A Free-Probabilistic State Evolution

and application to random matrix discrepancy

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Road map

I **Matrix discrepancy**

II **Free-probabilistic state evolution**

III **Application**

IV **Questions**

PART I

Matrix discrepancy

Matrix discrepancy

Let $A_1, \cdots, A_n$ be $n \times n$ symmetric with $\max_i \|A_i\|_{\mathrm{op}} \leq 1$, and let

$$\mathcal{A}(x) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i A_i \,.$$

Question (Zouzias 2012, Meka 2014):

Does there exist a constant C such that

$$\min_{x \in \{-1,+1\}^n} \|\mathcal{A}(x)\|_{\mathrm{op}} \leq C\,?$$

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If A_i's are **diagonal** $\implies$ set/vector discrepancy:

- Spencer's "six standard deviations suffice". Non-constructive.
- Large literature; existence + algorithms.
- Related: Beck-Fiala , Komlös conjectures.

"op" means $\ell_2 \to \ell_2$.

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- Random signs: $O(\sqrt{\log n})$ via non-commutative Khintchine inequality.
- Bansal, Jiang, Meka (2022): True if $\max_i \text{rank}(A_i) \leq n/(\log n)^3$.
- Akbas, Sra (2026) : True if $A_1, \ldots, A_n$ in a C^*-algebra of dimension $O(n)$.

Random Matrix discrepancy

Today:

$$A_1, \cdots, A_n \overset{\text{iid}}{\sim} \text{GOE}(d) : \qquad (A_i)_{jk} = (A_i)_{kj} \sim N(0, (1+\delta_{jk})/d).$$

$$\mathcal{A}(x) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} x_i A_i \, .$$

Question (Kunisky, Zhang 2023):

If

$$\frac{2n}{d^2} \longrightarrow \alpha \, ,$$

can we compress the spectrum :

$$\min_{x \in \{-1,+1\}^n} \|\mathcal{A}(x)\|_{\text{op}} < 2 \, ?$$

(Adaptive choice is required: for x fixed, $\mathcal{A}(x) \sim \text{GOE}(d)$.)

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$$\mathsf{OPT} := \min_{x \in \{-1,+1\}^n} \|\mathcal{A}(x)\|_{\mathrm{op}} .$$

Theorem (Maillard 2025):

Let $2n/d^2 \longrightarrow \alpha$.

1. For any $\kappa < 2$, if $\alpha < \alpha_0(\kappa)$ then OPT $> \kappa$ whp.
2. If α large enough (> 3),

$$\mathsf{OPT} < \kappa(\alpha) < 2 \quad \text{whp}$$

   with $\kappa(\alpha) \to 0$ as $\alpha \to \infty$.

Proof by first and second moments + LDP for norm-constrained GOE matrices.

Algorithm

Theorem (Chen, EA 2026+):

Let $2n/d^2 \longrightarrow \alpha$. One can algorithmically achieve $m \in \mathbb{R}^n$ with

$$\|\mathcal{A}(m)\|_{\mathrm{op}} \leq 2\sqrt{1 - \frac{2\sqrt{\alpha}}{\pi} B\left(\frac{1}{2}, \frac{3}{4}\right) + \frac{2\alpha}{\pi}} + o(1)\,,$$

