

Minicourse on arithmetic of K3 surfaces: Isogenies

ICERM workshop: Arithmetic, Geometry and Computations on
K3 surfaces

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Set up

S projective complex K3 surface

$$H^2(S, \mathbb{Z}) \simeq U^3 \oplus (-E_8)^2, \quad U \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with E_8 positive definite unimodular lattice associated to the Dynkin diagram

$\mathbb{C}\omega = \Gamma(S, \Omega_S^2) \subset H^2(S, \mathbb{C})$ the period of S

$\text{Pic}(S) = \omega^\perp \cap H^2(S, \mathbb{Z})$ with orthogonal complement $T(S)$

Torelli Theorem: If there is an isomorphism of lattices $H^2(S_1, \mathbb{Z}) \xrightarrow{\sim} H^2(S_2, \mathbb{Z})$ respecting periods then $S_1 \simeq S_2$.

Some remarks on lattices

A lattice L is non-degenerate if the natural homomorphism $L \rightarrow \text{Hom}(L, \mathbb{Z})$ is injective. It is an isomorphism iff L is unimodular.

For such a lattice L , the discriminant group $d(L) := \text{Hom}(L, \mathbb{Z})/L$, a finite abelian group of order equal to the absolute value of the determinant of the bilinear form. For even L , there is a quadratic form $q_L : d(L) \rightarrow \mathbb{Q}/2\mathbb{Z}$ induced by extending the quadratic form on L by linearity.

If $P, T \subset L$ are saturated non-degenerate sublattices of an even unimodular lattice with $T = P^\perp$ then $(d(P), q_P) \simeq (d(T), -q_T)$. This is induced by the inclusion $P \oplus T \subset L$.

Stabilization theorem of Nikulin

Suppose that P_1 and P_2 are even non-degenerate lattices with the same rank, signature, and discriminant invariants

$$(d(P_1), q_{P_1}) \simeq (d(P_2), q_{P_2}).$$

Then $P_1 \oplus U \simeq P_2 \oplus U$. For indefinite P_1 and P_2 with the same invariants, stabilization is unnecessary provided

$$\ell(d(P_i)) + 2 \leq \text{rank}(P_i)$$

where ℓ is the minimal number of generators.

Example: The lattices

$$\begin{pmatrix} 2 & 13 \\ 13 & 12 \end{pmatrix} \quad \begin{pmatrix} 8 & 15 \\ 15 & 10 \end{pmatrix}$$

have the same invariants, and are stably isomorphic but not isomorphic over $\mathbb{Z}$.

An equivalence relation on K3 surfaces

The K3 surfaces S_1 and S_2 are **D-equivalent** if

$$T(S_1) \simeq T(S_2)$$

as lattices, compatible with the periods. It follows that $\text{Pic}(S_1)$ and $\text{Pic}(S_2)$ have the same invariants whence

$$\text{Pic}(S_1) \oplus U \simeq \text{Pic}(S_2) \oplus U.$$

Note that S_1 and S_2 may be non-isomorphic. This definitely happens when stabilization is required i.e. $\text{Pic}(S_1) \not\simeq \text{Pic}(S_2)$. But it may also happen when they are isomorphic, i.e. when

$$\text{Pic}(S_1) \oplus T(S_1) \xrightarrow{\sim} \text{Pic}(S_2) \oplus T(S_2)$$

fails to extend to an isomorphism

$$H^2(S_1, \mathbb{Z}) \not\xrightarrow{\sim} H^2(S_2, \mathbb{Z}).$$

Mukai lattice of a K3

This is defined as the free abelian group

$$\tilde{H}(S, \mathbb{Z}) = H^0(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z})$$

along with the Mukai pairing $\langle, \rangle$ with the following properties:

- ▶ $\langle, \rangle$ is the intersection pairing on H^2 ;
- ▶ the sign of the intersection pairing is reversed for $H^0 \oplus H^4$;
- ▶ the remaining intersections are zero.

We may interpret $\tilde{H}(S, \mathbb{Z})$ as a weight-two Hodge structure by the shifting the weights of the outer summands.

