

AUTOMORPHISMS OF QUARTIC SURFACES AND CREMONA TRANSFORMATIONS II

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Arithmetic, Geometry and Computations on K3 surfaces
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AUTOMORPHISMS OF QUARTIC SURFACES AND CREMONA TRANSFORMATIONS II

Joint work with [Daniela Paiva](#) and [Sokratis Zikas](#)

(“*On Gizatullin’s Problem for quartic surfaces of Picard rank 2*” -
arXiv:2410.08415 - Algebraic Geometry, to appear)

(We always work over $\mathbb{C}$)

RECAP OF ANA'S TALK

$S \subset \mathbb{P}^3$ smooth quartic surface

QUESTION (GIZATULLIN)

Which automorphisms of S are induced by a Cremona transformation of the ambient space $\mathbb{P}^3$?

EXAMPLES (OGUIO 2012)

- $\text{Aut}(S) \cong \mathbb{Z}$, and no nontrivial automorphism of S is induced by a Cremona transformation of $\mathbb{P}^3$.
- $\text{Aut}(S) \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$, and every automorphism of S is induced by a Cremona transformation of $\mathbb{P}^3$.

RECAP OF ANA'S TALK

$S \subset \mathbb{P}^3$ smooth quartic surface

- $1 \leq \rho(S) \leq 20$
- If $\rho(S) = 1$, then $\text{Aut}(S) = \{1\}$
- If $\rho(S) = 2$ and S is general, then

$$\text{Aut}(S) \cong \begin{cases} \{1\} \\ \mathbb{Z}_2 \\ \mathbb{Z} \\ \mathbb{Z}_2 * \mathbb{Z}_2 \end{cases}$$

THEOREM (PAIVA-QUEDO 2025)

If S is general with $\rho(S) = 2$ and $|\text{disc}(S)| > 233$, then no nontrivial element of $\text{Aut}(S)$ is induced by a Cremona transformation of $\mathbb{P}^3$.

MAIN RESULT

$S \subset \mathbb{P}^3$ general smooth quartic surface with $\rho(S) = 2$

$$\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}W \qquad Q = \begin{pmatrix} 4 & b \\ b & 2c \end{pmatrix},$$

$r := -\det Q$ the **discriminant** of S

THEOREM (A-PAIVA-ZIKAS)

- ① If $r > 57$ or $r = 52$, then no nontrivial automorphism of S is induced by a Cremona transformation of $\mathbb{P}^3$.
- ② If $r \leq 57$, $r \neq 52$, then every automorphism of S is induced by a Cremona transformation of $\mathbb{P}^3$.
 - $r \in \{9, 12, 16, 24, 25, 33, 36, 44, 49, 57\}$ and $\mathrm{Aut}(S) = \{1\}$;
 - $r \in \{17, 41\}$ and $\mathrm{Aut}(S) \cong \mathbb{Z}_2$;
 - $r \in \{28, 56\}$ and $\mathrm{Aut}(S) \cong \mathbb{Z}_2 * \mathbb{Z}_2$; or
 - $r \in \{20, 32, 40, 48\}$ and $\mathrm{Aut}(S) \cong \mathbb{Z}$.

MAIN TOOL: BIRATIONAL GEOMETRY OF C-Y PAIRS

$S \subset \mathbb{P}^3$ smooth quartic surface

QUESTION (GIZATULLIN)

Which automorphisms of S are induced by a Cremona transformation of the ambient space $\mathbb{P}^3$?

$(\mathbb{P}^3, S)$ is a **Calabi-Yau pair**

DEFINITION (CALABI-YAU PAIR (X, D))

- X **terminal** projective variety
- D is a hypersurface $\sim -K_X$
- (X, D) is **log canonical**

BACKGROUND ON BIRATIONAL GEOMETRY

$$\mathrm{Bir}(\mathbb{P}^n) := \{ \varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n \text{ birational self-map} \}$$

EXAMPLE (THE STANDARD QUADRATIC TRANSFORMATION)

$$\begin{array}{ccc} \tau : & \mathbb{P}^2 & \dashrightarrow \mathbb{P}^2 \\ & (x : y : z) & \longmapsto \left(\frac{1}{x} : \frac{1}{y} : \frac{1}{z} \right) = (yz : xz : xy) \end{array}$$

