

Computing Lattices of $K3$

Lecture 3: Lower bounds: from p

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ICERM: Arithmetic, Geometry and Comp

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Slides available at [edgarcosta](https://edgarcosta.github.io)

The other direction

- Two lectures of upper bounds. Reduction gave $\rho \leq \rho_{\text{max}}$ upper bound on the rank.
- The rank alone does not determine the Picard lattice, nothing about the Galois action, and exhibits no curves.

Today: produce the classes. A lower bound is a curve. The set of curves with their intersections is the lattice itself.

For some approaches see [Bouyer-Costa-Festi-Nicholas], [Festi].

An analytic approach

Lefschetz (1,1) theorem

A homology class $\gamma \in H_2(X, \mathbf{Z})$ is in $\text{Pic } X$ if and only if $\int_{\gamma} \omega_X$ form ω_X on X , unique up to scaling.

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$$H_2(X, \mathbf{Z}) \longrightarrow \mathbf{C},$$

Hence, if $\Pi = [\int_\gamma \omega_X]_{\gamma \in H_2(X, \mathbf{Z})} \in \mathbf{C}^{22}$ represents this (saturated) lattice $\Lambda \subset H_2(X, \mathbf{Z})$ of solutions

$$\Pi R = 0, \qquad R \in H_2(X, \mathbf{Z})$$

An analytic approach

- $\Pi = [\int_{\gamma} \omega_X]_{\gamma \in H_2(X, \mathbf{Z})} \in \mathbf{C}^{22}$ can be computed:
 - rigorously as a ball via deformation for projective hyperplane sections
github.com/emresezgin
 - lefschetz-family (heuristic homology basis, Lairez-Pichon)
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 - Elsenhans-Jahnel implemented also an heuristic approach
- **Heuristically**, via lattice reduction algorithms, we can find $\Lambda \subset \Pi$
- There is no obvious way to **prove** that our guesses are actually correct
- Nonetheless, given Π as a ball, one can compute $B \gg 0$ such

$$\text{Pic}(X)_{|B} := \mathbf{Z} \langle \gamma \in \text{Pic } X \mid -\gamma_{\text{prim}}^2 < B \rangle$$

Our setup

Let C be a **nice** (smooth, projective, geometrically integral) curve over k . Let J be the Jacobian of C .

Goal

Given the equations of C , compute the endomorphism ring $\text{End}(J)$.

Heuristic solution

$J = \mathbf{C}^g / \Lambda_J$, the period lattice Λ_J computed numerical $H^0(C, \Omega_C)$.

By picking a k -basis for $H^0(C, \Omega_C)$, we have

$$\mathrm{End}(J) = \{T \in M_g(k)\}$$

Hence, if Π is a **period matrix** for C , i.e., $\Lambda_J = \Pi \mathbf{Z}^{2g}$, then the solutions (T, R) to

$$T\Pi = \Pi R, \qquad T \in M_g(k).$$

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Representing endomorphisms via corresp

$$\alpha_C : C \overset{AJ}{\longrightarrow} J \overset{\alpha}{\dashrightarrow} J \dashrightarrow \text{Sym}^g(C$$

$$P \mapsto \{Q_1, \dots, Q_g\} \iff \alpha([P$$

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THEOREM (C-MASCOT-SIJSLING-VOIGHT)

We give an algorithm for **nondegenerate** $\alpha \in \mathbb{M}_g(k)$

$$\alpha \mapsto \begin{cases} \text{true} & \text{if } \alpha \in \text{End } J, \\ \text{false} & \text{if } \alpha \notin \text{End } J \end{cases}$$

By interpolation via α_C or by locally solving a differential equation on $C \times C$

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- Our method works just as well for isogenies and projections.
- Try it: [github.com/edgarcosta/endomorphisms](#), putatively in M

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Today we will try to use the period map.

$$H_2(X, \mathbf{Z}) \longrightarrow \mathbf{C},$$

$$\Pi = \left[\int_{\gamma} \omega_X \right]_{\gamma \in H_2(X, \mathbf{Z})}$$

$$\text{Pic}(X^{al}) = \ker \left(H_2(X, \mathbf{Z}) \longrightarrow \mathbf{C} \right)$$

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- Matching upper bounds can be deduced by positive characteristic

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- Matching upper bounds can be deduced by positive characteristic methods
- The pencil has a symplectic $\mathbf{Z}/7\mathbf{Z}$ action.
- Its coinvariant lattice $\Omega_7 = (H^2(X, \mathbf{Z})^{\mathbf{Z}/7\mathbf{Z}})^\perp$ has rank 18 and

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- The pencil has a symplectic $\mathbf{Z}/7\mathbf{Z}$ action.
- Its coinvariant lattice $\Omega_7 = (H^2(X, \mathbf{Z})^{\mathbf{Z}/7\mathbf{Z}})^\perp$ has rank 18 and determinant $4 \cdot 7^3$
- $\text{Pic}(X^{al}) = \langle 4 \rangle \oplus \Omega_7$, of determinant $4 \cdot 7^3 = 1372$ and saturation index 1

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- The "smallest" non-trivial curves that appear are smooth rational curves

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- No small rational curves: There are no lines, no conics, no twisted cubics
- The "smallest" non-trivial curves that appear are smooth rational quartics
- Lattice computations with Λ predict that there are

133056

smooth rational quartics spanning Λ .

