Geometric/Combinatorial Viewpoint Helps Designing Algorithms in TDA

Tamal K. Dey
Department of Computer Science
Purdue University

Algorithms for Homological Persistence

Algorithm for Generalized Rank

Algorithm for vector-field barcode

Persistent homology

- **Topological signature for data**
- **Cornerstone for TDA**

Herbert Edelsbrunner, David Letscher, and Afra Zomorodian. **Topological persistence and simplification**. In Proceedings 41st Annual Symposium on Foundations of Computer Science, pages 454–463. IEEE, 2000.

Afra Zomorodian and Gunnar Carlsson. **Computing persistent homology**. Discrete & Computational Geometry, 33(2):249–274, 2005.

Algebraic viewpoint

1. **persistence module (indexed by $\mathbb{Z}$ *or* $\mathbb{N}$)**:

$$\mathbb{V} : V_0 \to V_1 \to \cdots \to V_{m-1} \to V_m = V_{m+1} = \cdots$$

2. **Interval module** $\mathbb{I}^{[b,d)}$:

$$\boxed{\begin{array}{c} [b, \quad b+1, \quad \cdots, \quad d-1] \\ 0 \to \cdots \to 0 \to \mathbf{k} \xrightarrow{=} \mathbf{k} \xrightarrow{=} \cdots \xrightarrow{=} \mathbf{k} \longrightarrow 0 \to \cdots \to 0 \end{array}}$$

3. **Interval decomposition**: $\Longleftrightarrow$ **Persistence barcode**: **[Gabriel 72]**

$$\mathbb{V} \cong \bigoplus_{\alpha \in \mathsf{N}} \mathbb{I}^{[b_\alpha, d_\alpha)} \qquad \mathsf{Pers}_p(\mathbb{V}) = \{[b_\alpha, d_\alpha) \mid \alpha \in \mathsf{N}\}$$

Peter Gabriel. Unzerlegbare Darstellungen I. Manuscripta Mathematica, 6(1):71–103, 1972.

Algebraic viewpoint

1. **Input modules by structural maps, matrices $[f_i]$:**

$$\mathbb{V} : V_0 \xrightarrow{[f_0]} V_1 \xrightarrow{[f_1]} \cdots \to V_{m-1} \xrightarrow{[f_{\{m-1\}}]} V_m = V_{m+1} = \cdots$$

2. **Compute intervals $[b, d)$**

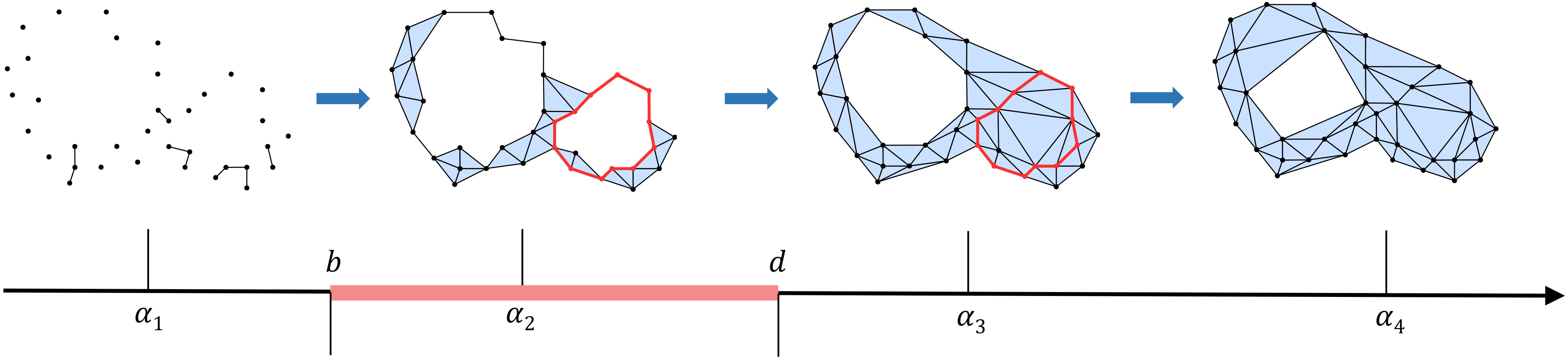
$$\boxed{\begin{array}{c} [b, \qquad b+1, \qquad \cdots, \qquad d-1] \\ 0 \to \cdots \to 0 \to \mathbf{k} \xrightarrow{=} \mathbf{k} \xrightarrow{=} \cdots \xrightarrow{=} \mathbf{k} \longrightarrow 0 \to \cdots \to 0 \end{array}}$$

