

On Cox rings of Calabi–Yau hypersurfaces of toric Fano varieties

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- 1 Cox rings and Mori dream spaces
- 2 Toric Fano varieties and CY hypersurfaces
- 3 Cox rings of embedded varieties
- 4 Birational automorphisms
- 5 Classification in low dimension

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Cox rings and Mori dream spaces

Cox rings

Let X be a normal projective variety over $\mathbb{C}$ with finitely generated divisor class group $\mathrm{Cl}(X)$. Assume that $\mathrm{Cl}(X)$ is free and fix a subgroup K of $\mathrm{WDiv}(X)$ such that $K \rightarrow \mathrm{Cl}(X)$ is an isomorphism.

The *Cox ring* of X is the K -graded algebra [ADHL15]:

$$\mathcal{R}(X) := \bigoplus_{D \in K} \Gamma(X, \mathcal{O}_X(D)).$$

Cox rings and Mori dream spaces

Characteristic morphism

Assume that $\mathcal{R}(X)$ is finitely generated and let $[H] \in \text{Cl}(X)$ be an ample class. The inclusion $R_H := \bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(nH)) \subseteq \mathcal{R}(X)$ induces a rational map:

$$\overline{X} := \text{Spec } \mathcal{R}(X) \dashrightarrow \text{Proj } R_H = X.$$

The indeterminacy locus of this map is the *irrelevant locus* $\text{Irr}(X)$.

$$\begin{array}{ccc} \hat{X} & \hookrightarrow & \overline{X} \\ p_X \downarrow & & \\ X & & \end{array}$$

Let $\hat{X} := \overline{X} \setminus \text{Irr}(X)$. The induced morphism $p_X : \hat{X} \rightarrow X$, called *characteristic morphism*, is a good quotient by the action of the torus $H_X := \text{Spec } \mathbb{C}[\text{Cl}(X)]$.

Cox rings and Mori dream spaces

Mori dream spaces

Theorem (Hu and Keel [HK00]). Let X be a normal, $\mathbb{Q}$ -factorial projective variety such that $\mathrm{Cl}(X)$ is finitely generated. Then $\mathcal{R}(X)$ is finitely generated if and only if X is a *Mori dream space*:

- ① $\mathrm{Nef}(X)$ is the convex hull of finitely many semiample classes;
- ② there is a finite collection of small birational maps $\{f_i : X \rightarrow X_i\}_{i \in I}$ such that X_i satisfy 1) and

$$\mathrm{Mov}(X) = \bigcup_{i \in I} f_i^* \mathrm{Nef}(X_i).$$

Cox rings and Mori dream spaces

Cox rings of toric varieties

Toric varieties. Let X be a toric variety defined by a fan $\Sigma \subseteq N_{\mathbb{R}}$ whose rays are generated by $v_1, \dots, v_r \in N$. Then

$$\mathcal{R}(X) = \mathbb{C}[x_1, \dots, x_r], \quad \deg(x_i) = [D_{v_i}] \in \text{Cl}(X).$$

A *primitive collection* for X is a subset $\mathcal{P} \subseteq \{v_1, \dots, v_r\}$ which is not contained in any cone of Σ . The irrelevant locus of X is

$$\text{Irr}(X) = \bigcup_{\mathcal{P} \text{ primitive}} V(x_i : v_i \in \mathcal{P}).$$

Cox rings and Mori dream spaces

Why Cox rings of Calabi-Yau hypersurfaces in toric Fano ambient?

- **Main goal:** start exploring the behavior of *Cox rings* of Calabi-Yau manifolds.
- Finite generation of the Cox ring often follows from positivity properties of the anticanonical class (Fano varieties and toric varieties have such property), Calabi-Yau manifolds are challenging since their canonical class is trivial.
- If the Cox ring is finitely generated, then the variety is a *Mori dream space*, thus its birational geometry becomes more explicit and computable.
- Toric ambient geometry makes the problem combinatorial and toric varieties have easy Cox rings.