Thus S_1 and S_2 are D -equivalent iff

$$\tilde{H}(S_1, \mathbb{Z}) \xrightarrow{\sim} \tilde{H}(S_2, \mathbb{Z})$$

compatibly with periods.

Example

Let S be a K3 surface with $\text{Pic}(S) = \mathbb{Z}h$ and h of degree $2d$. Suppose that σ is the number of distinct prime factors of d . Then there are $2^{\sigma-1}$ K3 surfaces in the D -equivalence class of S . Indeed, this is the cardinality of the quotient

$$\text{Aut}(d(\mathbb{Z}h), q_{\mathbb{Z}h}) / \pm 1$$

or the number of unimodular extensions of

$$\mathbb{Z}h \oplus T(S).$$

Hosono-Lian-Oguiso-Yau give many more formulas in this spirit.

Thus D -equivalence is non-trivial in rank one for degrees

$$12, 20, 24, 28, 30, \dots$$

Mukai's characterization of D -equivalence

Let S_1 and S_2 be K3 surfaces. Then the following are equivalent:

- ▶ S_1 and S_2 are D -equivalent;
- ▶ S_2 is a fine moduli space of simple sheaves on S_1 and *vice versa*;

The latter condition means there are Hodge classes

$$v_i = (r_i, f_i, s_i) \in \tilde{H}(S_i, \mathbb{Z})$$

such that $S_2 \simeq M_{v_1}(S_1)$, the moduli space of rank r_1 simple sheaves $\mathcal{E} \rightarrow S_1$ with $c_1(\mathcal{E}) = f_1$ and $\chi(\mathcal{E}) = r_1 + s_1$, and $S_1 \simeq M_{v_2}(S_2)$, with rank r_2 , $c_1 = f_2$, and $\chi = r_2 + s_2$.

Warning: We are simplifying here, ignoring stability/semistability and the role of polarizations in the construction of moduli.

In general, the vectors $v_i = (r_i, f_i, s_i)$ are isotropic

$$\langle v_i, v_i \rangle = 0, \quad f_i^2 = 2r_i s_i.$$

Example: Let (S_1, h_1) and (S_2, h_2) be D -equivalent (non-isomorphic) K3 surfaces of degree 12. Then

$$S_2 \simeq M_{(2, h_1, 3)}(S_1) \text{ and } S_1 \simeq M_{(2, h_2, 3)}(S_2).$$

Mukai has studied this case in detail; Kuznetsov develops it further as an instance of his theory of “homological projective duality”.

An arithmetic digression

Suppose that S_1 and S_2 are K3 surfaces over a general field K . Assume they are D -equivalent over K via the moduli space formulation.

Question

Does $S_1(K) \neq \emptyset$ imply $S_2(K) \neq \emptyset$?

This is valid over $\mathbb{R}$ and $\mathbb{F}_q$, and over local fields with hypothesis on the reduction types (with Tschinkel).

It **fails** for most of the possible extensions: Ascher - Dasartha - Perry - Zhou (twisted case) and Addington - Antieau - Frei (higher-dimensional case).

Twisted versions of D -equivalence

We turn to a weaker notion of isogeny, i.e.

$$T(S_1) \hookrightarrow T(S_2)$$

with cokernel cyclic of order $m > 1$. This group-theoretic data is encoded by

$$\alpha \in H^2(S_2, \mathbb{Q}) / (H^2(S_2, \mathbb{Z}) \oplus \text{Pic}(S_2)_{\mathbb{Q}}) = \text{Br}(S_2)$$

of order m , i.e.

$$T(S_1) = \alpha^{\perp} \subset T(S_2)$$

where intersections are taken modulo m .

This also has a moduli interpretation: $S_2 = M_V(S_1)$ a moduli space of simple sheaves that fails to be fine, i.e. α encodes the obstruction over moduli to a universal sheaf $\mathcal{E} \rightarrow S_1 \times M_V(S_1)$.