3. **Can be done in $O(n^3)$ time where $n = \sum_i size\,[f_i]$**

Persistent homology

Input: $\mathcal{F} : \varnothing = K_0 \xhookrightarrow{\sigma_1} K_1 \xhookrightarrow{\sigma_2} \cdots \xhookrightarrow{\sigma_{m-1}} K_{m-1} \xhookrightarrow{\sigma_m} K_m$

An interval $[b, d]$: starts and ends with **indices indexing the filtration**



A cycle gets born

A cycle dies

Barcode from filtration

Simplex-wise filtration:

$$\mathcal{F} : K_0 \xhookrightarrow{\sigma_1} K_1 \xhookrightarrow{\sigma_2} \cdots \xhookrightarrow{\sigma_{m-1}} K_{m-1} \xhookrightarrow{\sigma_m} K_m$$

$$\Downarrow$$

Induced module:

$$\mathsf{H}_p(\mathcal{F}) : \mathsf{H}_p(K_0) \to \mathsf{H}_p(K_1) \to \cdots \to \mathsf{H}_p(K_{m-1}) \to \mathsf{H}_p(K_m)$$

$$\Downarrow$$

Interval decomposition: **[Gabriel 72]**

$$\mathsf{H}_p(\mathcal{F}) = \bigoplus_{\alpha \in \mathcal{A}} \mathcal{I}^{[b_\alpha, d_\alpha)}$$

$$\Downarrow$$

p-th persistence barcode:

$$\mathsf{Pers}_p(\mathcal{F}) = \{[b_\alpha, d_\alpha] \mid \alpha \in \mathcal{A}\}$$

Peter Gabriel. Unzerlegbare Darstellungen I. Manuscripta Mathematica, 6(1):71–103, 1972.

Barcode from tower

Tower with simplicial maps:

$$\mathcal{F} : K_0 \xrightarrow{s_1} K_1 \xrightarrow{s_2} \cdots \xrightarrow{s_{m-1}} K_{m-1} \xrightarrow{s_m} K_m$$

$$\Downarrow$$

Induced module:

$$\mathsf{H}_p(\mathcal{F}) : \mathsf{H}_p(K_0) \xrightarrow{s_1^*} \mathsf{H}_p(K_1) \xrightarrow{s_2^*} \cdots \xrightarrow{s_{m-1}^*} \mathsf{H}_p(K_{m-1}) \xrightarrow{s_m^*} \mathsf{H}_p(K_m$$

$$\Downarrow$$

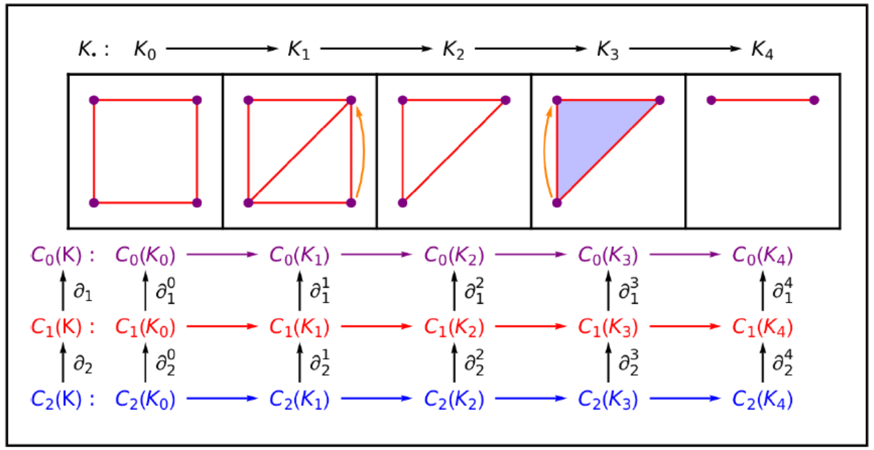
Interval decomposition:

$$\mathsf{H}_p(\mathcal{F}) = \bigoplus_{\alpha \in \mathcal{A}} \mathcal{I}^{[b_\alpha, d_\alpha)}$$

$$\Downarrow$$

***p*-th persistence barcode**:

$$\mathsf{Pers}_p(\mathcal{F}) = \{[b_\alpha, d_\alpha] \,|\, \alpha \in \mathcal{A}\}$$