Outline

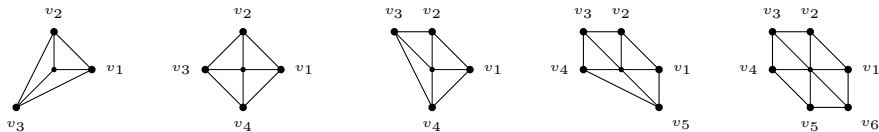
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Toric Fano varieties and CY hypersurfaces

Toric varieties

Let $N \cong \mathbb{Z}^n$ and $M = \text{Hom}(N, \mathbb{Z})$ be its dual. Consider

- $\Delta \subset N_{\mathbb{R}}$ a *smooth Fano polytope* of dimension $n \geq 3$: $0 \in \text{int}(\Delta)$, and each facet's vertices form a lattice basis.
- $Z = Z_{\Delta}$ the associated smooth toric Fano variety, defined by the fan over the faces of Δ .
- In this case the Cox ring is $\mathcal{R}(Z_{\Delta}) = \mathbb{C}[x_{\rho} : \rho \in \text{Vert}(\Delta)]$.
- There is a bijection between $\Delta^{\circ} \cap M$ and the set of monomials of degree $-K_Z$ in $\mathcal{R}(Z_{\Delta})$.



Toric Fano varieties and CY hypersurfaces

CY hypersurfaces

Let $\Delta \subseteq N_{\mathbb{R}}$ be a smooth Fano polytope and $Z = Z_{\Delta}$ be the associated smooth Fano toric variety.

Theorem.

A general (very general if $n = 3$) $X \in |-K_Z|$ is a Calabi-Yau manifold, i.e. $K_X \sim 0$ and $h^i(X, \mathcal{O}_X) = 0$ for $0 < i < n - 1$, and $\text{Cl}(Z) \cong \text{Cl}(X)$ by restriction.

Specific goals:

- decide whether X is a Mori dream space in terms of the combinatorics of Δ ;
- compute $\mathcal{R}(X)$ in terms of the defining equation of X in $\mathcal{R}(Z) = \mathbb{C}[x_1, \dots, x_r]$.

Toric Fano varieties and CY hypersurfaces

Primitive pairs

Given a smooth Fano polytope Δ , a *primitive pair* is a pair of vertices $\{v_i, v_j\}$ of Δ not contained in a common proper face.

Main properties:

- primitive pairs $\{v_i, v_j\}$ correspond bijectively to codimension two components $V(x_i, x_j)$ of the irrelevant locus of Z_Δ ;
- one of the following relations holds:

$$v_i + v_j = 0 \text{ (degree 2),} \quad v_i + v_j = v_k \text{ (degree 1);}$$

- the degree coincides with the vanishing order of a general anticanonical hypersurface of Z_Δ along $V(x_i, x_j)$.

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Cox rings of embedded varieties

A first result

Theorem (Jow [Jow11]; -, Laface [AL12])

Let Z be a smooth Mori dream space of dimension $n \geq 3$ whose irrelevant locus has codimension ≥ 3 and $X \subseteq Z$ be a general (very general if $n = 3$) smooth ample divisor defined by $f \in \mathcal{R}(Z)$. Then

$$\mathcal{R}(X) \cong \mathcal{R}(Z)/(f),$$

in particular X is a Mori dream space.

In particular this holds for a general anticanonical hypersurface X of Z_Δ when Δ is a smooth Fano polytope without primitive pairs.

Example

A general anticanonical hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$ is a Mori dream space.

Cox rings of embedded varieties

Theorem 1

Theorem (-, Laface, Ugaglia 2026)

Let Δ be a smooth reflexive polytope of dimension $n \geq 3$ with vertices $v_1, \dots, v_r$. If the primitive pairs of Δ are those in the following table, then X is a MDS and its Cox ring is as follows.

