Illumination Bodies and Affine Surface Areas

supported by a grant from the National Science Foundation
and by a Simons Fellowship

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The (convex) floating body (Barany&Larman; Schütt&W).

Let $\delta \in \mathbb{R}$, $\delta \geq 0$

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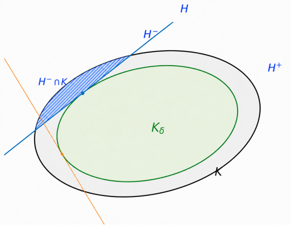

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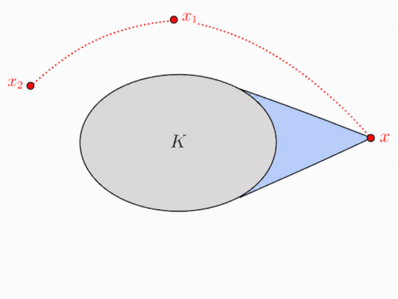
Definition [W]

Let K be a convex body in $\mathbb{R}^n$ and $\delta \geq 0$. The illumination body K^δ is defined by

$$K^\delta = \{x \in \mathbb{R}^n : \mathrm{vol}_n([x, K] \setminus K) \leq \delta\}$$



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Proof

Follows from [e.g. Mordhorst&W]

$$\mathrm{vol}_n([z,K]\setminus K) = -\frac{1}{2}\mathrm{vol}_n(K) + \frac{1}{2n}\int_{\partial K} |(z-x)\cdot N_K(x)|\; d\mu_K(x)$$

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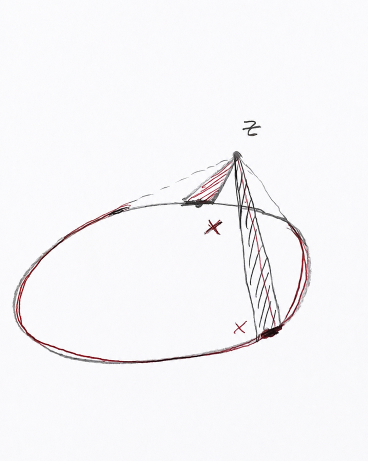

Affine Surface Area [Blaschke]

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- useful and important quantity $\implies$ we want it for **all** convex bodies.
- why the name

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- Analog of affine surface area in $Sp^n(\lambda)$

Let $K \subset Sp^n(\lambda)$ be a convex body (a compact, geodesically convex set with nonempty interior) and $\kappa^\lambda(K, \cdot)$ the Gauss curvature. Define

$$\Omega^\lambda(K) = \int_{\partial K} \kappa^\lambda(K, x)^{\frac{1}{n+1}} \, \mathrm{vol}_{\partial K}^\lambda(\mathrm{d}x)$$

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We call $\Omega^\lambda(K)$ the Floating Area of K

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Definition [Besau&W]

Let $K \subset \mathbb{S}^n$ be a proper convex body and $\delta > 0$. The (spherical) convex floating body $K_{[\delta]}$ is the intersection of all closed hemispheres $\mathbb{S}^-$ such that the hyperspheres $\mathbb{S}^+$ cut off a set of volume less or equal δ from K

$$K_{[\delta]} = \bigcap \left\{ \mathbb{S}^- : \mathrm{vol}_n \left(K \cap \mathbb{S}^+ \right) \leq \delta \right\}$$

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- $Sp^n(\lambda)$: Assouline & Besau & W
- Riemannian surfaces: Assouline & Schütt & W
- R-ball convex bodies: Oliva & W & Yalikun

The case $Sp^n(\lambda)$

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Definition [Assouline&Besau&W]

Let $\lambda \in \mathbb{R}$ and $n \geq 2$. For $K \subset Sp^n(\lambda)$ and $\delta > 0$, the illumination body $\mathcal{I}_\delta^\lambda(K)$ is defined by

$$\mathcal{I}_\delta^\lambda(K) = \left\{ \mathbf{x} \in Sp^n(\lambda) : \mathrm{vol}_n^\lambda\left([\mathbf{x}, K] \setminus K\right) \leq \delta \right\},$$

where

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- Illumination bodies in $Sp^n(\lambda)$ are star bodies. They need not be convex.

