

Computing Lattices of K3

Lecture 2: p-adic Hodge-theoretic

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ICERM: Arithmetic, Geometry and Computation

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Slides available at edgarcosta.github.io

Where we got to yesterday

- $T = T(X)_{\mathbb{Q}}; E = \text{End}_{\text{Hdg}}(T); d = [E : \mathbb{Q}]; m = \dim$

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$$\rho(X_p^{\text{al}}) \geq \begin{cases} \rho(X^{\text{al}}) & \text{if } E \text{ is CM} \\ \rho(X^{\text{al}}) + d & \text{if } E \text{ is total}$$

Equality occurs infinitely often (density 1 after some finite extension)
 If E is totally real and m is odd, infinitely many good ordinary primes
 $\rho(X^{\text{al}}) + d$ and

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Jumping is frequent; extra endomorphisms complicate the picture
 It is very hard to prove RM!

We are now not only asking to find algebraic cycles in X but

Picard lattice, over finite fields

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Since Frob_q acts semisimply, we have:

$$\rho(X_{p, \mathbf{F}_{q^m}}) = \sum_{\zeta^m=1} \text{ord}_{t=q\zeta} P_2(t)$$

$$\rho(X_p^{\text{al}}) = \sum_{\zeta} \text{ord}_{t=q\zeta} P_2(t)$$

where ζ runs over all roots of unity.

Note: $\rho(X_p^{\text{al}}) \equiv 0 \pmod{2}$

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The Hasse-Weil zeta function $Z_{X_p}(t)$ can be written as

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One may deduce P_2 from the point counts $\#X_p(\mathbf{F}_{q^m})$ for $m \leq b_2/2 + 1$

Reduction to finite characteristic

In this lecture we will focus on projective hypersurfaces, which are more generic, as at some point we will need to do explicit calculations.

Take $f \in \mathbf{Z}[x, y, z, w]$ and $X := Z(f) \subset \mathbf{P}_{\mathbf{Z}}^3$.

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If X and X_p are smooth then the specialization map is injective

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$$\text{and } \rho(X_{\mathbf{Q}^{\mathrm{al}}}) = \rho(X_{\mathbf{Q}_p^{\mathrm{al}}}) \leq \rho(X_p^{\mathrm{al}}).$$

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The specialization map has torsion-free cokernel for $p \neq 2$.

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Can we use it without computing $\mathrm{Pic}(X_p^{\mathrm{al}})$?

The lifting question

Goal

For a given f and p , improve the inequality $\rho(X_{\mathbf{Q}^{\text{al}}}) \leq \rho(X_p^{\text{al}})$.

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Not "which prime" but "which classes": lifting, not

1st ingredient: cohomology

Choose a finite extension $K/\mathbf{Q}_p$, with residue field $k = \mathbf{F}_{p^m}$, so $\rho(X_K) =$

Over characteristic zero we have:

- $H := H_{\mathrm{dR}}^2(X/\mathbf{Q}_p) = F^0 \supset F^1 \supset F^2$, the Hodge filtration
- $\mathrm{Pic}(X^{\mathrm{al}})_{\mathbf{Q}} \hookrightarrow F_K^1 := F^1 \otimes_{\mathbf{Q}_p} K$
- For $d = 4$, $\dim F^i(X) = 22, 21, 1$.

Over characteristic p we have:

- $\mathrm{Pic}(X_p^{\mathrm{al}})_{\mathbf{Q}} \hookrightarrow H_{\mathrm{crys}}^2(X_p \times_{\mathbf{F}_p} k/W(k)) \otimes_{W(k)} K \simeq H$

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Theorem: C lifts to $X^{\text{al}} \iff C \in F_K^1$.

Abelian surface

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$$\text{Frob} \mid_{H^1_{\text{dR}}(A/\mathbf{Q}_p)} \equiv \begin{array}{rrr} 31 \cdot 641 & 31 \cdot 241 & 10 \\ 31 \cdot 341 & 31 \cdot 472 & 9 \\ 31 \cdot 844 & 31 \cdot 515 & 4 \\ 31 \cdot 811 & 31 \cdot 404 & 13 \end{array}$$

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$$\det(t - 31^{-1} \text{Frob} \mid H^2_{\text{dR}}(A/\mathbf{Q}_p)) = (t - 1)^2(3$$

Thus, $\rho\big(A_{\mathbf{F}_p^{\text{al}}}\big) = 2$.

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Since the basis of H^1 respects the Hodge filtration, the inc

Abelian surface, continued

SageMath package: `crystalline_obstruction`.