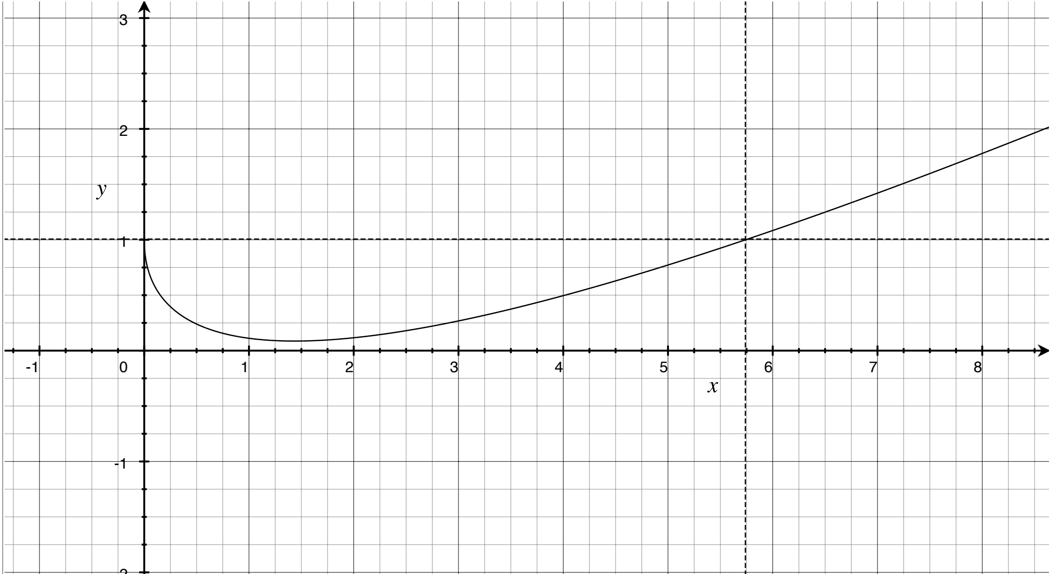
with

$$\frac{1}{n}\mathsf{dist}_{\ell_2}(m, \{-1, +1\}^n)^2 = o(1)\,.$$

More generally, one can realize every variance between

$$\sigma_{\pm}^2(\alpha) = 1 \pm \frac{2\sqrt{\alpha}}{\pi} B\left(\frac{1}{2}, \frac{3}{4}\right) + \frac{2\alpha}{\pi}\,, \qquad B\left(\frac{1}{2}, \frac{3}{4}\right) = 2.396280469...$$

$$\alpha \mapsto 1 - \frac{2\sqrt{\alpha}}{\pi} B\left(\frac{1}{2}, \frac{3}{4}\right) + \frac{2\alpha}{\pi}$$



PART II

Free-probabilistic state evolution

The Gaussian operator

Let

$$H_n = \big(\mathbb{R}^n, \langle \cdot, \cdot \rangle_n\big), \qquad \langle x, y \rangle_n = n^{-1} x^\top y\,,$$

$$K_d = \big(\mathrm{Sym}_d, \langle \cdot, \cdot \rangle_d\big)\,, \qquad \langle M, N \rangle_d = d^{-1}\,\mathrm{Tr}(MN)\,.$$

$$\mathcal{A} : H_n \longrightarrow K_d\,, \qquad \mathcal{A}(x) = \frac{1}{\sqrt{n}} \sum_i x_i A_i\,.$$

$$\mathcal{A}^* : K_d \longrightarrow H_n\,, \qquad \mathcal{A}^*(M) = \sqrt{n}(\langle A_i, M \rangle_d)_{i=1}^n\,.$$

Approximate Message Passing Iteration

Initial $u^0 \in \mathbb{R}^n$. Choose functions $g_t : \mathbb{R}^{t+1} \to \mathbb{R}, t \geq 0$, and scalars $(a_{t,s})_{0\leq s\leq t}$.

$$m_i^t = g_t(u_i^0, \ldots, u_i^t), \qquad M^t = \sum_{s=0}^{t} a_{t,s} V^s,$$

$$V^t = \mathcal{A}(m^t) - \underbrace{\alpha \sum_{r=0}^{t} d_{t,r} M^{r-1}}_{\mathrm{ons}_V^t}, \qquad u^{t+1} = \mathcal{A}^*(M^t) - \underbrace{\sum_{r=0}^{t} a_{t,r} m^r}_{\mathrm{ons}_u^t},$$

$$d_{t,r} = \frac{1}{n} \sum_{i=1}^{n} \partial_r g_t(u_i^0, \ldots, u_i^t), \qquad M^{-1} = 0_{d\times d}.$$

The 'ons' corrections: The reuse of $\mathcal{A}$ creates correlations with the past. The two subracted terms 'cancel' this self-interaction.

