Gaussian Conjugate Rogers–Shephard Inequality

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Gaussian Correlation Inequality

Gaussian Correlation Inequality, Royen '14

$\forall$ convex $K = -K$ and $L = -L$ in $\mathbb{R}^n$, $\gamma(K \cap L) \geq \gamma(K)\gamma(L)$.

Prior partial results:

- Khatri, Šidák '67 - $K = [-a, a] \times \mathbb{R}^{n-1}$.
- Pitt '77 - confirmed $n = 2$, inspired conjecture.
- Folklore - convex K, L are unconditional.
- Schechtman–Schlumprecht–Zinn '98 - K, L centered ellipsoids; convex sets of small measure.
- Hargé '99, Cordero-Erausquin '02 - K centered ellipsoid.
- Royen '14 - resolved conjecture. By origin-symmetry $(X_i^2) \sim \chi^2$ and Laplace transform; more general Γ-distributions.
- Latała–Matlak '17 - detailed exposition.
- Eskenazis–Nayar–Tkocz '18 - extended GCI to Gaussian mixtures $\sim S \cdot Z$. E.g. symmetric p-stable laws, $p \in (0, 2)$.

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$\forall$ convex $K = -K$ and $L = -L$ in $\mathbb{R}^n$, $\gamma(K \cap L) \geq \gamma(K)\gamma(L)$ $(*)$.

Recent developments:

- M. '25 - gave new proof using extension of Nakamura-Tsuji's Inverse Brascamp–Lieb inequality (Gaussian saturation).
- NT '25 - $(*)$ holds for Gaussian centered K, L: $\int_{K,L} \vec{x} d\gamma(x) = \vec{0}$.
  Equality in $(*)$ iff $K = K_0 \times E_m, L = E_m^{\perp} \times L_0$ up to null sets.

Tehranchi's Conjecture

Conjecture (Tehranchi '17)

$\forall$ convex $K = -K$ and $L = -L$ in $\mathbb{R}^n$, $\gamma(K)\gamma(L) \leq \gamma(K \cap L)\gamma(K + L)$.

- Would be strengthening of GCI since $\gamma(K + L) \leq 1$.
- Stability: if $\gamma(K \cap L) \leq (1 + \epsilon)\gamma(K)\gamma(L)$ then $\gamma(K + L) \geq \frac{1}{1+\epsilon}$.
- Motivation: γ appears twice on both sides, $\exists$ scaling limit.
  In scaling limit: $|K||L| \leq |K \cap L||K + L|$ is Rogers–Shephard Inq.
- Tehranchi '17 - holds with C^n factor.
- Assoulin–Chor–Sadovsky '24 - True for unconditional K, L,
  False if $K + L \leftrightarrow \mathrm{Conv}(K \cup L)$.

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Rogers–Shephard Inequality

(Henceforth, all bodies assumed convex):

- Rogers–Shephard '58: $|K \cap L||K - L| \leq \binom{2n}{n}|K||L|$.
- $C \subset \mathbb{R}^N$, F is a k-dim subspace: $|C \cap F|_k |P_{F^\perp} C|_{N-k} \leq \binom{N}{k}|C|_N$.
- When $C = -C$, RS method yields $|C|_N \leq |C \cap F|_k |P_{F^\perp} C|_{N-k}$. Spingarn '93: enough to have C centered.
- Applying to $C = K \times L \subset \mathbb{R}^{2n}$, $F = \{(x, x) : x \in \mathbb{R}^n\}$, $F^\perp = \{(x, -x) : x \in \mathbb{R}^n\}$, $\sqrt{2}^n$ Jacobian factor cancels, obtain:

  RSS Inq: K, L convex centered $\rightsquigarrow$ $|K||L| \leq |K \cap L||K - L|$.
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Main Theorem

Thm (Gaussian Conjugate RS Inq = GCRSI, MNT '26)

K, L convex, Gaussian centered $\rightsquigarrow \gamma(K)\gamma(L) \le \gamma(K \cap L)\gamma(K + L)$.
Equality iff $K = K_0 \times E_m, L = E_m^{\perp} \times L_0$ up to null sets.