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Thus, $\rho\big(A_{\mathbf{F}_p^{\text{al}}}\big) = 2$.

Compute 2 eigenvectors

$$v_1 \equiv (0, 839, 568, 649, 1, 372)$$

$$v_2 \equiv (1, 136, 618, 206, 0, 0)$$

The last coordinate of the vectors above gives the projection to H^2/F^1 .

Therefore, $v_1 \notin F^1$ and the corresponding algebraic cycle cannot lift to $\mathbf{Q}_p$.

Thus, we improved $\text{rank NS}(A_{\mathbf{Q}^{\text{al}}}) \leq 2$ to $\text{rank NS}(A_{\mathbf{Q}^{\text{al}}}) \leq 1$, and the van Luijk's method would have succeeded in this example by using a second p

What Frobenius acts on

Via the isomorphism $H^2_{\text{crys}}(X_p/\mathbf{Z}_p) \otimes \mathbf{Q}_p \simeq H^2_{\text{dR}}(X/\mathbf{Q}_p)$

$$\text{Frob}_p : H^2_{\text{dR}}(X/\mathbf{Q}_p) \rightarrow H^2_{\text{dR}}(X/\mathbf{Q}_p)$$

The obstruction map [C-Sertöz]

Compute a p -adic approximation of the obstruction map

$$\pi : \ker(\mathrm{Frob}_p^m - p^m \mathrm{id} \mid H)$$

If $\pi(C) \neq 0$, then $C \notin \mathrm{sp}(\mathrm{Pic}(X^{\mathrm{al}})_{\mathbf{Q}})$. (cf. $\mathrm{Pic}(X_{\mathbf{C}})$)

1. compute a p -adic approximation of Frob_p
2. compute an approximation of $\ker(\mathrm{Frob}_p^m - p^m \mathrm{id} \mid H) =$
the distinct cyclotomic factors of $p^{-22} P_2(pt)$.
3. compute an approximation of $\pi : \ker(\mathrm{Frob}_p^m - p^m \mathrm{id} \mid H)$
4. $\rho(X^{\mathrm{al}}) \leq \dim_{\mathbf{Q}_p} \ker \pi_{\mathbf{Q}_p}$

By picking a basis that respects the Hodge filtration, the map

$$H^2_{\mathrm{dR}}(X/\mathbf{Q}_p) \rightarrow H^2_{\mathrm{dR}}(X/\mathbf{C})$$

is a coordinate projection;

$\pi_{\mathbf{Q}_p} = \pi$ is its restriction to $\ker(\mathrm{Frob}_p^m - p^m \mathrm{id} \mid H)$.

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- What matters here is the shape of the output: an app

Why finite precision still proves something

- An approximate Frobenius gives an approximate eigenvalue.
- But $\pi(C) \neq 0$ is an *open* condition.
- Establishing it to finite precision establishes it.
- Every dimension the method removes is removed rigorously.

Raising N can only remove more. It never puts a dimension back.

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factor of $p^{-1} \operatorname{Frob}_p$	dimension	lifts
$t - 1$	1	1
$(t - 1)^4$	4	3
$t + 1$	1	0
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at $p = 89$ the obstruction improves 10 to 4, and 4 is sharp

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		$t - 1$	1
		$(t - 1)^2$	2
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At $p = 29: 5 \rightarrow 3$, but no further at any precision.

The involution permits only one condition per 2-dimension

What the three examples say

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Does this always work?

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1. at $p = 89$ the obstruction improves 10 to 4, and 4 is
2. at $p = 31$ the reduction bound is already the sharp
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We would also love to combine this with methods over $\mathbf{C}$.

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$$X : w^2 = (-y^2/8 + yz - z^2)(7x^2/8 + 5$$

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- The two extra classes span the piece $t^2 + 1$, irreducible over $\mathbf{C}$.
 $\rho(X_{\mathbf{Q}^{\text{al}}}) = 16$, and X has RM by $\mathbf{Q}(\sqrt{-2})$.
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 $\rho(X_{\mathbf{Q}^{\text{al}}}) = 16$, and X has RM by $\mathbf{Q}(\sqrt{-2})$.
- An obstruction on the $t^2 + 1$ block would prove $\rho(X_{\mathbf{Q}^{\text{al}}}) = 14$.
- After base change to $\mathbf{Q}(\sqrt{-2})$, the residue degree at 83 is 2. Use $F^2 = -1$ on the extra piece. A kernel line is Frobenius-stable.

Given an approximation to Frob_p we will obstruct 2 extra cycles.
 We don't yet know how to compute such approximation!

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What conditions on RM would we need for this not to fail?