THEOREM (NOETHER-CASTELNUOVO 1870-1901)

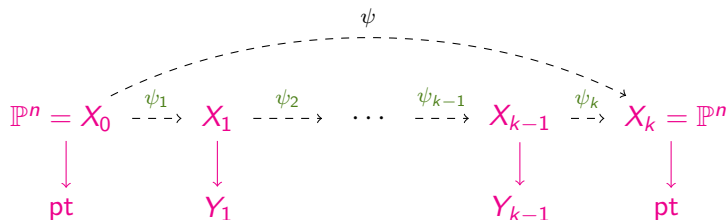
$$\mathrm{Bir}(\mathbb{P}^2) = \langle \mathrm{Aut}(\mathbb{P}^2), \tau \rangle$$

THEOREM (HILDA HUDSON 1927)

For $n \geq 3$, $\mathrm{Bir}(\mathbb{P}^n)$ cannot be generated by elements of bounded degree.

THE SARKISOV PROGRAM

THEOREM (CORTI 1995, HACON-MCKERNAN 2013)



The $X_i \rightarrow Y_i$'s are Mori fiber spaces

The ψ_i 's are elementary links

BIRATIONAL GEOMETRY OF CALABI-YAU PAIRS

DEFINITION (C-Y PAIR (X, D))

- X **terminal** projective variety
- D effective Weil divisor

$$K_X + D \sim 0$$

- (X, D) is **log canonical**

REMARK

$\exists \omega_D$ (unique up to scaling)

$$\operatorname{div}(\omega_D) = -D$$

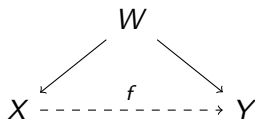
DEFINITION (VOLUME PRESERVING MAP $(X, D_X) \dashrightarrow (Y, D_Y)$)

$f : X \dashrightarrow Y$ birational map $\Leftrightarrow f : \Omega^n \dashrightarrow \Omega^n$

BIRATIONAL GEOMETRY OF CALABI-YAU PAIRS

REMARK

$f : X \dashrightarrow Y$ birational map is **volume preserving** $\iff \forall$



$\forall E \subset W,$

$$a(E, K_X + D_X) = a(E, K_Y + D_Y)$$

EXAMPLE

If $D \subset \mathbb{P}^n$ is a **smooth** hypersurface of degree $n + 1$, and $f : X \rightarrow \mathbb{P}^n$ is a **volume preserving** divisorial contraction with smooth center Z , then

$$Z \subset D \quad \text{and} \quad \text{codim}_{\mathbb{P}^n}(Z) = 2.$$

BIRATIONAL GEOMETRY OF CALABI-YAU PAIRS

DEFINITION

(X, D) Calabi-Yau pair

$$\mathrm{Bir}(X, D) := \left\{ \varphi \in \mathrm{Bir}(X) \mid \varphi : (X, D) \dashrightarrow (X, D) \text{ is volume preserving} \right\}$$

REMARK

When (X, D) is **canonical**

$$\mathrm{Bir}(X, D) = \left\{ \varphi \in \mathrm{Bir}(X) \mid \varphi|_D \in \mathrm{Bir}(D) \right\}$$

EXAMPLE $((\mathbb{P}^3, S))$

$S \subset \mathbb{P}^3$ quartic surface with at worst Du Val singularities

BIRATIONAL GEOMETRY OF CALABI-YAU PAIRS

$S \subset \mathbb{P}^3$ smooth quartic surface

QUESTION (GIZATULLIN)

Which automorphisms of S are induced by a Cremona transformation of the ambient space $\mathbb{P}^3$?

QUESTION

What is the image of the restriction homomorphism

$$\mathrm{Bir}(\mathbb{P}^3, S) = \left\{ \varphi \in \mathrm{Bir}(\mathbb{P}^3) \mid \varphi|_S \in \mathrm{Bir}(S) \right\} \longrightarrow \mathrm{Aut}(S) ?$$

VOLUME PRESERVING SARKISOV PROGRAM

THEOREM (CORTI-KALOGHIROS 2016)

A volume preserving birational map between Mori fibered Calabi-Yau pairs is a composition of **volume preserving Sarkisov links**.