Algorithms
[DFW14,KS19]

T. K. Dey, F. Fan, Y. Wang. Computing topological persistence for simplicial maps. In Proceedings 30th Annual Symposium on Computational Geometry, pages 345–354, 2014.

M. Kerber, H. Schreiber. Barcodes of towers and a streaming algorithm for1440persistent homology. Discrete & Computational Geometry, 61:852–879, 2019.

Zigzag persistence

Zigzag persistence

Zigzag modules:

$$\mathcal{V} : V_0 \xleftrightarrow{[f_0]} V_1 \xleftrightarrow{[f_1]} \cdots \xleftrightarrow{[f_{m-2}]} V_{m-1} \xleftrightarrow{[f_{m-1}]} V_m$$

$$\Downarrow$$

Interval decomposition:

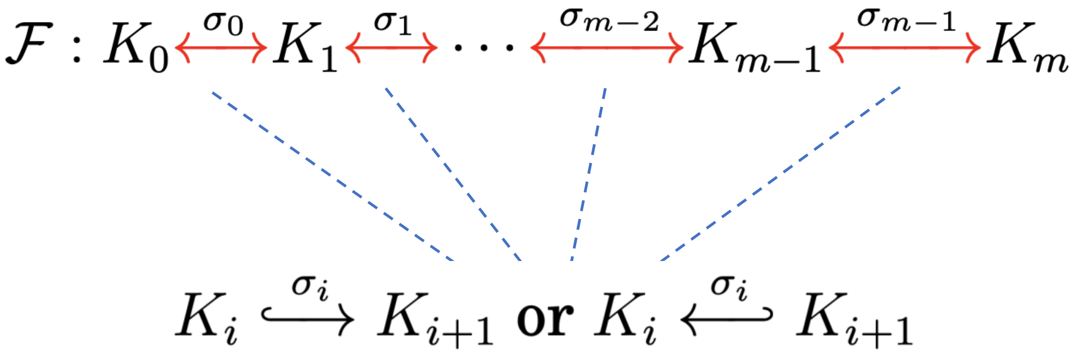
$$\mathcal{V} = \bigoplus_{\alpha \in \mathcal{A}} \mathcal{I}^{[b_\alpha, d_\alpha]}$$

$$\Downarrow$$

p-th persistence barcode:

$$\mathsf{Pers}_p(\mathcal{V}) = \{[b_\alpha, d_\alpha] \mid \alpha \in \mathcal{A}\}$$

Zigzag filtration:



$$\mathcal{F} : K_0 \xleftrightarrow{\sigma_0} K_1 \xleftrightarrow{\sigma_1} \cdots \xleftrightarrow{\sigma_{m-2}} K_{m-1} \xleftrightarrow{\sigma_{m-1}} K_m$$

$$K_i \xhookrightarrow{\sigma_i} K_{i+1} \textbf{ or } K_i \xhookleftarrow{\sigma_i} K_{i+1}$$

Gunnar Carlsson and Vin de Silva. **Zigzag persistence**.
Foundations of Computational Mathematics, 10(4):367–405, 2010.