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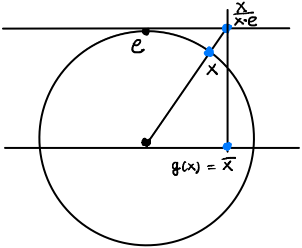
gnomonic projection $g(x) = \bar{x} = \frac{x}{x \cdot \mathbf{e}} - \mathbf{e}$



• any convex body $\overline{K}$ in $\mathbb{R}^2$ determines a unique spherical convex body K in the open half-space of $Sp^2(1) \cong \mathbb{S}^2$ and vice versa

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Let S be a (spherical) square determined by $\overline{S} = [-1, 1]^2, \quad S = g^{-1}(\overline{S})$

Then $\mathcal{I}_\delta^\lambda(S)$is not convex for all $\delta > 0$.

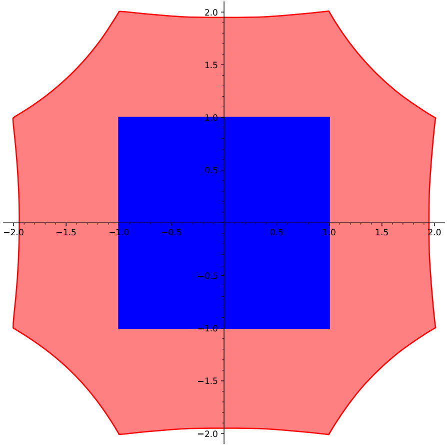

Figure: The spherical illumination body (red part) of a spherical square (blue part) in the gnomonic projection. The illumination body of the square is not geodesically convex

Theorem 1 [Assouline&Besau&W]

Let $n \geq 2$ and $\lambda \in \mathbb{R}$. If $K \subset Sp^n(\lambda)$, then the right derivative of $\mathrm{vol}_n^\lambda(\mathcal{I}_\delta^\lambda(K))$ at $\delta = 0$ exists.

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$$\begin{aligned} \lim_{\delta \to 0^+} \frac{\mathrm{vol}_n^\lambda(\mathcal{I}_\delta^\lambda(K)) - \mathrm{vol}_n^\lambda(K)}{\delta^{\frac{2}{n+1}}} &= d_n\, \Omega^\lambda(K) \\ &= d_n \int_{\partial K} \kappa^\lambda(K,x)^{\frac{1}{n+1}}\, \mathrm{vol}_{\partial K}^\lambda(\mathrm{d}x) \end{aligned}$$

Let K be a convex body in $\mathbb{R}^n$ and let $U \supset K$ be an open set. Let $\varphi : U \to (0, \infty)$ be a continuous, integrable function. Let $A \subset \mathbb{R}^n$ be a measurable set. We define a measure vol_n^φ on $\mathbb{R}^n$ by

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For $\delta > 0$, the φ-weighted illumination body $\mathcal{I}_\delta^{\varphi}(K)$ is

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- The φ-illumination body need not be convex or bounded
- If $0 \in \mathrm{int}(K)$, $\mathcal{I}_\delta^\varphi(K)$ is star-shaped
- For sufficiently small δ, $K \subset \mathcal{I}_\delta^\varphi(K) \subset U$, so $\mathcal{I}_\delta^\varphi(K)$ is a *star body*

Theorem 3 [Assouline&Besau&W]

Let K be a convex body in $\mathbb{R}^n$, $U \supset K$ open and let $\varphi : U \to (0, +\infty)$ be continuous.

$$\lim_{\delta \to 0^+} \frac{\mathrm{vol}_n(\mathcal{I}_\delta^\varphi(K)) - \mathrm{vol}_n(K)}{\delta^{\frac{2}{n+1}}} = d_n \int_{\partial K} \kappa(K, \mathbf{x})^{\frac{1}{n+1}} \varphi(\mathbf{x})^{-\frac{2}{n+1}} \, d\mu_K(x)$$

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To do so, we use

- a projective model for the space forms $Sp^n(\lambda)$

$$\mathbb{B}^n_\lambda := \begin{cases} \operatorname{int} B_0 \ \left(1/\sqrt{-\lambda}\right) & \text{if } \lambda < 0 \ \text{(Beltrami-Klein model)} \\ \mathbb{R}^n & \text{if } \lambda \geq 0 \end{cases}$$

