Minicourse on arithmetic of K3 surfaces: Density of rational points on special K3s

ICERM workshop: Arithmetic, Geometry and Computations on K3 surfaces

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Motivation and context

K field of characteristic zero

S K3 surface over K with rational points $S(K)$

Question

Is $S(K)$ Zariski dense? Potentially dense? This means $S(K')$ is Zariski dense for some finite extension K'/K.

It makes sense to consider density in other topologies, e.g. when $K = \mathbb{R}, \mathbb{C}$ or $\mathbb{Q}_p$, but our focus will be the Zariski case.

The mantra "Geometry determines arithmetic" suggests that varieties with trivial canonical class – K3 surfaces, abelian surfaces – should share Diophantine properties after an appropriate finite extension.

First sample theorem

Theorem (Harris-Tschinkel 2000)

Let $S \subset \mathbb{P}^3$ be a smooth quartic surface over a number field K and $\ell \subset S$ a line defined over K.

- *rational points of S are potentially dense;*
- *assuming ℓ does not meet six or more other lines in S (over $\overline{K}$) then $S(K)$ is Zariski dense.*

An example is the Fermat surface

$$w^4 + x^4 = y^4 + z^4, \quad \ell = \{w = y,\ x = z\}.$$

Of course, this contains many more lines!

Strategy:

Project from the line to get a genus-one (elliptic) fibration

$$\pi_\ell : S \to \mathbb{P}^1.$$

The line ℓ is a trisection of the fibration, meeting the generic fiber in three points. Basechange

$$S \times_{\mathbb{P}^1} \ell \to \ell$$

to get an elliptic fibration with two sections: ℓ and its trace. The difference has infinite order, unless we are unlucky. Since $\ell \simeq \mathbb{P}^1$ we get lots of rational points on the non-trivial section and its multiples.

This is subtle!

Euler considered the equation

$$x^4 + y^4 + z^4 = w^4$$

and conjectured in 1772 that the only integer solutions were trivial, i.e. with some coordinates zero. Elkies showed the rational points are Zariski dense in 1988!

Logan-McKinnon-van Luijk have density results for many surfaces

$$ax^4 + by^4 + cz^4 + dw^4 = 0, \quad abcd = \text{square},$$

over $\mathbb{Q}$. They require a "seed" rational point with nonzero coordinates, not sitting on the 48 distinguished lines.

Automating the argument

At the cost of taking field extensions, we can extend this to a larger class of K3 surfaces.

Consider elliptic K3 surfaces S, those having a fibration

$$\pi : S \to \mathbb{P}^1$$

with genus one fibers. (We don't insist on a section!) Recall basic facts:

- a complex projective K3 surface S is elliptic iff the Picard group $\mathrm{Pic}(S)$ represents zero, i.e. there is a nonzero divisor class f on S with $f \cdot f = 0$;
- when $\mathrm{Pic}(S)$ has rank $\geqslant 5$ then S is necessarily elliptic.

Theorem (Bogomolov-Tschinkel 2000)

Elliptic K3 surfaces have potentially dense rational points.

The basic strategy is to find a rational multisection

$$\mathbb{P}^1 \simeq M \subset S \xrightarrow{\pi} \mathbb{P}^1$$

of degree $m > 1$, so that the base-change

$$S \times_{\mathbb{P}^1} M \to M$$

has a section of infinite order. (Take the diagonal to be the origin.)

This is straightforward if M is simply branched over $\mathbb{P}^1$ or π is not isotrivial. More work is required with degenerate singular fibers.

Second sample theorem

Let $S \subset \mathbb{P}^2 \times \mathbb{P}^2$ be a smooth complete intersection of $(1,1)$ and $(2,2)$ hypersurfaces

$$\mathrm{Pic}(S) \supset \begin{array}{c|cc} & f_1 & f_2 \\ \hline f_1 & 2 & 4 \\ f_2 & 4 & 2 \end{array}$$

The projections $\pi_i : S \to \mathbb{P}^2$ give covering involutions ι_1 and ι_2 and $\iota_1 \circ \iota_2$ has infinite order. By producing an appropriate rational curve $M \subset S$, with dense orbit, we can establish:

S has potentially dense rational points.