Zigzag persistence

Zigzag filtration:

$$\mathcal{F} : K_0 \xleftrightarrow{\sigma_0} K_1 \xleftrightarrow{\sigma_1} \cdots \xleftrightarrow{\sigma_{m-2}} K_{m-1} \xleftrightarrow{\sigma_{m-1}} K_m$$

$$\Downarrow$$

Induced module:

$$\mathsf{H}_p(\mathcal{F}) : \mathsf{H}_p(K_0) \leftrightarrow \mathsf{H}_p(K_1) \leftrightarrow \cdots \leftrightarrow \mathsf{H}_p(K_{m-1}) \leftrightarrow \mathsf{H}_p(K_m)$$

$$\Downarrow$$

Interval decomposition: **[Gabriel 72]**

$$\mathsf{H}_p(\mathcal{F}) = \bigoplus_{\alpha \in \mathcal{A}} \mathcal{I}^{[b_\alpha, d_\alpha]}$$

$$\Downarrow$$

p-th persistence barcode:

$$\mathsf{Pers}_p(\mathcal{F}) = \{[b_\alpha, d_\alpha] \mid \alpha \in \mathcal{A}\}$$

Peter Gabriel. Unzerlegbare Darstellungen I. Manuscripta Mathematica, 6(1):71–103, 1972.

New algorithm FastZigzag

- **Input Zigzag Filtration**: ($\mathcal{F}$)

$$\mathcal{F} : \varnothing = K_0 \overset{\sigma_0}{\longleftrightarrow} K_1 \overset{\sigma_1}{\longleftrightarrow} \cdots \overset{\sigma_{m-1}}{\longleftrightarrow} K_m = \varnothing$$

- Convert to a **non-zigzag filtration** of **same length**
  - Linear time • Very Fast

$$\mathcal{F}' : \varnothing = K'_0 \overset{\sigma'_0}{\hookrightarrow} K'_1 \overset{\sigma'_1}{\hookrightarrow} \cdots \overset{\sigma'_{m-1}}{\hookrightarrow} K'_m$$

- Compute barcode for **non-zigzag filtration** $\mathcal{F}$
  - Fast software [Gudhi, Phat etc.]

- Convert barcode of $\mathcal{F}'$ to that of $\mathcal{F}$
  - $O(1)$ conversion per bar

T. K. Dey and T. Hou. **Fast Computation of Zigzag Persistence**.
European Symposium on Algorithms, ESA, 2022.

Non-repetitive Zigzag

- **A simplex is added at most one time**

$$\mathcal{F} : \varnothing = K_0 \cdots \overset{\sigma}{\hookrightarrow} \cdots\cdots \overset{\sigma}{\hookleftarrow} \cdots\cdots \overset{\sigma}{\hookrightarrow} \cdots \xleftrightarrow{\sigma_{m-1}} K_m = \varnothing$$ **Repetitive**

- 1. Convert input **zigzag** to a **non-repetitive zigzag filtration** of **same length**
- 2. Convert a **non-repetitive zigzag** to an **up-down filtration** of **same length**
- 3. Convert **up-down** filtration to an **extended filtration** of **same length**

Conversions 1,2,3 are all done by a single simple linear scan of the input filtration