- These counts push close to known maximums [Degtyarev].

Reconstructing isolated curves from their

Turns out one can compute a bit more for hypersurfaces

$$\varphi: H_2(X, \mathbf{Z}) \times H_{\mathrm{dR}}^2(X/k) \rightarrow \mathbf{C}$$

Note, if $\gamma \in \mathrm{Pic} X^{al}$, then $\frac{1}{2\pi i} \int_{\gamma} \omega \in k^{al}$ for $\omega \in F^1 H_{\mathrm{dR}}^2(X/k)$.

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THEOREM (MOVASATI-SERTÖZ)

If $\gamma = [C] \in H_2(X, \mathbf{Z})$ for a curve $C \subset X$ then from $\frac{1}{2\pi i} \left(\int_{\gamma} \right)$
 $I(C) \subsetneq I_{\gamma}$.

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For a smooth rational quartic curve $C \subset X$ we have that the equ
generates $I_{[C],2}$, i.e., $I(C)_2 = I_{[C],2}$.

Reconstructing quadric surfaces

$$X : x^4 + xyzw + y^3z + yw^3$$

$$\mathrm{Pic}(X)|_B \subseteq \Lambda \overset{?}{\subseteq}$$

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 $133056 = 336 + 1008 + 1176 + 3528 \cdot 3 + 4704 \cdot 3$

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$$133056 = 336 + 1008 + 1176 + 3528 \cdot 3 + 4704 \cdot 3$$
- For each quartic curve $C \subset X$, we can compute

$$I_{[C],2} = \langle a_0x^2 + \dots$$

that defines a quadric surface Q , such that $Q \cap X = C \cup C$ containing a pair of quartics.

- We aim reconstruct the ten (algebraic!) coefficients of these qu

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- The minimal polynomials have large height about 9k characters

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- We are presented with same 168 degree field L in 9 different w

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In our case, we have all the **compatible** embeddings

$$\sigma_i : \mathbf{Q}(a_k) \hookrightarrow L$$

Thus the isomorphism is given by the solution of the following

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In practice it is faster to refine the complex embeddings iteratively: their heights

Intersecting the quadric surfaces with the

$$Q : a_0x^2 + a_1xy + \cdots + a_9w^2 = 0 \subset \mathbf{P}^3,$$

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Show that $Q \cap X$ decomposes into two quartic curves.

- It suffices to show that the singular locus S of $Q \cap X$ consists

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Hence, S is zero-dimensional and reduced, and $\deg S \leq 10$.

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- We conclude $\deg S = 10$ via Gotzmann regularity theorem, by checking that $\dim L[x, y, z, w]_{\bullet} / I_{\bullet} = 10$ for $\bullet = 6, 7$.

Certifying $\text{Pic } X = \Lambda$

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The inclusion $\Lambda_Q \subseteq \Lambda$ is not explicit!

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Hence, it is sufficient to show that $\text{rank } \Lambda_Q = \text{rank } \Lambda = 19$.

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- Compute the intersections of these 336 curves with each other over $\mathbf{C}$.

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We can do this in two ways:

- Compute the intersections of these 336 curves with each other over $\mathbf{F}_2$.
- Certify that these correspond to the original classes.
Showing that there are at most 66528 distinct quadrics. Can be done by counting the number of quadrics that intersect a given curve in $\mathbf{F}_2$.
This establishes a bijection between quadric surfaces and the 168 p

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- Compute the intersections of these 336 curves with each other over $\mathbf{F}_2$.
- Certify that these correspond to the original classes.
Showing that there are at most 66528 distinct quadrics. Can be done by counting.
This establishes a bijection between quadric surfaces and the 168 p

$$\text{Pic } X = \Lambda$$

Computing the Galois action

$$Q : a_0x^2 + a_1xy + \cdots + a_9w^2 = 0 \subset \mathbf{P}^3,$$

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For Q smooth, $K = L(\text{disc } Q)$ [Costa-Sertöz].

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We guess that $K = F(\sqrt[14]{u})$ where $[F:\mathbf{Q}] = 24$ and $\text{Gal}(F/\mathbf{Q})$

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1. Can we compute $\text{Gal}(K/\mathbf{Q})$?
2. $\text{Gal}(K/\mathbf{Q}) \stackrel{?}{=} \text{Aut } \Lambda$?
3. $H^1(\text{Gal}(\bar{k}/k), \text{Pic } X) = ?$

Summary

THEOREM (C-SERTÖZ)

The quartic surface $X : x^4 + xyzw + y^3z + yw^3 + z^3w =$
over a quadratic extension of $L := \mathbf{Q}(\{a_i\}_i)$.

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Do you have a challenge K