \leqslant (1+\epsilon)\log_{1+\alpha} N$$

- Estimating the ball size in the subset of typical vectors is easier than in $\mathcal{S}_{q,N}$

Randomized constructions, II

- ▷ (Uniform sampling from $\mathcal{S}_{q,N}$) With high probability, this yields asymptotically good ℓ_1 codes with distance guarantees and coordinates bounded as $O(\log N)$.

Uniform sampling from $\mathcal{S}_{q,N}$ is easy:

Sample numbers $b_1 < b_2 < \cdots < b_{q-1}$ uniformly without replacement from the set $\{1, \ldots, N+q-1\}$.

A uniform random vector $(x_1, \ldots, x_q)$ in $\mathcal{S}_{q,N}$:

$x_1 = b_1 - 1, x_i = b_i - b_{i-1} - 1, i = 2, \ldots, q-1, x_q = (N+q-1) - b_{q-1}$

- ▷ (Independent coordinates) Let $\mathsf{Y}_1, \ldots, \mathsf{Y}_\mathsf{N} \sim \mathsf{Unif}\{0, 1, \ldots, \mathsf{q}-1\}$ be i.i.d. RVs . Define a random vector $\mathsf{X} \in \mathcal{S}_{\mathsf{q},\mathsf{N}}$:

$$\mathsf{X}_\mathsf{i} = \sum_{\mathsf{j}=1}^{\mathsf{N}} \mathbb{1}_{\{\mathsf{Y}_\mathsf{j}=\mathsf{i}\}}, \quad \mathsf{i} = 0, \ldots, \mathsf{q}-1 \tag{3}$$

Sampling L independent vectors according to this procedure yields an asymptotically good ℓ_1 code with coordinates at most $O(\log N/\log\log N)$

Asymptotically good Fock state codes

Theorem (Exact asymptotically good Fock state codes)

For any $0 < \delta < 1$, $\varepsilon > 0$ *and* $\alpha > 0$ *there exists a sequence of Fock state codes* $(Q_N)_N$, $Q_N \subseteq \mathcal{H}_{\alpha N, N}$ *with distance at least* δN*, local excitation* $B = (1+\varepsilon)\log_{1+\alpha} N$ *and dimension* K_N *as long as*

$$\frac{1}{N}\log_2 K_N \leqslant R^*(\delta, \alpha) - (\alpha+\delta)h_2\Big(\frac{\alpha}{\alpha+\delta}\Big) + o(1),$$

for some fixed $o(1)$ *function.*

The bounded per-mode occupancy (local excitation) contributes to the implementation side of Fock state codes:

- ▷ Generation of Fock states with high photon numbers in practice remains a challenge
- ▷ The coherence lifetime of a Fock state $|n\rangle$ scales as $1/n$, so low per-mode occupancy increases stability of the system.

AQEC

- A channel $\mathscr{N} : L(\mathcal{H}) \to L(\mathcal{H}')$ defined by a set of Kraus error operators $\{A_k\}_k$

$$\rho \mapsto \mathscr{N}(\rho) = \sum_k A_k \rho A_k^\dagger$$

- A code $Q \subset \mathcal{H}$ corrects an error set $\mathcal{E}$ if there exists a decoding operation $\mathscr{D} : L(\mathcal{H}') \to L(\mathcal{H})$ such that for any channel $\mathscr{N}$ with $\{A_k\}_k \subset \mathcal{E}$ $\mathscr{D} \circ \mathscr{N}|_{L(C)} = I_{L(C)}$

- The starting point in many studies of AQECCs is the BÉNY-ORESHKOV formalism (2010). The performance of a code is measured by the worst-case entanglement fidelity

$$\mathcal{F}_e(\mathscr{N}, \mathscr{M}) \triangleq \inf_{\rho \in D(\mathcal{H})} \mathcal{F}(\mathscr{N} \otimes I_{L(\mathcal{H})}(|\psi_\rho\rangle\langle\psi_\rho|), \mathscr{M} \otimes I_{L(\mathcal{H})}(|\psi_\rho\rangle\langle\psi_\rho|)),$$

where $\mathscr{N}, \mathscr{M} : L(\mathcal{H}) \to L(\mathcal{H}')$ are two quantum channels and $|\psi_\rho\rangle$ is any purification of ρ.

- A code $Q \subseteq \mathcal{H}$ is called an ε-AQECC for a channel $\mathscr{N}$ if there exists a decoding operation $\mathscr{D}$ such that

$$d(\mathscr{D} \circ \mathscr{N}|_{L(Q)}, I_{L(Q)}) := \sqrt{1 - \mathcal{F}_e(\mathscr{D} \circ \mathscr{N}|_{L(Q)}, I_{L(Q)})} \leqslant \varepsilon.$$

Approximate KL conditions

Proposition:

Let $\mathscr{N}$ *be a quantum channel with Kraus operators* $\mathcal{E} = \{A_k\}_{k=0}^{M-1}$*, and let* $Q \subseteq H$ *be a quantum code with an orthonormal basis* $\{|c_i\rangle\}_{i=0}^{K-1}$ *that satisfies*

$$\langle c_i| A_k^\dagger A_l |c_j\rangle = 0 \quad \text{if } i \neq j \text{ or } \ell \neq k.$$