Primitive pairs	f	$\mathcal{R}(X)$
(i) $v_1 + v_2 = v_3$	$f_1x_1 + f_2x_2$	$\frac{\mathbb{C}[x_1, \dots, x_r, s]}{\langle f_2 - sx_1, f_1 + sx_2 \rangle}$
(ii) $v_1 + v_2 = 0$	$f_1x_1^2 + f_2x_1x_2 + f_3x_2^2$	$\frac{\mathbb{C}[x_1, \dots, x_r, s_1, s_2]}{\langle f_3 - x_1s_1, f_2 + x_1s_2 + x_2s_1, f_1 - x_2s_2 \rangle}$
(iii) $v_1 + v_2 = 0, v_1 + v_4 = v_3, v_2 + v_3 = v_4$	$f_1x_1^2x_3 + f_2x_1x_2 + f_3x_2^2x_4$	$\frac{\mathbb{C}[x_1, \dots, x_r, s_1, s_2]}{\langle f_3 - x_1s_1, f_2 + x_1x_3s_2 + x_2x_4s_1, f_1 - x_2s_2 \rangle}$
(iv) $v_1 + v_2 = v_5, v_3 + v_4 = v_6$	$f_1x_1x_3 + f_2x_1x_4 + f_3x_2x_3 + f_4x_2x_4$	$\frac{\mathbb{C}[x_1, \dots, x_r, s_1, s_2]}{\left\langle \begin{array}{l} f_1f_4 - f_2f_3 + s_1s_2, x_2s_2 + x_3f_1 + x_4f_2 \\ x_1s_2 - x_3f_3 - x_4f_4, x_1f_2 + x_2f_4 - x_3s_1 \\ x_1f_1 + x_2f_3 + x_4s_1 \end{array} \right\rangle}$

Cox rings of embedded varieties

Idea of proof

Let $R := \mathcal{R}(Z)/(f)$, $\tilde{X} = \operatorname{Spec}(R) \subseteq \overline{Z}$, $\hat{X} = \tilde{X} \cap \hat{Z}$ and $\tilde{X}_1 \subseteq \tilde{X}$ be the open subset obtained by removing the union of all codimension-one subvarieties of $\tilde{X} \setminus \hat{X}$:

$$\hat{X} \subseteq \tilde{X}_1 \subseteq \tilde{X}.$$

Such subvarieties are exactly codimension two components of the irrelevant locus of Z .

Theorem (Herrera-Laface-Ugaglia [HLU24])

Let $\{m_1, \dots, m_k\} \subseteq \mathcal{R}(Z)$ be a homogeneous basis of the defining ideal of $\tilde{X} \setminus \tilde{X}_1$. If $\tilde{X}_1$ is normal and $\operatorname{Cl}(Z) \xrightarrow{\sim} \operatorname{Cl}(X)$ then

$$\mathcal{R}(X) = \bigcap_{i=1}^k (\mathcal{R}(Z)/(f))_{m_i}.$$

Cox rings of embedded varieties

Idea of proof

In case (i) the only codimension two component of $\text{Irr}(Z_\Delta)$ is $V(x_1, x_2)$ and the equation of a general Calabi-Yau hypersurface is

$$f = f_1x_1 + f_2x_2, \quad f_2 \in \mathbb{C}[x_2, \dots, x_r].$$

Observe that

$$s := \frac{f_2}{x_1} = -\frac{f_1}{x_2} \in \mathcal{R}(X) = \mathcal{R}(Z)/(f)_{x_1} \cap \mathcal{R}(Z)/(f)_{x_2}.$$

Let $I = (f_2 - sx_1, f_1 + sx_2)$. The condition $\text{cod}_{V(I)} V(x_1, x_2) \geq 2$ guarantees that $\mathcal{R}(X) = \mathcal{R}(Z)[s]/I$.