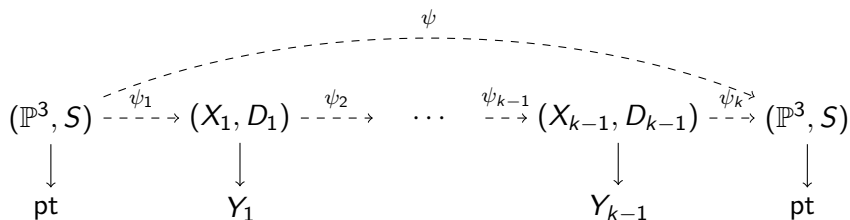
$$\begin{array}{ccccccc} & & \psi & & & & \\ & \nearrow \text{---} & & \nwarrow \text{---} & & & \\ (X_0, D_0) & \xrightarrow{\psi_1} & (X_1, D_1) & \xrightarrow{\psi_2} & \cdots & \xrightarrow{\psi_{k-1}} & (X_{k-1}, D_{k-1}) & \xrightarrow{\psi_k} & (X_k, D_k) \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ Y_0 & & Y_1 & & & & Y_{k-1} & & Y_k \end{array}$$

IDEA OF PROOF

$S \subset \mathbb{P}^3$ general smooth quartic surface with $\rho(S) = 2$ and discriminant r

THEOREM (A-PAIVA-ZIKAS)

① If $r > 57$ or $r = 52$, then $\text{Bir}(\mathbb{P}^3, S) = \{1\}$

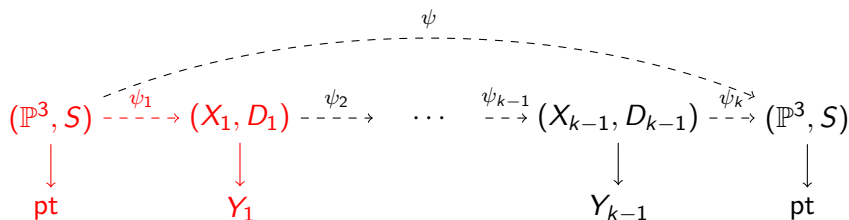


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The **Sarkisov link** ψ_1 starts by blowing up a curve $C \subset S$

PROPOSITION

Classification of curves $C \subset S$ initiating a Sarkisov link in terms of (g, d) :

$(0, 1), (0, 2), (0, 3), (0, 4), (0, 5), (0, 6), (0, 7), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7),$
 $(2, 5), (2, 6), (2, 7), (2, 8), (3, 6), (3, 7), (3, 8), (4, 6), (4, 7), (4, 8), (5, 7), (5, 8),$
 $(6, 8), (6, 9), (7, 8), (7, 9), (8, 9), (9, 9), (10, 9), (10, 10), (11, 10), (14, 11)$

$$\mathbb{Z}H \oplus \mathbb{Z}C \subset \text{Pic}(S) \implies r \mid \text{disc}(\mathbb{Z}H \oplus \mathbb{Z}C)$$

$$\text{disc}(\mathbb{Z}H \oplus \mathbb{Z}C) = \left\{ \begin{array}{l} 9, 12, 17, 24, 33, 44, 57, 9, 16, 25, 36, 49, \\ 17, 28, 41, 56, 20, 33, 48, 12, 25, 40, 17, 32, \\ 24, 41, 16, 33, 25, 17, 9, 28, 20, 17 \end{array} \right\}$$



THEOREM (A-PAIVA-ZIKAS)

② If $r \leq 57$, $r \neq 52$, then

$$\mathrm{Bir}(\mathbb{P}^3, S) \xrightarrow{\sim} \mathrm{Aut}(S) \rightarrow 1$$

EXAMPLE ($r = 56$)

$$\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C \quad Q = \begin{pmatrix} 4 & 8 \\ 8 & 2 \end{pmatrix}$$

$$\mathrm{Aut}(S) \cong \mathbb{Z}_2 * \mathbb{Z}_2 \quad \text{generated by} \quad \begin{pmatrix} -1 & 0 \\ 8 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 31 & 120 \\ -8 & 31 \end{pmatrix}$$