- When $K = -K$, $L = -L$, this confirms Tehranchi's conjecture.
- Fuses GCI and RSI into single statement, recovering both.
- "Conjugate" bc/ we obtain $K + L$, not $K - L$ (don't know how).

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More General Results - Motivation

Assume K, L convex and Gaussian centered in $\mathbb{R}^n$.

V. Milman–Pajor '00: $a^2 + b^2 = 1 \rightsquigarrow \gamma(K)\gamma(L) \leq \gamma(\frac{1}{b}K \cap \frac{1}{a}L)\gamma(aK - bL)$.

Proof: when $K = -K$, $L = -L$, repeat RS proof for $K \times L$ with $F = \{(bx, ax) : x \in \mathbb{R}^n\}$, use γ rotation invariant & product.

No homogeneity like for $|\cdot|$, hence $a^2 + b^2 = 1$ (e.g. $a = b = \frac{1}{\sqrt{2}}$).

Question: when does $\gamma(K)\gamma(L) \leq \gamma(\alpha K \cap \beta L)\gamma(aK + bL)$?

- $\Leftrightarrow \quad \gamma(K)\gamma(L) \leq \gamma(|\alpha| K \cap |\beta| L)\gamma(|a| K + \sigma |b| L)$, $\sigma \in \{-1, +1\}$. So we'll assume $\alpha, \beta, a, b > 0$, $\sigma \in \{-1, 1\}$.
- Testing $K = \mathbb{R}^n$ or $L = \mathbb{R}^n \rightsquigarrow \beta, \alpha \geq 1$.
- Scaling limit and homogeneity $\rightsquigarrow \alpha b, \beta a \geq 1$.
- So best one could hope for: for which $a, b > 0$, $\sigma \in \{-1, +1\}$,

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Thm (MNT '26)

True for $\sigma = +1$ (conjugate), $3\min(a^2, b^2) + \max(a^2, b^2) \geq 1$.
Characterization of equality (depends on $a^2, b^2 \geq\leq 1$).
False for $a + b < 1$ (already for $n = 1$).

Particular cases:

- Case $a = b = 1$ ($\sigma = +1$) is the GCRSI.
- Conjugate Milman–Pajor Inq (CMPI):
  $a^2 + b^2 = 1 \rightsquigarrow \gamma(K)\gamma(L) \leq \gamma(\frac{1}{b}K \cap \frac{1}{a}L)\gamma(aK + bL)$.
  Equality (when $a, b \neq 0$) iff $K = L = \mathbb{R}^n$.
- When $|a| = |b| = \lambda$, we obtain a characterization:

$$\gamma(K)\gamma(L) \leq \gamma(\frac{1}{\lambda}(K \cap L))\gamma(\lambda(K + L)) \quad \forall K, L \text{ convex } \gamma\text{-centered}$$

iff $\lambda \in [\frac{1}{2}, 1]$: $\lambda = 1$ is GCRSI; $\lambda = \frac{1}{\sqrt{2}}$ is CMPI; $\lambda = \frac{1}{2}$ extremal.

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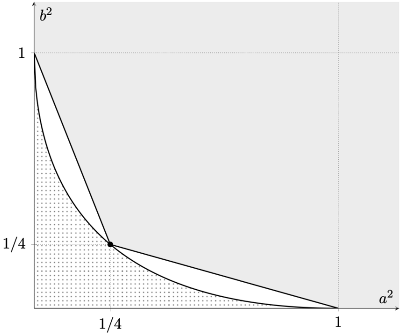

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Equivalent Functional Formulation

Let $f_i : \mathbb{R}^n \to \mathbb{R}_+$ be quasi-concave w/ Gaussian-centered level-sets.

$$\text{GCI} \;\Leftrightarrow\; \forall f_1, f_2 \quad \int f_1 d\gamma \int f_2 d\gamma \leq \int f_1 f_2 d\gamma.$$

What is an equivalent functional formulation of GCRSI? Multiplying RHS by $\int \sup_{z=x+y} f_1(x) f_2(y) d\gamma(z)$ would violate homogeneity...