Then Q *is an* ε*-AQECC with respect to* $\mathcal{E}$ *if there exist non-negative numbers* $\{\lambda_l\}_{l=0}^{M-1}$*,* $\sum_{l=0}^{M-1} \lambda_l = 1$ *such that*

$$\max_{i,l} |\langle c_i| A_l^\dagger A_l |c_i\rangle - \lambda_l| \leqslant \frac{\varepsilon^2}{KM}.$$

Approximate Fock state codes

We define a truncated AD channel $\mathscr{N}_{\leqslant t}$ as follows:

$$\mathscr{N}_{\leqslant t}(\rho) = \frac{1}{p_{N,t}} \sum_{\substack{\underline{r} \in \mathcal{S}_{q,r} \\ r \leqslant t}} A_{\underline{r}} \rho A_{\underline{r}}^{\dagger}, \qquad \text{where } p_{N,t} = \mathbb{P}(\text{Binom}(N, \gamma) \leqslant t)$$

Definition

A sequence of AQECCs $\mathcal{Q} = (Q_N)_N$, $Q_N \subseteq \mathcal{H}_{q_N,N}$, is called asymptotically good if there exists $\delta > \gamma > 0$ such that $Q_N, N \geqslant 1$ is an ε_N-AQECC for the channel $\mathscr{N}_{\leqslant \delta N}$, where $\varepsilon_N \downarrow 0$, and the sequence $\mathcal{Q}$ has a positive asymptotic rate, $R(\mathcal{Q}) > 0$.

> CLAUDE CRÉPEAU, DANIEL GOTTESMAN & ADAM SMITH (2005)
> “The connection between correcting general errors and erasure errors breaks down for approximate QECCs. This suggests there is no sensible notion of distance for an approximate quantum error-correcting code.”

There are no exact/approximate Fock state codes

Theorem

Assume that $Q \subseteq \mathcal{H}_{q,N}$ *is an* ε*-AQECC for the truncated channel* $\mathscr{N}_{\leqslant t}$*, then* Q *is an* ε'*-AQECC for the AD channel* $\mathscr{N}$*, where*

$$\varepsilon' = \sqrt{1-(1-\varepsilon^2)p_{N,t}}\,.$$

Conversely, if $Q \subseteq \mathcal{H}_{q,N}$ *is an* ε*-AQECC for the AD channel* $\mathscr{N}$*, then it is also* ε'*-AQECC for the truncated channel* $\mathscr{N}_{\leqslant t}$ *with*

$$\varepsilon' = \frac{\varepsilon}{\sqrt{p_{N,t}}}.$$

Remark: Here we construct the code Q relying on an arbitrary partition of the classical ℓ_1 code into equal parts.

Corollary

A sequence of codes $\mathcal{Q} = (Q_N)_N$ *is asymptotically good (in the approximate or exact sense) if and only if it is an AQECC code sequence for the AD channel with a sufficiently small (constant) loss parameter* $\gamma > 0$*.*

The constructions we present give rise to asymptotically good AQECC Fock state codes. They rely on arbitrary partitions of classical codes, bypassing the need for Tverberg-type arguments

Qudit PI codes

Dicke states:

$$|D_{\underline{n}}\rangle = \frac{1}{\sqrt{\binom{N}{\underline{n}}}} \sum_{\substack{\boldsymbol{x}\in\mathcal{Q}^N \\ \mathsf{C}(\boldsymbol{x})=\underline{n}}} |\boldsymbol{x}\rangle$$

where $\mathcal{Q} = \{0, 1, \ldots, q-1\}$ and $\mathsf{C}(\boldsymbol{x}) = (n_0, n_1, \ldots, n_{q-1})$ is the *composition* of $\boldsymbol{x}$:

$$n_i = \#\{j : x_j = i\}$$

For instance,

$$D_{(1,3)} \propto |1110\rangle + |1101\rangle + |1011\rangle + |0111\rangle$$

A K-dimensional qudit PI code is defined by the basis

$$|c_i\rangle = \sum_{\underline{n}\in S_{q,N}} \alpha_{\underline{n}}^{(i)} |D_{\underline{n}}\rangle\,, \quad i = 0, 1, \ldots, K-1.$$

Asymptotically good PI codes

PI codes were introduced by M.B. Ruskai (2000) and studied by Y. Ouyang (2015–) and in the works with Arda Aydin (AAB '24, AAB'26)

From Fock state codes to PI codes: Let $C_N \in \mathcal{S}_{q,N}$ be a classical ℓ_1 code with distance $t+1$. Let Q be a K-dimensional Fock state code with total excitation N constructed from C_N. Define the mapping

$$|\underline{n}\rangle_b \underset{h^{-1}}{\overset{h}{\Longleftrightarrow}} |D_{\underline{n}}\rangle .$$

Applying this mapping to a Fock state code Q yields a PI code with alphabet size q, length N, and distance $d = t+1$ defined by the basis

$$|c_i\rangle = \sum_{\underline{n} \in I_i} \alpha_{\underline{n}}^{(i)} \cdot |D_{\underline{n}}\rangle , \quad i \in \{0, 1, \ldots, K-1\}.$$

This link yields a family of asymptotically good qudit PI codes with rate and distance similar to the Fock state codes in $\mathcal{H}_{\alpha N, N}$