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Thus

$$\lim_{\delta \to 0^+} \frac{\mathrm{vol}_n^\lambda(\mathcal{I}_\delta^\lambda(K)) - \mathrm{vol}_n^\lambda(K)}{\delta^{\frac{2}{n+1}}} = \lim_{\delta \to 0^+} \frac{\mathrm{vol}_n(\mathcal{I}_\delta^{\varphi_\lambda}(\overline{K})) - \mathrm{vol}_n(\overline{K})}{\delta^{\frac{2}{n+1}}}$$

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Now we use

- $\mathrm{vol}_{\partial K}^\lambda(\mathrm{d}\mathbf{x}) = \sqrt{\frac{1+\lambda\,(N_{\overline{K}}(\mathbf{x})\cdot\mathbf{x})^2}{(1+\lambda\,\|\mathbf{x}\|^2)^n}}\,\mu_{\overline{K}}(\mathbf{x})$
- $\kappa^\lambda(K,\mathbf{x}) = \kappa(\overline{K},\mathbf{x})\left(\frac{1+\lambda\|\mathbf{x}\|^2}{1+\lambda\,(N_{\overline{K}}(\mathbf{x})\cdot\mathbf{x})^2}\right)^{\frac{n+1}{2}}$

R-Ball convex bodies

Let $C = R\, B_2^n$ be fixed.

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(Artstein-Avidan, Bezdek, Chor, Connelly, Csikos, Drach, Florentin, Fodor, Kabluchko, Langi, Marynych, Molchanov, Nazodi, Pach, Tatarko, ...)

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Definition [Schütt&W&Yalikun]

Let $\delta \geq 0$. Let K be an R-ball convex body. The R-ball floating body K_δ^R is

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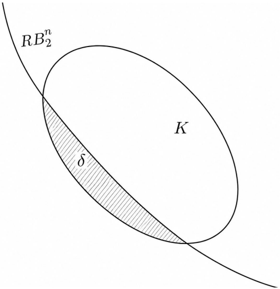

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- The convex body $R\,B_2^n$ replaces the hyperplanes of the definition of the classical convex floating body:
  When $R \to \infty$, then $K_\delta^R \to K_\delta$

Let $\kappa_i(K,x)$, $1 \leq i \leq n-1$, be the (generalized) principal curvatures, i.e. $\kappa(K,x) = \prod_{i=1}^{n-1} \kappa_i(K,x)$

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Theorem [Schütt&W&Yalikun, 2025]

Let K be an R-ball convex body such that for all $x \in \partial K$, for all i, $\kappa_i(K,x) > \frac{1}{R}$, and let K_δ^R be its R-ball floating body. Then

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Definition [Schütt&W&Yalikun, 2025]

Let K be an R-ball convex body. Then the relative surface area of K is

$$as^R(K) = \int_{\partial K} \prod_{i=1}^{n-1} \left(\kappa_i(K,x) - \frac{1}{R} \right)^{\frac{1}{n+1}} d\mu_K(x).$$

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We need the R-convex hull of a R-ball convex body K and a point x:

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Let $\delta \geq 0$. Let K be a R-ball convex body. The R-ball illumination body $I_R^\delta(K)$ of K is

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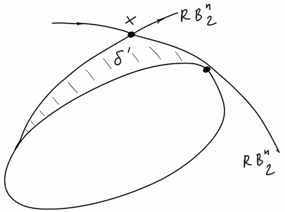

- The R-ball illumination body of an R-ball convex body is not R-ball convex in general:

  $R\, B_2^n$ is R-ball convex. Its R-ball illumination body is

  $$(R + \alpha_n\, \delta^{\frac{2}{n+1}})B_2^n$$

  which is not R-ball convex but $\tilde{R}$-ball convex for all $\tilde{R} \geq (R + \alpha_n\, \delta^{\frac{2}{n+1}})$

Proposition [Oliva&W&Yalikun, 2026+]

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- Let K be an R-ball convex body in $\mathbb{R}^2$ that is C_+^2. Then for δ small $I_R^\delta(K)$ is $\tilde{R}$-ball convex with

$$\tilde{R} = R + d_2\delta^{\frac{2}{3}} \max_{\theta\in[0,2\pi]} \left[\left(\kappa(K,\theta) - \frac{1}{R}\right)^{\frac{1}{3}} + \left(\left(\kappa(K,\theta) - \frac{1}{R}\right)^{\frac{1}{3}}\right)'' \right]$$

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