Using 4-functions Thm, can show:

$$\text{GCRSI} \;\Leftrightarrow\; \forall f_1, f_2 \quad \int f_1 d\gamma \int f_2 d\gamma \leq \int \max{}_0(f_1, f_2) d\gamma \int f_1 \,\square\, f_2 d\gamma,$$

$$\max{}_0(p,q) = \begin{cases} \max(p,q) & p,q > 0 \\ 0 & pq = 0 \end{cases}, \quad f_1 \,\square\, f_2(z) = \sup_{z=x+y} \min(f_1(x), f_2(y)).$$

Note: $\max(\mathbf{1}_K, \mathbf{1}_L) = \mathbf{1}_{K \cup L}$ but $\max{}_0(\mathbf{1}_K, \mathbf{1}_L) = \mathbf{1}_{K \cap L}$.

This gives us a hint regarding how to establish GCRSI:

$$\begin{aligned} f_1(x_1) f_2(x_2) &= \max{}_0(f_1(x_1), f_2(x_2)) \min(f_1(x_1), f_2(x_2)) \\ &\leq \max{}_0(f_1(x_1), f_2(x_2)) f_1 \,\square\, f_2(x_1 + x_2). \end{aligned}$$

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$$\begin{aligned} f_1(x_1) f_2(x_2) &= \max{}_0(f_1(x_1), f_2(x_2)) \min(f_1(x_1), f_2(x_2)) \\ &\leq \max{}_0(f_1(x_1), f_2(x_2)) f_1 \,\Box\, f_2(x_1 + x_2). \end{aligned}$$

GCRSI for log-concave functions

Thm (GCRSI for log-concave functions, MNT '26)

Let $f_1, f_2, h_2 : \mathbb{R}^n \to \mathbb{R}_+$, $h_1 : \mathbb{R}^{2n} \to \mathbb{R}_+$ log-concave, such that $\int \vec{x} f_i(x) d\gamma(x) = \vec{0}$, $i = 1, 2$ (no centering assumption on h_j).

If $$f_1(x_1) f_2(x_2) \leq h_1(x_1, x_2) h_2(x_1 + x_2) \quad \forall x_1, x_2 \in \mathbb{R}^n,$$

Then $$\int_{\mathbb{R}^n} f_1 d\gamma^n \int_{\mathbb{R}^n} f_2 d\gamma^n \leq \int_{\mathbb{R}^n} h_1(y, y) d\gamma^n(y) \int_{\mathbb{R}^n} h_2 d\gamma^n.$$

In addition, characterization of equality case (e.g. implies $h_2 \equiv C$).

Applying to $f_1 = \mathbf{1}_K$, $f_2 = \mathbf{1}_L$, $h_1 = \mathbf{1}_{K \times L}$, $h_2 = \mathbf{1}_{K+L}$ yields GCRSI:

$$\gamma^n(K)\gamma^n(L) \leq \gamma^n(K \cap L)\gamma^n(K + L).$$

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Main Tool

γ_Γ = Gaussian prob. w/ covariance $\Gamma \geq 0$, $g_A(x) = \exp(-\frac{1}{2}\langle Ax, x\rangle)$.

Thm (Gaussian Forward Reverse BL Inq, MNT '26)

$E = \oplus_{i=1}^{I} \mathbb{R}^{n_i}$, $L_j : E \to \mathbb{R}^{m_j}$ surjective, $j = 1, \ldots, J$, $\{c_i\}, \{d_j\} > 0$.
$Q : E \to E$ (any signature), $\Sigma_i > 0$ on $\mathbb{R}^{n_i}$, $\Gamma_j \geq 0$ on $\mathbb{R}^{m_j}$ (degenerate?)