Example of Mukai

Let (S_1, h) be a generic polarized K3 surface of degree eight, realized as a complete intersection of quadrics

$$S_1 = \{Q_0 = Q_1 = Q_2 = 0\} \subset \mathbb{P}^5 \quad \mathcal{Q} \rightarrow \mathbb{P}(\Gamma(\mathcal{I}_{S_1}(2))) \simeq \mathbb{P}^2$$

denote the resulting net of quadric forms and $F_2(\mathcal{Q}/\mathbb{P}^2)$ the relative variety of isotropic planes in quadric fourfolds. Stein factorization

$$F_2(\mathcal{Q}/\mathbb{P}^2) \rightarrow S_2 \rightarrow \mathbb{P}^2$$

is an étale $\mathbb{P}^3$ bundle over a double cover branched over the sextic plane curve $\det(t_0 Q_0 + t_1 Q_1 + t_2 Q_2) = 0$. Let α denote the Brauer class classifying this bundle i.e. $\alpha \in \mathrm{Br}(S_2)[2]$. We have

$$T(S_1) \hookrightarrow T(S_2)$$

as an index-two sublattice and $S_2 \simeq M_{(2,h,2)}(S_1)$.

Generalization

Let (S_1, h) be polarized of degree $2\ell^2 d$ for some $\ell \in \mathbb{N}$. Then $S_2 \simeq M_{(\ell, h, \ell d)}(S_1)$ has degree $2d$ and comes naturally with an element $\alpha \in \text{Br}(S_2)[\ell]$. This class is represented by a projective bundle of relative dimension $\chi - 1 = \ell(d + 1) - 1$ over S_2 , i.e. the push forward of the twisted universal bundle.

These are examples of “geometric realizations” of Brauer classes alluded to in the last lecture.

Question

Over number fields, can these obstruct the Hasse principle?

Doubly twisted version of isogeny

Suppose that S_1 and S_2 share a common sublattice, e.g.

$$T(S_1) \leftarrow L \hookrightarrow T(S_2)$$

where both cokernels are cyclic of prime order ℓ . Here we get two Brauer classes

$$\alpha_1 \in \mathrm{Br}(S_1)[\ell], \quad \alpha_2 \in \mathrm{Br}(S_2)[\ell].$$

These have moduli space avatars as well, studied by Căldăraru, Huybrechts, Stellari, and Yoshioka.

Buskin and Huybrechts, inspired by Shafarevich, have analyzed more general finite index sublattices. These typically have interpretations via twisted sheaves or hyperkähler geometry.

Examples

Consider a general double cover $X \rightarrow \mathbb{P}^2 \times \mathbb{P}^2$ branched along a divisor of bidegree $(2, 2)$. It has two quadric surface bundles

$$\pi_i : X \rightarrow \mathbb{P}^2$$

degenerating along sextic plane curves. Stein factorization yields degree-two K3 surfaces $S_i \rightarrow \mathbb{P}^2$ and Brauer classes $\alpha_i \in \mathrm{Br}(S_i)[2]$. The non-algebraic cohomology gives inclusions

$$L(-1) := \langle h_1^2, h_1 h_2, h_2^2 \rangle^\perp \quad T(S_1) \supset L \subset T(S_2)$$

where the inclusions both have index two.

This construction yields counterexamples to the Hasse principle witnessed by transcendental Brauer classes! Iliev - Kapustka - Kapustka - Ranestad relate this to a hyperkähler fourfold.

What is next?

Hot off the presses:

Theorem (Frei - Petok - Várilly-Alvarado 2026)

Let S be a very general K3 surface of degree $2d$, $n \in \mathbb{N}$, and $\alpha \in \text{Br}(S)[n]$. Then there is an étale $\mathbb{P}^{n-1}$ -bundle $E \rightarrow S$ with a natural embedding into a hyperkähler manifold X of dimension $\leq 4n - 2$. Moreover, X admits a contraction collapsing E to S .

Interestingly, E is not always canonical; the possible choices are also parametrized by a hyperkähler manifold.

They prove much more - stay tuned!

Previous examples along these lines were analyzed by McKinnie, Sawon, Tanimoto, and Várilly-Alvarado.