Approximate Message Passing Iteration

Example:

$g_t(u^0, \ldots, u^t) = u^t$, $a_{t,s} = \delta_{t,s}$:

$$V^t = \mathcal{A}(u^t) - \alpha V^{t-1}\,, \qquad\qquad u^{t+1} = \mathcal{A}^*(V^t) - u^t\,.$$

Similar to power iteration + 'normalization by subtraction'.

Approximate Message Passing Iteration

AMP trajectory: $u^0, \ldots, u^t \in \mathbb{R}^n, \quad V^0, \ldots, V^t \in \mathrm{Sym}_d$.

We want to understand the joint empirical distribution of vectors $u^0, \ldots, u^t$, and the joint spectral distribution of the matrices $V^0, \ldots, V^t$.

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The vector side:

Track the empirical measures

$$\frac{1}{n}\sum_{i=1}^{n} \delta_{u_i^0, \cdots, u_i^t}$$

tested against pseudo-Lipschitz functions $\psi : \mathbb{R}^{t+1} \to \mathbb{R}$. Standard empirical convergence.

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The matrix side:

Track traces and norms of polynomial evaluations

$$\frac{1}{d} \operatorname{Tr} P(V^0, \ldots, V^t), \quad \|P(V^0, \ldots, V^t)\|.$$

$\to$ Notion of **strong convergence** in free probability.

Strong convergence

Let $\tau_d(X) = (1/d)\,\mathrm{Tr}\,X$. Let $(\mathcal{N}, \tau)$ be a C^*-probability space (C^* algebra of bounded self-adjoint operators, closed under $\|\cdot\|$-topology + faithful trace τ).

Strong convergence

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Definition: A tuple $(X_1, \cdots, X_k)$ of self-adjoint $d \times d$ matrices converges strongly in noncommutative distribution to $(s_1, \cdots, s_k)$ of self-adjoint elements in $\mathcal{N}$:

$$(X_1, \cdots, X_k) \xrightarrow{\text{str}} (s_1, \ldots, s_k)\,,$$

if for every noncommutative polynomial P,

$$\tau_d(P(X_1, \ldots, X_k)) \longrightarrow \tau(P(s_1, \ldots, s_k))\,, \quad \|P(X_1, \ldots, X_k)\|_{\mathrm{op}} \longrightarrow \|P(s_1, \ldots, s_k)\|\,.$$

Strong convergence rules out spectral outliers.

Voiculescu, Mingo-Speicher, Nica-Speicher,...

Semicircular variables

A centered semicircular family $(s_1, \ldots, s_k)$ is characterized by its covariance $(\tau(s_i s_j))$ and vanishing free cumulants above order two.

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Properties:

- If $s \sim \mathrm{SC}(0,1)$, then $\tau(s^k) = \int x^k \mu_{\mathsf{sc}}(\mathrm{d}x)$, $k \geq 0$.
- If $s \sim \mathrm{SC}(0,\sigma^2)$, then $\|s\| = 2\sigma$. (variance computation $\implies$ operator-norm guarantee.)
- A semicircular family is **free** iff $\tau(s_i s_j) = 0$ for $i \neq j$.

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Freeness: "alternating products have vanishing trace". Sub-algebras $\mathcal{A}_1, \dots, \mathcal{A}_s$ are free if $\tau(a_1 a_2 \dots a_r) = 0$ whenever $a_i \in \mathcal{A}_{j_i}$, $j_1 \neq j_2$, $j_2 \neq j_3, \dots, j_{r-1} \neq j_r$.

Voiculescu 1992. Mingo, Speicher 2017.

The target bi-process

Let $(U^1, \dots, U^{T+1})$ be centered jointly Gaussian, independent of U^0, and let $(S^0, \dots, S^T)$ be a centered correlated semicircular family.