Cox rings of embedded varieties

Example

Let $Z = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))$:

$$V = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & -1 \end{bmatrix}, \quad \mathcal{R}(Z) = \mathbb{C}[x_1, \dots, x_5], \quad \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 2 \end{bmatrix}.$$

The equation of an anticanonical hypersurface is of the form

$$f_0(x_1, x_2, x_4)x_3^2 - f_1(x_1, x_2, x_4)x_3x_5 + f_2(x_1, x_2, x_4)x_5^2 = 0.$$

By Theorem 1 ii) the Cox ring of X is finitely generated and given by

$$\mathcal{R}(X) = \frac{\mathbb{C}[x_1, \dots, x_5, s_1, s_2]}{\langle f_0 + x_5s_2, f_1 + x_3s_2 + x_5s_1, f_2 + x_3s_1 \rangle}.$$

The projection to (x_1, x_2, x_4) defines a double cover $X \rightarrow \mathbb{P}^2$ branched along the smooth sextic $f_1^2 - 4f_0f_2 = 0$. The line $f_2 = 0$ and the quintic $f_0 = 0$ split in the double cover, giving the new generators s_1, s_2 .

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Birational automorphisms

Theorem 2

Theorem (-, Laface, Ugaglia 2026)

Let Δ be a smooth reflexive polytope of dimension $n \geq 3$. If Δ contains four distinct vertices v_1, v_2, v_3, v_4 such that one of the following holds:

- i) $v_1 + v_2 = v_3 + v_4 = 0$, or*
- ii) $v_1 + v_2 = 0$ and $v_3 + v_4 = v_1$,*

then $\text{Bir}(X) = \text{PsAut}(X)$ has an action of infinite order on $\text{GL}(\text{Pic}(X)_{\mathbb{Q}})$. In particular X is not a Mori dream space.

Birational automorphisms

Idea of proof

- A primitive pair of a *smooth* reflexive polytope with $v_i + v_j = 0$ yields a toric conic bundle on Z_Δ .
- Restriction to X gives a double cover and thus a small birational involution on X .
- In case (i) X has two involutions, computing their action on $\text{Pic}(X)$ one shows that their composition has infinite order.
- In case (ii) there is a fibration $U = Z_{\Delta'} \times (\mathbb{C}^*)^{n-2} \rightarrow (\mathbb{C}^*)^{n-2}$ where $Z_{\Delta'}$ is a blow up of $\mathbb{F}_1$ at $k \in \{0, 1, 2\}$ points. Its restriction to $X \cap U$ defines an elliptic fibration π with a section defined by $x_1 = 0$. We prove that the Mordell-Weil group of π is infinite.
- In both cases we conclude that $\text{Bir}(X) = \text{PsAut}(X)$ has an action of infinite order on $\text{GL}(\text{Pic}(X)_{\mathbb{Q}})$, obstructing the finite generation of $\mathcal{R}(X)$ (by Hu and Keel theorem).

Birational automorphisms

Example–Proposition

Proposition (Ottem [Ott15])

Let X be a smooth Calabi-Yau hypersurface of dimension ≥ 3 in $Z = (\mathbb{P}^1)^m \times \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k}$, where $n_1, \dots, n_k \geq 2$.

- If $m = 0$, then X is a MDS and $\text{Eff}(X) = \text{Mov}(X) = \text{Nef}(X) = \text{Eff}(Z)$;
- if $m = 1$, then X is a MDS and $\text{Eff}(X) \subset \text{Eff}(Z)$;
- if $m > 1$, then X is not a MDS, the image of $\text{PsAut}(X)$ in $\text{GL}(\text{Pic}(X))$ is infinite and $\text{Mov}(X)$ is not rational polyhedral.

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Classification in low dimension

K3 surfaces

Theorem (-, Hausen, Laface [AHL10]; McKernan [McK10])

The following are equivalent for a K3 surface:

- $\mathcal{R}(X)$ is finitely generated;
- $\text{Bir}(X) = \text{Aut}(X)$ is finite;
- $\text{Eff}(X) \subseteq \text{Pic}(X)_{\mathbb{Q}}$ is polyhedral.

Moreover, this property only depends on the lattice-structure of the Picard group of the surface. In the case of K3 hypersurfaces of smooth toric Fano threefolds, Theorem 1 and Theorem 2 cover all cases.

Classification in low dimension

CY 3-folds

Theorem (McKernan [McK10])

A Calabi-Yau threefold is a Mori dream space if and only if its effective cone is rational polyhedral.

