Minicourse on arithmetic of K3 surfaces: Obstructions to rational points on K3 surfaces

ICERM workshop: Arithmetic, Geometry and Computations on K3 surfaces

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Local/global principles, Brauer groups, and obstructions

Set up

K number field with Galois group $\Gamma = \mathrm{Gal}(\overline{K}/K)$ and ring of integers $\mathfrak{o}_K$

S/K K3 surface with $\overline{S} = S_{\overline{K}}$

$\mathrm{Pic}(\overline{S}) \simeq \mathbb{Z}^\rho$ geometric Picard group and

$$\mathrm{Br}(\overline{S}) \simeq \mathrm{Hom}(T(\overline{S}), \mathbb{Z}/\mathbb{Q}) \simeq (\mathbb{Q}/\mathbb{Z})^{22-\rho}$$

geometric Brauer group, with Γ actions

$\mathcal{S} \to \mathrm{Spec}(\mathfrak{o}_K)$ flat proper model of S over the ring of integers

v a place of K with completion K_v, with $\mathfrak{p}$-adic ring $\mathfrak{o}_{K,\mathfrak{p}}$, residue characteristic p, and reduction $\mathcal{S} \pmod{\mathfrak{p}}$ for $v = \mathfrak{p}$ non-archimedean

Local obstructions

$S(K) \neq \emptyset \Rightarrow S(K_v) \neq \emptyset$ for each v

If $\mathcal{S} \pmod{\mathfrak{p}}$ is smooth (good reduction) then it almost always has $\mathfrak{o}_{K,\mathfrak{p}}/\mathfrak{p}$-points unless this field has small cardinality. Similar analysis yields smooth points even when the reduction has rational double points. Hensel's Lemma guarantees these lift to K_v.

At the other extreme, if the fiber is an Enriques surface with multiplicity two then $S(K_v) = \emptyset$ (residue characteristic not two).

To summarize, calculations are needed for a finite number of $\mathfrak{p}$: the places of bad reduction and those of small cardinality.

Deconstructing the Brauer group of S: Step 0

The Hochschild-Serre spectral sequence

$$H^p(\Gamma, H^q(\overline{S}, \mathbb{G}_m)) \Rightarrow H^{p+q}(S, \mathbb{G}_m), \quad H^2(S, \mathbb{G}_m) = \mathrm{Br}(S)$$

has several constituent parts.

The arrow $\mathrm{Br}(K) \to \mathrm{Br}(S)$ has kernel equal to the cokernel of

$$\mathrm{Pic}(S) \to H^0(\Gamma, \mathrm{Pic}(\overline{S})).$$

Elements may be represented by embeddings $S \hookrightarrow V$ into a non-split Brauer-Severi variety over K. These always obstruct rational points!

Algebraic versus transcendental

The arrow

$$\ker(H^1(\Gamma, \mathrm{Pic}(\overline{S})) \to H^3(\Gamma, K^*)) \hookrightarrow \mathrm{Br}(S)/\mathrm{Br}(K)$$

yields information similar to what one gets for rational surfaces. This is the algebraic part of the Brauer obstruction.

The arrow

$$\mathrm{Br}(S) \to H^0(\Gamma, \mathrm{Br}(\overline{S}))$$

captures the transcendental Brauer group, which survives on passage to $\overline{K}$. It is harder to write these classes down, as they contain geometric information beyond Galois cohomology.

Good news: $\mathrm{Br}(S)/\mathrm{Br}(K)$ is finite! (Skorobogatov-Zarhin)

Deconstructing the Brauer group of S: Step 1

The crux is the evaluation pairing

$$\begin{gathered}\mathrm{Br}(S) \times S(K_v) \to \mathrm{Br}(K_v) \hookrightarrow \mathbb{Q}/\mathbb{Z} \\ (\alpha, s_v) \mapsto s_v^*\alpha.\end{gathered}$$

Class field theory gives an exact sequence

$$0 \to \mathrm{Br}(K) \to \oplus_v \mathrm{Br}(K_v) \to \mathbb{Q}/\mathbb{Z} \to 0.$$

Since $\mathrm{Br}(\mathfrak{o}_{K,\mathfrak{p}}) = 0$, the $s_v^*\alpha$ are trivial at places of good reduction where α is unramified; we need only evaluate at infinite places, where $\mathcal{S}$ has bad reduction, and where α ramifies. Hence

$$\sum_v s_v^*\alpha \in \mathbb{Q}/\mathbb{Z}$$

is well-defined; the exact sequence shows it vanishes on $\alpha \in \mathrm{Br}(K)$ and when the s_v all come from $s \in S(K)$.