Silverman will explore such examples in more detail!

Automorphisms - extending the argument

Theorem (Bogomolov-Tschinkel 2000)

K3 surfaces with infinite automorphisms have potentially dense rational points.

This covers K3 surfaces S with

$$\mathrm{Pic}(S) = \begin{array}{c|cc} & f_1 & f_2 \\ \hline f_1 & 2 & n \\ f_2 & n & 2 \end{array}, \quad n \geqslant 4.$$

However, $n = 3$ is an exception:

$$\begin{array}{c|cc} & f_1 & f_2 \\ \hline f_1 & 2 & 3 \\ f_2 & 3 & 2 \end{array} \simeq \begin{array}{r|cc} & f_1 & f_2 - f_1 \\ \hline f_1 & 2 & 1 \\ f_2 - f_1 & 1 & -2 \end{array}$$

It has one involution and a Picard-Lefschetz reflection.

Summary of potential density results

The following classes of K3 surfaces S have potentially dense rational points over all fields of characteristic zero:

- $\mathrm{rank}(Pic(S)) \geqslant 5$;
- $\mathrm{rank}(Pic(S)) \geqslant 2$ provided $\mathrm{Pic}(S)$ does not represent -2.

This includes all but finitely many lattices of Picard rank 3 or 4. The largest exception is

$$\begin{array}{c|cccc} & f & E_1 & E_2 & E_3 \\ \hline f & 2 & 1 & 1 & 1 \\ E_1 & 1 & -2 & 0 & 0 \\ E_2 & 1 & 0 & -2 & 0 \\ E_3 & 1 & 0 & 0 & -2 \end{array}$$

And what about K3 surfaces of rank one?

Question

Is there a K3 surface S over a number field K with geometric rank one and dense rational points?

I know no published examples! Here is one alternative framing:

Theorem (H-Tschinkel 2000)

Let S be a K3 surface. For some n the punctual Hilbert scheme $S^{[n]}$ has potentially dense rational points.

The key construction is to show that $S^{[n]}$ has an abelian fibration

$$\pi : S^{[n]} \dashrightarrow \mathbb{P}^n.$$

for some n. If $\deg(S) = 2g - 2$ then $n = g$ works.

An embarrassing question

Question

Let X be a smooth projective manifold deformation equivalent to $S^{[n]}$ for a K3 surface S. Assume that either

- *X is birational to an abelian fibration; or*
- *the birational automorphism group of X is infinite.*

Does X have potentially dense rational points when it is defined over a number field?

This would be interesting even on restriction to X birational to $S^{[n]}$ for some K3 surface S.

This question is more salient now because results of Bayer-Macrì, Markman, Oguiso, and others allow us to read off from $\mathrm{Pic}(X)$ whether these conditions are satisfied.

Other potential density results – in rank one

Let X be the variety of lines on a smooth cubic fourfold Y; Beauville-Donagi have shown that X is deformation equivalent to $S^{[2]}$ for a K3 surface S.

Voisin has constructed a rational self-map

$$\phi : X \dashrightarrow X$$

defined as follows: For a general line $\ell \subset Y$, the normal bundle $N_{\ell/Y} \simeq \mathcal{O}^2 \oplus \mathcal{O}(1)$. Let $\Pi \supset \ell$ be the plane associated with the positive summand so that $Y \cap \Pi = \ell + 2\ell'$. Write $\phi(\ell) = \ell'$ which yields a degree-16 mapping.

Theorem (Amerik-Voisin 2007)

Assume that X is defined over a number field and $\mathrm{Pic}(X_{\mathbb{C}}) = \mathbb{Z}$. Then rational points on X are potentially dense.

As far as I can see, the argument **only** works in Picard rank one! It does not prove potential density for any Hilbert scheme $S^{[2]}$, even though these are topologically dense in the moduli space. Symmetric squares contain many rational surfaces arising from rational curves $\mathbb{P}^1 \to S$. Do none of these have dense orbits?

Question

Are there any examples of X of higher Picard rank where iterates of ϕ on one of these surfaces is dense?