For all log-concave $f_i : \mathbb{R}^{n_i} \to \mathbb{R}_+$ w/ $\int \vec{x} f_i(x) d\gamma_{\Sigma_i}(x) = \vec{0}$,
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If
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Then
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where $\mathrm{FR}_{LC}^{(\mathcal{G})}$ denotes best constant when testing centered Gaussians $f_i = g_{A_i}$ and $h_j = g_{B_j}$ ($A_i, B_j \geq 0$), namely:

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Proof Idea in Simplest Case

Write $f_i \ggcurly g_{Q_i}$ if $f_i = g_{Q_i} \cdot \text{log-concave}$, $f_i \llcurly g_{Q_i}$ if $f_i = g_{Q_i} \cdot \text{log-convex}$.

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By compactness $\exists$ maximizers f_i, h_j, normalize $\int f_i dx_i = \int h_j dy_j = 1$,

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Set

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By CLT, $\lim_{N\to\infty} f_i^{(N)} = \gamma_{\mathrm{Cov}(f_i)}$, $\lim_{N\to\infty} h_j^{(N)} = \gamma_{\mathrm{Cov}(h_j)} \rightsquigarrow$ obtain Gaussian extremizers in $(*)$ (this is the dual formulation of FRBL).

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Proof Idea in Simplest Case

Write $f_i \ggcurly g_{Q_i}$ if $f_i = g_{Q_i} \cdot$ log-concave, $f_i \llcurly g_{Q_i}$ if $f_i = g_{Q_i} \cdot$ log-convex.

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By compactness $\exists$ maximizers f_i, h_j, normalize $\int f_i dx_i = \int h_j dy_j = 1$,

$$\prod_{i=1}^{I} f_i(x_i)^{c_i} \leq \frac{1}{\mathrm{FR}} e^{-\frac{1}{2}\langle \mathbf{Q}x, x\rangle} \prod_{j=1}^{J} h_j(L_j x)^{d_j} \quad \forall x \in E. \quad (*)$$

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Self Convolution Preserves Inequality $(*)$

Write, for each fixed $\overline{x} \in \bigoplus_{i=1}^{I} \mathbb{R}^{n_i}$, $\overline{y} \in \bigoplus_{j=1}^{J} \mathbb{R}^{m_j}$:

$$f_i^{(\overline{x}_i)}(x_i) := f_i\left(\frac{\overline{x}_i + x_i}{\sqrt{2}}\right) f_i\left(\frac{\overline{x}_i - x_i}{\sqrt{2}}\right), \quad h_j^{(\overline{y}_j)}(y_j) := h_j\left(\frac{\overline{y}_j + y_j}{\sqrt{2}}\right) h_j\left(\frac{\overline{y}_j - y_j}{\sqrt{2}}\right)$$

Then $g_{A_i^\flat} \ll f_i^{(\overline{x}_i)} \ll g_{A_i^\sharp}$, $g_{B_j^\flat} \ll h_j^{(\overline{y}_j)} \ll g_{B_j^\sharp}$ even functions, and:

$$\prod_{i=1}^{I} f_i^{(\overline{x}_i)}(x_i)^{c_i} = \prod_{i=1}^{I} f_i\left(\frac{\overline{x}_i + x_i}{\sqrt{2}}\right)^{c_i} f_i\left(\frac{\overline{x}_i - x_i}{\sqrt{2}}\right)^{c_i}$$

$$\leq \frac{1}{\mathrm{FR}^2} \cdot e^{-\frac{1}{2}\langle \mathbf{Q}\frac{\overline{x}+x}{\sqrt{2}}, \frac{\overline{x}+x}{\sqrt{2}}\rangle} e^{-\frac{1}{2}\langle \mathbf{Q}\frac{\overline{x}-x}{\sqrt{2}}, \frac{\overline{x}-x}{\sqrt{2}}\rangle} \prod_{j=1}^{J} h_j\left(L_j\left(\frac{\overline{x}+x}{\sqrt{2}}\right)\right)^{d_j} h_j\left(L_j\left(\frac{\overline{x}-x}{\sqrt{2}}\right)\right)^{d_j}$$