Let

$$G^t = g_t(U^0, \dots, U^t), \qquad W^t = \sum_{s=0}^{t} a_{t,s} S^s.$$

Recursive definition of the correlations:

$$\tau(S^r S^s) = \mathbb{E}[G^r G^s].$$

scalar covariance $\to$ free covariance

$$\mathbb{E}[U^{r+1} U^{s+1}] = \alpha\, \tau(W^r W^s).$$

free covariance $\to$ scalar covariance

Free State Evolution

$$m_i^t = g_t(u_i^0, \ldots, u_i^t), \qquad M^t = \sum_{s=0}^{t} a_{t,s} V^s,$$

$$V^t = \mathcal{A}(m^t) - \alpha \sum_{r=0}^{t} d_{t,r} M^{r-1}, \qquad u^{t+1} = \mathcal{A}^*(M^t) - \sum_{r=0}^{t} a_{t,r} m^r,$$

Theorem (Chen, EA 2026+):

Fix T. Assume $2n/d^2 \to \alpha$, an independent convergent initialization u^0, regular g_t's, and fixed bounded $a_{t,s}$. Then

$$(u_i^{0:T+1}, m_i^{0:T})_{i \le n} \xrightarrow{\text{emp}} (U^{0:T+1}, G^{0:T}),$$

and

$$(V^{0:T}, M^{0:T}) \xrightarrow{\text{str}} (S^{0:T}, W^{0:T}).$$

empirical convergence of vectors, strong noncommutative convergence of matrices.

Proof by induction using Bolthausen conditioning

Collect past messages in $Q : \mathbb{R}^t \to H_n\,,\ Qe_s = m^s$ and $R : \mathbb{R}^t \to K_d\,,\ Re_s = M^s$, $0 \le s < t$:

$$Y = \mathcal{A}Q, \qquad X = \mathcal{A}^* R, \qquad G_Q = Q^* Q, \quad G_R = R^* R.$$

Conditionally on the alternating observations generated so far,

$$\mathcal{A} \overset{d}{=} \underbrace{Y G_Q^{-1} Q^* + R G_R^{-1} X^* P_Q^{\perp}}_{\text{regression determined by the past}} + \underbrace{P_R^{\perp} \widetilde{\mathcal{A}} P_Q^{\perp}}_{\text{fresh independent Gaussian block}}$$

Adaptive observations reveal only finitely many input and output directions. On the orthogonal complements, the Gaussian operator keeps its original law.

Bolthausen 2014. Berthier, Montanari, Nguyen 2017.

Onsager corrections remove the conditional bias

Project the new scalar message onto its past:

$$m^t = Q_{t-1}\beta_t + h_t, \qquad h_t \perp \mathrm{Ran}(Q_{t-1}).$$

Gaussian conditioning and integration by parts identify the conditional mean.

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After the Onsager subtraction,

$$V^t = \sum_{k<t} \beta_{t,k} V^k + \sqrt{\bar{q}_t}\, Z_t^{\mathrm{GOE}} + o_{\mathrm{op}}(1),$$

with Z_t^{GOE} independent of the past, $\bar{q}_t = \|h_t\|_2^2/n$.

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The adjoint update has the parallel form

$$u^{t+1} = \sum_{s<t} \gamma_{t,s} u^{s+1} + \sqrt{\alpha q_t}\, z^{t+1} + o_n(1),$$

where z^{t+1} has iid standard Gaussian coordinates. [Where $M^t = R_{t-1}\gamma_t + k_t$, $q_t = \tau_d(k_t^2)$.]

Joint strong asymptotic freeness

The **crucial input** beyond classical state evolution:

Theorem (Fan–Sun–Wang, 2021):

If

$$(B_1, \ldots, B_\ell) \xrightarrow{\text{str}} (b_1, \ldots, b_\ell)$$

and $Z_1, \ldots, Z_k$ are independent GOE matrices, independent of the B-array, then

$$(Z_1, \ldots, Z_k, B_1, \ldots, B_\ell) \xrightarrow{\text{str}} (s_1, \ldots, s_k, b_1, \ldots, b_\ell),$$

where the s_j's are freely independent semicircular variables, free from the past.

Following breakthrough by Haagerup–Thorbjornsen (2001), extentions by Schultz (2005), and Male (2011),...