Computational challenge:

Let S be a K3 surface of degree 6 or 14, so that $S^{[2]}$ is birational to the variety of lines of a cubic fourfold. Specifically, these arise from nodal or Pfaffian cubic fourfolds $Y \subset \mathbb{P}^5$.

Challenge

Describe the map $\phi : S^{[2]} \dashrightarrow S^{[2]}$ *explicitly and its action on rational curves.*

Question

Are there any other such endomorphisms among hyperkähler manifolds?

And why don't these happen for K3 surfaces?

The following statement has been posted to the arXiv (twice) and published in 2025 by Karzhemanov and Konovalov:

> *Let S be a complex projective K3 surface with $\mathrm{Pic}(S) = \mathbb{Z}$. Then there exists no rational map $\phi : A \dashrightarrow S$ of degree $\geqslant 2$ from a surface with trivial canonical class.*

Challenge

Formalize this statement in Lean.

Voisin has shown that a general quartic K3 surface cannot be dominated by an abelian surface.

Examples of non-trivial rational maps

Let $\pi : S \to \mathbb{P}^1$ be an elliptic K3 surface with multisection M of degree $m > 1$. Write $J(S) \to \mathbb{P}^1$ for its Jacobian elliptic fibration; the multisection gives an isomorphism $\mathrm{Pic}^m(S/\mathbb{P}^1) \xrightarrow{\sim} S$ over $\mathbb{P}^1$. Multiplication by m

$$\phi : S \dashrightarrow J(S)$$

gives a rational map over the base of degree m^2; it is regular where π is smooth, away from the singularities of the degenerate fibers.

In the generic case – where π has 24 degenerate fibers of Kodaira type I_1 – the indeterminacy is supported on the 24 nodes. The number of blowups needed grows quickly with m.

Dedieu's Theorem

This example is suggestive in light of:

Theorem (Dedieu 2007)

Let S be a complex projective K3 surface of Picard rank one and degree $2g-2$. Let $\phi : S \dashrightarrow S$ be a rational map of degree $\geqslant 2$; write $\phi^*\mathcal{O}_S(1) = \mathcal{O}_S(l)$ with $l > 1$. Then $\deg(\phi) = \lambda^2$ where λ is an integer such that $(2g-2)|(l-\lambda)$.

Dedieu says much more about the structure of these maps and how they act on rational curves.

A random remark:

Let X parametrize the lines on a cubic fourfold; Voisin gives a degree 16 self-map $\phi : X \dashrightarrow X$.

There is a specialization $X \rightsquigarrow S^{[2]}$ where (S, f) is a degree two K3 surface. These are limits associated with determinantal families

$$\det \begin{pmatrix} A & B & C \\ B & D & E \\ C & E & F \end{pmatrix} + \epsilon P(A, B, C, D, E, F) = 0.$$

The linear series $|f|$ gives an abelian fibration with section

$$S^{[2]} \overset{\sim}{\dashrightarrow} \mathrm{Pic}^2(|f|) \xrightarrow{\pi} \mathbb{P}^2.$$

Multiplication by two is a degree 16 $\varphi : S^{[2]} \dashrightarrow S^{[2]}$.
Are these related?

One last challenge

For a complex projective K3 surface S, write

$$T(S) = \mathrm{Pic}(S)^{\perp} \subset H^2(S, \mathbb{Z})$$

for the transcendental cohomology.

Recently, van Geemen and Schütt constructed one-parameter families of K3 surfaces S with the following properties:

- a totally real field F with $\deg(F/\mathbb{Q}) = 7$ acts on $T(S)_{\mathbb{Q}}$;
- $\dim_F T(S)_{\mathbb{Q}} = 3$.

With Bayer-Fluckiger, they show that any such F arises. Typically

$$\dim_{\mathbb{Q}} T(S)_{\mathbb{Q}} = 3 \cdot 7 \Rightarrow \mathrm{Pic}(S) \simeq \mathbb{Z}$$

but the surfaces are parametrized by a compact Shimura curve.

Challenge

Show that such surfaces admit no rational maps $\phi : S_1 \dashrightarrow S_2$ *with* $\deg(\phi) > 1$.

Of course, this follows from the general results on rational self-maps. But I find these the most surprising situations where they apply!