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$$\prod_{i=1}^{I} \left(\int_{\mathbb{R}^{n_i}} f_i^{(\overline{x}_i)} \, d\gamma_{\Sigma_i}(x_i) \right)^{c_i} \leq \frac{\mathrm{FR}}{\mathrm{FR}^2} e^{-\frac{1}{2}\langle \mathbf{Q}\overline{x}, \overline{x}\rangle} \prod_{j=1}^{J} \left(\int_{\mathbb{R}^{m_j}} h_j^{(L_j\overline{x})} \, d\gamma_{\Gamma_j}(y_j) \right)^{d_j} \quad \square$$

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Remaining Steps

Steps:

1. Even f_i, h_j, Lebesgue integration, $0 < A_i^\flat \leq A_i^\sharp < \infty$, $0 < B_j^\flat \leq B_j^\sharp < \infty$.
2. Change to Gaussian integration w.r.t. $\gamma_{\Sigma_i}, \gamma_{\Gamma_j}$, $\Sigma_i, \Gamma_j > 0$.
3. Remove evenness assumption.
   (Idea: $f_i^{(\overline{x}_i)}, h_j^{(\overline{y}_j)}$ are even for each $\overline{x}, \overline{y}$, so $\frac{1}{\mathrm{FR}} \leq \frac{\mathrm{FR}^{(e)}}{\mathrm{FR}^2} < \infty$).
4. Approximate degenerate $\Gamma_j \geq 0$ by non-degenerate ones.
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Reduction to Gaussians

Recall:

Thm (GCRSI for log-concave functions, MNT '26)

Let $f_1, f_2, h_2 : \mathbb{R}^n \to \mathbb{R}_+$, $h_1 : \mathbb{R}^{2n} \to \mathbb{R}_+$ log-concave, such that $\int \vec{x} f_i(x) d\gamma(x) = \vec{0}$, $i = 1, 2$ (no centering assumption on h_j).

If $$f_1(x_1) f_2(x_2) \leq h_1(x_1, x_2) h_2(x_1 + x_2) \quad \forall x_1, x_2 \in \mathbb{R}^n,$$

Then $$\int_{\mathbb{R}^n} f_1 d\gamma^n \int_{\mathbb{R}^n} f_2 d\gamma^n \leq \int_{\mathbb{R}^n} h_1(y, y) d\gamma^n(y) \int_{\mathbb{R}^n} h_2 d\gamma^n.$$

Proof: GFRBL Thm reduces the proof to showing that for all $A_1, A_2, C \in \mathrm{Sym}_{\geq 0}(n)$, $B \in \mathrm{Sym}_{\geq 0}(2n)$, denoting $\Delta = \begin{pmatrix} \mathrm{Id}_n & \mathrm{Id}_n \\ \mathrm{Id}_n & \mathrm{Id}_n \end{pmatrix}$,

$$g_{A_1}(x_1) g_{A_2}(x_2) \leq g_B(x_1, x_2) g_C(x_1 + x_2)$$
$$\Rightarrow \quad \int_{\mathbb{R}^n} g_{A_1} d\gamma \int_{\mathbb{R}^n} g_{A_2} d\gamma \leq \int_{\mathbb{R}^{2n}} g_B d\gamma_\Delta \int_{\mathbb{R}^n} g_C d\gamma.$$