How hard is this evaluation?

we need only evaluate at infinite places, where $\mathcal{S}$ has bad reduction, and where α ramifies

- Infinite places are easy: $\mathrm{Br}(\mathbb{C}) = 0$ and $\mathrm{Br}(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$ so evaluation depends on the connected components of $S(K_v)$.
- Places of good reduction where α ramifies are not a big deal, provided p does not divide the order of α. Local invariants are constant on $S(K_v)$ (Colliot-Thélène and Skorobogatov).
- Places of "mild" reduction, with rational singularities and simply-connected smooth locus, can be handled similarly (Várilly-Alvarado).
- The remaining places can be vexing – we need a convenient model of (S, α)!

When can this be carried out?

For algebraic α, once we have a reasonable model $\mathcal{S} \to \mathrm{Spec}(\mathfrak{o}_K)$ of S, we can usually carry out these computations using Galois cohomology.

For transcendental α, usually we need a geometric construction of α to control the ramification. Examples have been worked out by Ieronymou and with Várilly-Alvarado for two-torsion elements of degree-two K3 surfaces

$$S = \{w^2 = F(x, y, z)\} \quad F \in K[x, y, z]_6.$$

One geometric source of Brauer classes

Consider K3 surfaces S with

$$\mathrm{Pic}(S) = \begin{array}{c|cc} & f & M \\ \hline f & 0 & m \\ M & m & 2s \end{array}, \quad -1 \leqslant s < m-1, m \geqslant 2,$$

which have elliptic fibrations $\pi : S \to \mathbb{P}^1$ with multisection of degree m. The Jacobian $J(S)$ has

$$\mathrm{Pic}(J(S)) = \begin{array}{c|cc} & f & \sigma \\ \hline f & 0 & 1 \\ \sigma & 1 & -2 \end{array},$$

and comes with an $\alpha \in \mathrm{Br}(J(S))[m]$ (Ogg).

Challenge

Use these to obstruct weak approximation, especially where the moduli space is unirational.

Complex geometry digression

Challenge

Let $K = \mathbb{C}(C)$ where C is a curve, S a K3 surface over K, and $\alpha \in \mathrm{Br}(S)$. Define and construct "good" models

$$(\mathcal{S}, \mathcal{A}) \to C$$

where $\mathcal{S} \to C$ is flat and proper with generic fiber S and $\mathcal{A}$ restricts to α. How about after base change $C' \to C$ making the local monodromy unipotent?

Geometric complications

Review of rational varieties

Let S be a geometrically rational surface over $K = \mathbb{C}(C)$ where C is a curve. We have that $S(K) \neq \emptyset$.

More generally, we have

Theorem (Graber-Harris-Starr 2003)

If S is rationally connected over K then $S(K) \neq \emptyset$.

For proper S, the conclusion is equivalent to every proper flat model $f : \mathcal{S} \to C$ having a section.

In positive characteristic, we have

Theorem (de Jong-Starr 2003)

Let $f : \mathcal{S} \to C$ be a flat proper morphism to a curve over an algebraically closed field. Assume that the geometric generic fiber is normal and separably rationally connected. Then f has a section.

"Separably rationally connected" means there exists a very free curve in the smooth locus:

$$f : \mathbb{P}^1 \to \overline{S}^{\mathrm{sm}}, \quad f^* T_{\overline{S}} \text{ ample.}$$

This is equivalent to "rationally connected" for smooth projective varieties of characteristic zero.

Examples of geometrically SRC varieties include:

- geometrically rational smooth varieties;
- generic Fano hypersurfaces $S \subset \mathbb{P}^n$ (Yi Zhu 2024);
- Fano smooth complete intersections of two quadrics;
- "most" reductions mod p of rationally connected varieties over number fields.

Let k be the algebraic closure of a finite field and K the function field of a curve over k. Classical results of Lang and Nagata give that Fano complete intersections S over K **always** have rational points, regardless of their singularities.

Conclusion

Let $K = \mathbb{F}_q(C)$ be the function field of a curve over a finite field. Assuming S Fano or separably rationally connected over K, it is reasonable to expect rational points after extending scalars to $\mathbb{F}_{q^n}$ for some n.