C and $C' \sim 4H - C$ curves with $(g, d) = (2, 8)$

$$\begin{array}{ccc} X & \xrightarrow{\quad \tilde{\varphi} \quad} & X \\ \downarrow \mathrm{Bl}_C & & \downarrow \mathrm{Bl}_C \\ \mathbb{P}^3 & \xrightarrow{\quad \varphi \quad} & \mathbb{P}^3 \end{array}$$

THEOREM (A-PAIVA-ZIKAS)

② If $r \leq 57$, $r \neq 52$, then

$$\mathrm{Bir}(\mathbb{P}^3, S) \xrightarrow{\sim} \mathrm{Aut}(S) \rightarrow 1$$

EXAMPLE ($r = 48$)

$$\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C \quad Q = \begin{pmatrix} 4 & 8 \\ 8 & 4 \end{pmatrix}$$

$$\mathrm{Aut}(S) \cong \mathbb{Z} \quad \text{generated by} \quad \begin{pmatrix} 209 & 56 \\ -56 & -15 \end{pmatrix}$$

C and $C' \sim 4H - C$ curves with $(g, d) = (3, 8)$

$$\begin{array}{ccc} X & \xrightarrow{\quad \tilde{\varphi} \quad} & X_1 \\ \downarrow Bl_C & & \downarrow \\ \mathbb{P}^3 & \xrightarrow{\quad \varphi \quad} & \mathbb{P}^3 \end{array}$$

THEOREM (A-PAIVA-ZIKAS)

② If $r \leq 57$, $r \neq 52$, then

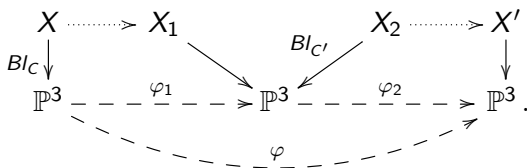
$$\mathrm{Bir}(\mathbb{P}^3, S) \xrightarrow{\sim} \mathrm{Aut}(S) \rightarrow 1$$

EXAMPLE ($r = 48$)

$$\mathrm{Pic}(S) = \mathbb{Z}H \oplus \mathbb{Z}C \quad Q = \begin{pmatrix} 4 & 8 \\ 8 & 4 \end{pmatrix}$$

$$\mathrm{Aut}(S) \cong \mathbb{Z} \text{ generated by } \begin{pmatrix} 209 & 56 \\ -56 & -15 \end{pmatrix}$$

C and $C' \sim 4H - C$ curves with $(g, d) = (3, 8)$



HIGHER PICARD RANK

$S \subset \mathbb{P}^3$ smooth quartic surface with $\rho(S) = 3$

EXAMPLES

- There are surfaces S for which no nontrivial automorphism is induced by a Cremona transformation of $\mathbb{P}^3$:

$$Q = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -40 & 0 \\ 0 & 0 & -40 \end{pmatrix}$$

- (Oguiso's example) $\text{Aut}(S) \cong \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$, and every automorphism of S is induced by a Cremona transformation of $\mathbb{P}^3$.

HIGHER PICARD RANK

$S \subset \mathbb{P}^3$ smooth quartic surface with $\rho(S) = 3$

EXAMPLE

$$Q = \begin{pmatrix} 4 & 0 & 0 \\ 0 & -12 & 0 \\ 0 & 0 & -20 \end{pmatrix}$$

- S contains no rational curves nor elliptic curves $\implies$
 $\text{Aut}(S) = \pi_1(\Sigma_g)$, for a Riemann surface Σ_g of genus $g \geq 2$.
- There are only 2 curve classes in S initiating a Sarkisov link, with
 $(g, d) = (3, 8) \implies \text{Im}\left(\text{Bir}(\mathbb{P}^3, S) \rightarrow \text{Aut}(S)\right) \cong \mathbb{Z}$
- (ChatGPT) $g = 9$ and 18 explicit generators (!)

Thank you!