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$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \geq \begin{pmatrix} B_1 & B_2 \\ B_2^* & B_4 \end{pmatrix} + \begin{pmatrix} C & C \\ C & C \end{pmatrix} \quad (*)$$
$$\Rightarrow \det(\mathrm{Id}_n + A_1)\det(\mathrm{Id}_n + A_2) \geq \det(\mathrm{Id}_n + \bar{B})\det(\mathrm{Id}_n + C).$$

Not so trivial, e.g. adding Id_{2n} to $(*)$, it is FALSE ($n = 1$) that:

$$\det\left(\begin{pmatrix} \mathrm{Id}_n & 0 \\ 0 & \mathrm{Id}_n \end{pmatrix} + \begin{pmatrix} B_1 & B_2 \\ B_2^* & B_4 \end{pmatrix} + \begin{pmatrix} C & C \\ C & C \end{pmatrix} \right) \geq \det(\mathrm{Id}_n + \bar{B})\det(\mathrm{Id}_n + C).$$

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Gaussian Computation

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \geq \begin{pmatrix} B_1 & B_2 \\ B_2^* & B_4 \end{pmatrix} + \begin{pmatrix} C & C \\ C & C \end{pmatrix} \quad (*)$$

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Lemma

$A, B \in \mathrm{Sym}_{\geq 0}(n) \quad \rightsquigarrow \quad \det(\mathrm{Id}_n + A)\det(\mathrm{Id}_n + B) \geq \det(\mathrm{Id}_n + A + B)$.

Proof: trivial for commuting A, B, in general nice exercise.
Note this is GCI for Gaussians: $\int g_A d\gamma \int g_B d\gamma \leq \int g_A g_B d\gamma$.

Corollary

$\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathrm{Sym}_{\geq 0}(n) \quad \rightsquigarrow \quad \det(\mathcal{C} + \mathcal{A})\det(\mathcal{C} + \mathcal{B}) \geq \det(\mathcal{C})\det(\mathcal{C} + \mathcal{A} + \mathcal{B})$.

Proof of $(*)$: Since $A_1, A_2 \geq C$, apply Corollary to $\mathcal{C} = \mathrm{Id}_n + C$, $\mathcal{A} = A_1 - C \geq 0$, $\mathcal{B} = A_2 - C \geq 0$:

$$\det(\mathrm{Id}_n + A_1)\det(\mathrm{Id}_n + A_2) \geq \det(\mathrm{Id}_n + C)\det(\mathrm{Id}_n + A_1 + A_2 - C)$$

$$(\text{use } A_1 + A_2 \geq \bar{B} + 4C) \geq \det(\mathrm{Id}_n + C)\det(\mathrm{Id}_n + \bar{B} + 3C) \qquad \square$$

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$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \geq \begin{pmatrix} B_1 & B_2 \\ B_2^* & B_4 \end{pmatrix} + \begin{pmatrix} C & C \\ C & C \end{pmatrix} \quad (*)$$

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Lemma

$A, B \in \mathrm{Sym}_{\geq 0}(n) \quad \rightsquigarrow \quad \det(\mathrm{Id}_n + A)\det(\mathrm{Id}_n + B) \geq \det(\mathrm{Id}_n + A + B)$.

Proof: trivial for commuting A, B, in general nice exercise.

Note this is GCI for Gaussians: $\int g_A d\gamma \int g_B d\gamma \leq \int g_A g_B d\gamma$.

Corollary

$\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathrm{Sym}_{\geq 0}(n) \quad \rightsquigarrow \quad \det(\mathcal{C} + \mathcal{A})\det(\mathcal{C} + \mathcal{B}) \geq \det(\mathcal{C})\det(\mathcal{C} + \mathcal{A} + \mathcal{B})$.

Proof of $(*)$: Since $A_1, A_2 \geq C$, apply Corollary to $\mathcal{C} = \mathrm{Id}_n + C$, $\mathcal{A} = A_1 - C \geq 0$, $\mathcal{B} = A_2 - C \geq 0$:

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Thanks for you attention!