To prove polyhedrality of the effective cone in some cases we apply an algorithm based on the following result.

Proposition

Let $F \subseteq \text{Eff}(X)$ be a cone and assume there is a smooth irreducible surface $i : S \rightarrow X$ such that $i^\text{Eff}(X) \subseteq \text{Eff}(S)$ and the intersection form on $\langle i^*(F) \rangle$ is negative semi-definite (non trivial). Then F is contained in a facet of $\text{Eff}(X)$.*

Classification in low dimension

Corollary

MDS	Method	Dimension 3	Dimension 4
Yes	No primitive pairs	23	44, 70, 141, 146, 147
	Thm.1 (i)	19	40, 60, 69, 137, 138
	Thm.1 (ii)	7, 20, 22	25, 41, 64, 139, 144, 145
	Thm.1 (iii)	6, 16	24, 36, 65, 127, 128
	Thm.1 (iv)		53, 55, 97, 109
	Alg.		33, 34, 35, 38, 54, 93, 94, 104, 110, 132, 133
	Ex.		117
No	Thm. 2 (i)	21	37, 42, 52, 59, 61, 62, 63, 66, 126, 140, 142, 143
	Thm. 2 (ii)	11, 12, 18	29, 30, 31, 39, 43, 47, 49, 50, 58, 68, 74, 75, 83, 96, 105, 108, 111, 114, 115, 116, 129, 134, 135, 136
	Thm. 2(i) and (ii)	8, 9, 10, 13, 14, 15, 17	26, 27, 28, 32, 45, 46, 48, 51, 56, 57, 67, 71, 72, 73, 76, 77, 78, 79, 80, 81, 82, 84, 85, 86, 87, 88, 89, 90, 91, 92, 95, 98, 99, 100, 101, 102, 103, 106, 107, 112, 113, 118, 119, 120, 121, 122, 123, 124, 125, 130, 131

Classification in low dimension

Corollary

Corollary

If $X \subseteq Z_\Delta$ has dimension two or three, then either X is a MDS or $\text{Bir}(X)$ is infinite.

Conclusions and open problems

- Calabi–Yau hypersurfaces in smooth toric Fano varieties form a natural testbed for Cox rings at the boundary of positivity.
- Primitive pairs of the reflexive polytope Δ provide effective, combinatorial criteria.
- In dimensions 2 and 3 one obtains a complete classification and a sharp dichotomy:

$$\mathcal{R}(X) \text{ finitely generated} \quad \text{or} \quad \text{Bir}(X) \text{ infinite.}$$

- In higher dimension the combinatorics of primitive pairs is more complicated and we are missing a complete classification.
- We would like to explore further the structure of $\text{Bir}(X)$ for this class of examples and the consequences of our results for the Morrison-Kawamata cone conjecture.

References

- [ADHL15] I. Arzhantsev, U. Derenthal, J. Hausen, and A. Laface. *Cox rings*. Camb. Stud. Adv. Math., vol. 144. Cambridge: Cambridge University Press, 2015.
- [AHL10] M. Artebani, J. Hausen, and A. Laface. On Cox rings of K3 surfaces. *Compos. Math.* 146 (4):964–998, 2010.
- [AL12] M. Artebani and A. Laface. Hypersurfaces in Mori dream spaces. *J. Algebra* 371:26–37, 2012.
- [HK00] Y. Hu and S. Keel. Mori dream spaces and GIT. *Mich. Math. J.* 48:331–348, 2000.
- [HLU24] C. Herrera, A. Laface, and L. Ugaglia. *The Cox ring of an embedded variety*, 2024. arXiv:2411.17370.
- [Jow11] S.-Y. Jow. A Lefschetz hyperplane theorem for Mori dream spaces. *Math. Z.* 268 (1-2):197–209, 2011.
- [McK10] J. McKernan. Mori dream spaces. *Jpn. J. Math.* (3) 5 (1):127–151, 2010.
- [Ott15] J. C. Ottem. Birational geometry of hypersurfaces in products of projective spaces. *Math. Z.* 280 (1-2):135–148, 2015.