PART III

Incremental AMP and application

Stochastic controls and their free-probabilistic lift

IAMP for matrix discrepancy

Goal: Find $x \in \{-1,+1\}^n$ such that $\|\mathcal{A}(x)\|_{\mathrm{op}} \leq 2\sigma < 2$.

Recall from AMP:

$$m_i^t = g_t(u_i^0, \ldots, u_i^t), \qquad M^t = \sum_{s=0}^{t} a_{t,s} V^s, \qquad \mathcal{A}(m^t) = V^t + \alpha \sum_{r=0}^{t} d_{t,r} M^{r-1}.$$

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Orthogonal updates: Let g_t and (a_{ts}) be such that

$$m_i^t = \sum_{s<t} b_s(u_i^{0:s})(u_i^{s+1} - u_i^s)\,, \qquad M^t = \sum_{s<t} a_s(V^{s+1} - V^s)\,.$$

Subag 2018. Montanari 2019.

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Subag 2018. Montanari 2019.

Lemma (independent/free increments): If $U^0 = 0$, $g_0(u) = \sqrt{\delta}$, then

$$U^{t+1} - U^t \perp U^{0:t}, \qquad S^{t+1} - S^t \perp^{\text{free}} S^{0:t}, \quad t \ge 0\,.$$

IAMP gives independent increments

Variance recursions: From State Evolution we get

$$\mathbb{E}[(U^{t+1} - U^t)^2] = \alpha a_{t-1}^2 \tau((S^t - S^{t-1})^2)\,,$$

$$\tau((S^t - S^{t-1})^2) = \mathbb{E}[b_{t-1}^2] \cdot \mathbb{E}[(U^t - U^{t-1})^2]\,.$$

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Impose constraint:

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Impose constraint:

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$\implies$

U-increments have constant variance δ. Standard Brownian motion.

S-increments have variance $\mathbb{E}[b_t^2]\delta$. (Time changed) **free** Brownian motion.

Output spectral control

Recall:

$$m_i^t = \sum_{s<t} b_s(u_i^{0:s})(u_i^{s+1} - u_i^s)\,, \qquad M^t = \sum_{s<t} a_s(V^{s+1} - V^s)\,.$$

with

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with

$$\alpha a_t^2 \mathbb{E}[b_t^2] = 1 \quad \forall t\,.$$

Spectral control: Let $\mu_t = \mathbb{E}[b_t]$, $r_t = \mathbb{E}[b_t^2]$, $a_t = -1/\sqrt{\alpha r_t}$. One obtains by calculus + SE

$$\begin{aligned} \mathcal{A}(m^t) &= \sum_{r=0}^{t} \left[1 + \alpha\mu_r a_r\right](V^{r+1} - V^r) \\ &\xrightarrow{\text{str}} \sum_{r=0}^{t} \left[1 + \alpha\mu_r a_r\right](S^{r+1} - S^r) \sim \mathrm{SC}(0, \sigma_t^2)\,, \end{aligned}$$

with $\sigma_t^2 = \delta \sum_{r=0}^{t} (\sqrt{r_s} + \eta\sqrt{\alpha}\mu_r)^2$.

Output vector; need to design b_t

A stochastic control approach: Fix $\gamma : [0,1) \to \mathbb{R}_+$. Let Φ_γ solve PDE

$$\partial_t \Phi_\gamma + \frac{1}{2}\left\{\gamma(t)(\partial_x \Phi_\gamma)^2 + \partial_{xx}\Phi_\gamma\right\} = 0, \qquad \Phi_\gamma(1,x) = |x|.$$

Drive

$$\mathrm{d}X_t = \gamma(t)\partial_x\Phi_\gamma(t,X_t)\,\mathrm{d} + \mathrm{d}B_t, \qquad \boxed{b_t = \partial_{xx}\Phi_\gamma(t,X_t).}$$

Itô's formula gives

$$m_t = \partial_x \Phi_\gamma(t,X_t), \qquad \mathrm{d}m_t = b_t\,\mathrm{d}B_t,$$

and therefore

$$\boxed{m_1 = \mathrm{sign}(X_1) \in \{-1,+1\}}.$$

Montanari 2019, El Alaoui–Montanari–Sellke 2021.