Thus it makes sense to expect tools from Galois cohomology to control rational points.

Presumably, number fields behave analogously.

Why are other classes of varieties harder?

Theorem (Graber-Harris-Mazur-Starr 2005)

Fix a complex variety B and $d \in \mathbb{N}$. Then there exists a bounded family $\mathcal{H}_d$ of morphisms $h : C \to B$ from smooth irreducible curves with the following property: For every proper morphism $\pi : \mathcal{X} \to B$ of relative dimension $\leqslant d$, if $h : C \to B$ is very general in $\mathcal{H}_d$ then the pullback

$$\mathcal{X} \times_B C \to C$$

has a section iff the generic fiber of π contains a (geometrically) rationally connected subvariety.

Thus the GHS theorem characterizes varieties uniformly admitting rational points over function fields of complex curves: They must contain rationally connected subvarieties.

And where do K3 surfaces fit in?

There are plenty of K3 surfaces over $K = \mathbb{C}(t)$ without rational points e.g.

$$S = \{w^4 + tx^4 + t^2y^4 + t^3z^4 = 0\}.$$

This goes back to Lang's 1951 theory of "normic forms".

Let $B \subset \mathbb{P}(\mathbb{C}[w, x, y, z]_4) \simeq \mathbb{P}^{34}$ be the complement of the discriminant and $\pi : \mathcal{X} \to B$ the universal family of smooth quartic surfaces. The example above is a twisted cubic curve in this space. Smaller degree $C \to \mathbb{P}^{34}$ always yield quartic surfaces $\mathcal{X} \times_B C \to C$ with sections!

Thus $\mathcal{X} \to B$ has no rational sections, nor rational curves defined over B. Of course, the geometric fibers have lots of rational curves but these are permuted without fixed points.

Interlude: Density examples

Consider a very general pencil

$$\mathcal{S} = \{sF + tG = 0\} \subset \mathbb{P}^3 \times \mathbb{P}^1 \xrightarrow{f} \mathbb{P}^1, \quad F, G \in \mathbb{C}[w, x, y, z]_4.$$

The base locus $\{F = G = 0\}$ gives constant sections. This is the first of an infinite series of one-parameter families of sections that are dense in $\mathcal{S}$ (with Tschinkel).

Now look at a very general

$$\mathcal{S} = \{s^2F + stG + t^2H\} \subset \mathbb{P}^3 \times \mathbb{P}^1 \xrightarrow{f} \mathbb{P}^1, \quad F, G, H \in \mathbb{C}[w, x, y, z]_4$$

a Calabi Yau fibered in K3 surfaces. It has 64 constant sections $\{F = G = H = 0\}$; rigid sections are dense in $\mathcal{S}$ (Zhiyuan Li).

Can these ever be defined over a number field? Countable field?

The punchline for the obstruction package

For most curves $C \to B \subset \mathbb{P}^{34}$ of sufficiently large degree, the base changes

$$\pi_C : \mathcal{X} \times_B C \to C$$

admit no sections. However, the fibers are either smooth or have a single ordinary A_1 singularity; there are no local obstructions to sections over points $p \in C$. And the Brauer group of $\mathbb{C}(C)$ vanishes, so there is no Brauer-Manin obstruction!

> *Many K3 surface fibrations over complex curves lack sections, yet still admit local sections and no Brauer-Manin obstruction.*

Complex geometry questions

Challenge

Suppose that $\mathcal{X} \to B$ in the GHMS theorem is defined over a number field. Show there exists a "certificate curve" C defined over $\overline{\mathbb{Q}}$.

Question

Consider a family of K3 surfaces over a curve $\varpi : \mathcal{S} \to C$, defined over $\overline{\mathbb{Q}}$. Is there a reasonable algorithm for deciding whether ϖ has a section?

Is there **any** *invariant certifying the non-existence of sections when local obstructions vanish?*

Finite field questions

Fix a finite field $\mathbb{F}_q$ with algebraic closure k, along with a curve C over $\mathbb{F}_q$. Presumably, questions over $K = \mathbb{F}_q(C)$ and $k(C)$ are no easier!

Challenge

Produce curves $C \to U \subset \mathbb{P}^{34}$ such that the resulting quartic surface over K has local points and trivial Brauer group.

Question

Can we show that any of these lack $k(C)$ (or $\mathbb{F}_q(C)$) rational points? Is "very general" is needed in the GHMS converse theorem?