The lower endpoint gives binary discrepancy

With the choice $\gamma(t) = \Gamma\,\mathbf{1}_{\{t \geq q\}}$. The construction approaches

$$\boxed{\|\mathcal{A}(m^{\lfloor 1/\delta \rfloor})\|_{\mathrm{op}} \leq 2\sqrt{1 - \frac{2\sqrt{\alpha}}{\pi} B\left(\frac{1}{2}, \frac{3}{4}\right) + \frac{2\alpha}{\pi}} + o(1)\,.}$$

Algorithmic complexity: The controls are computed offline in one dimension. The high-dimensional cost is the number of applications of $\mathcal{A}$ and $\mathcal{A}^*$.

PART IV

Questions

Questions

1. Improve previous construction:

Optimized choice of γ. More general non-linearities on the matrix side:

$$M^t = F_t(V^0, \dots, V^t)\,, \quad \text{instead of} \quad M^t = \sum_{s=0}^{t} a_{t,s} V^s\,.$$

E.g. $F_t(\dots) = \sum_{s<t} \quad a_s(V^{s+1} - V^s)a_s$, $a_s \in \mathrm{Sym}_d$ predictible.

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E.g. $F_t(\ldots) = \sum_{s<t} \sum_k a_s^k (V^{s+1} - V^s) a_s^k$, $a_s^k \in \mathrm{Sym}_d$ predictible.

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E.g. $F_t(\ldots) = \sum_{s<t} C_s \#(V^{s+1} - V^s)$, $C_s : \mathrm{Sym}_d \to \mathrm{Sym}_d$ predictible.

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E.g. $F_t(\ldots) = \sum_{s<t} C_s\#(V^{s+1} - V^s)$, $C_s : \mathrm{Sym}_d \to \mathrm{Sym}_d$ predictible.

Leads to genuine **free** stochastic calculus (Biane–Speicher 1998):

$$\mathcal{A}(m^t) \xrightarrow{\text{str}} \int_0^t (1 + \alpha\mu_r C_r\#)\mathrm{d}S_r\,.$$

Q: Design C_r to achieve a target operator norm / spectral distribution...

Questions

2. What is the value of

$$\min_{x \in \{-1,+1\}^n} \|\mathcal{A}(x)\|_{\mathrm{op}} \quad ?$$

A “free” Parisi formula??

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A "free" Parisi formula??

3. Which spectral densities are possible (algorithmically)?

Given target density ρ, find a signing x such that

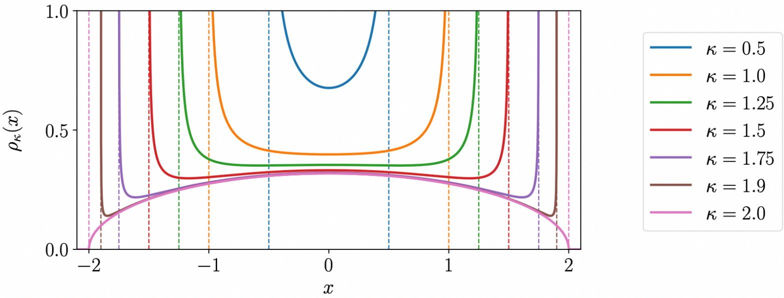
$$\frac{1}{d}\sum_{i=1}^{d} \delta_{\lambda_i(\mathcal{A}(x))} \simeq \rho \ ?$$

Constrained GOE spectrum

Theorem (Maillard 2025):

Let $\kappa < 2$. Let $A \sim \mathrm{GOE}(d)$ conditioned on $\|A\|_{\mathrm{op}} \leq \kappa$. Then spectral density

$$\mu_A \xrightarrow{w} \rho_\kappa(x) = \frac{4 + \kappa^2 - 2x^2}{4\pi\sqrt{\kappa^2 - x^2}} .$$



Moreover, a typical satisfiable signing x leads $\mathcal{A}(x)$ to have the above spectrum.

Thank you!