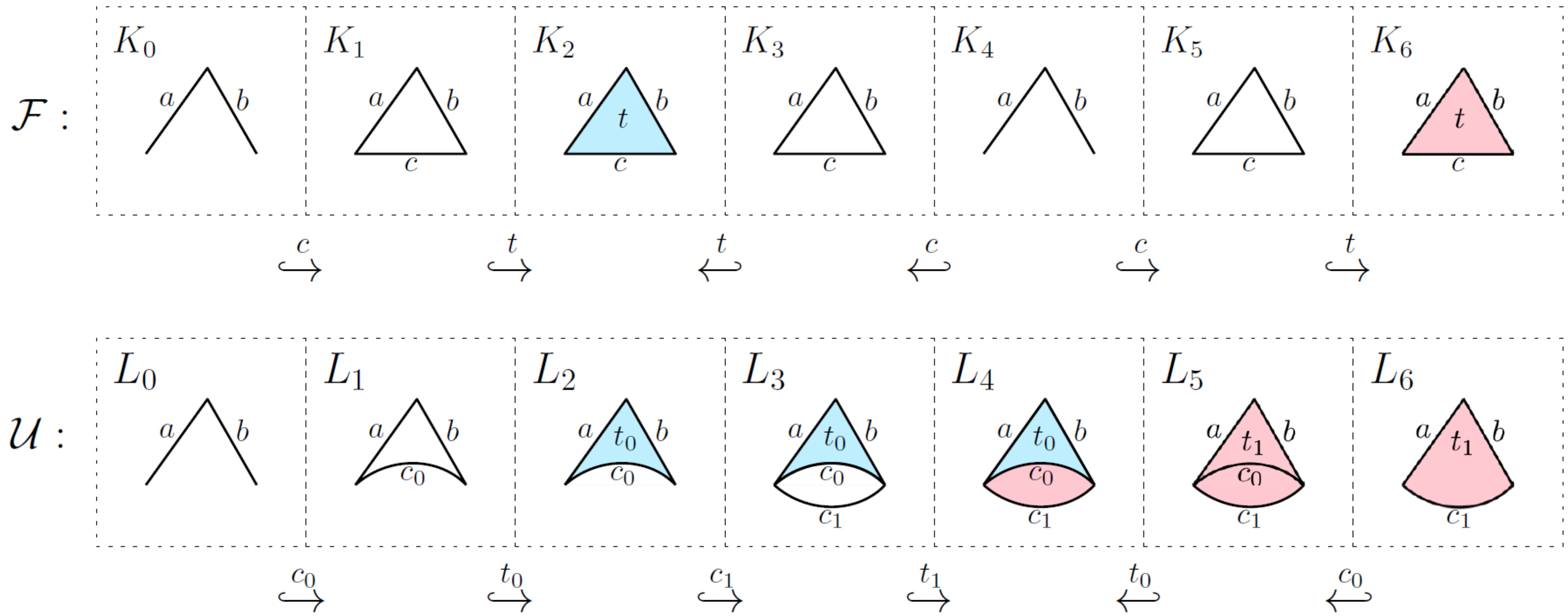
- Linear time • Very Fast

Repetitive to Non-repetitive Zigzag

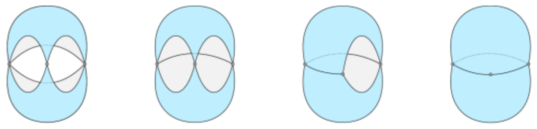
- **Treat each new occurrence of simplex σ as a new "cell"**

$$\mathcal{F} : \varnothing = K_0 \cdots \xhookrightarrow{\sigma} \cdots\cdots \xhookleftarrow{\sigma} \cdots\cdots \xhookrightarrow{\sigma} \cdots \xleftrightarrow{\sigma_{m-1}} K_m = \varnothing$$

$$\hat{\mathcal{F}} : \varnothing = K_0 \cdots \xhookrightarrow{\downarrow\sigma} \cdots\cdots \xhookleftarrow{\uparrow\sigma} \cdots\cdots \xhookrightarrow{\downarrow\hat{\sigma}} \cdots \xleftrightarrow{\sigma_{m-1}} K_m = \varnothing$$



Treat $\hat{\mathcal{F}}$ as cell-wise $\triangle$-complex Filtration



Up-down to Extended to Non-zigzag

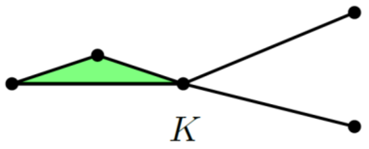
- **up-down** to **extended filtration: Use Mayer-Vietoris Pyramid [CdSM09]**

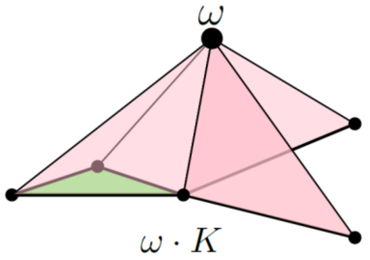
$$\mathcal{U} : \varnothing = L_0 \xhookrightarrow{\tau_0} \cdots \xhookrightarrow{\tau_{n-1}} L_n \xhookleftarrow{\tau_n} \cdots \xhookleftarrow{\tau_{2n-1}} L_{2n} = \varnothing$$

$$\mathcal{E} : \varnothing = L_0 \hookrightarrow \cdots \hookrightarrow L_n = (\hat{K}, L_{2n} = \varnothing) \hookrightarrow (\hat{K}, L_{2n-1}) \hookrightarrow \cdots \hookrightarrow (\hat{K}, L_n) = (\hat{K}, \hat{K})$$

- **Use Coning [CEH06]**

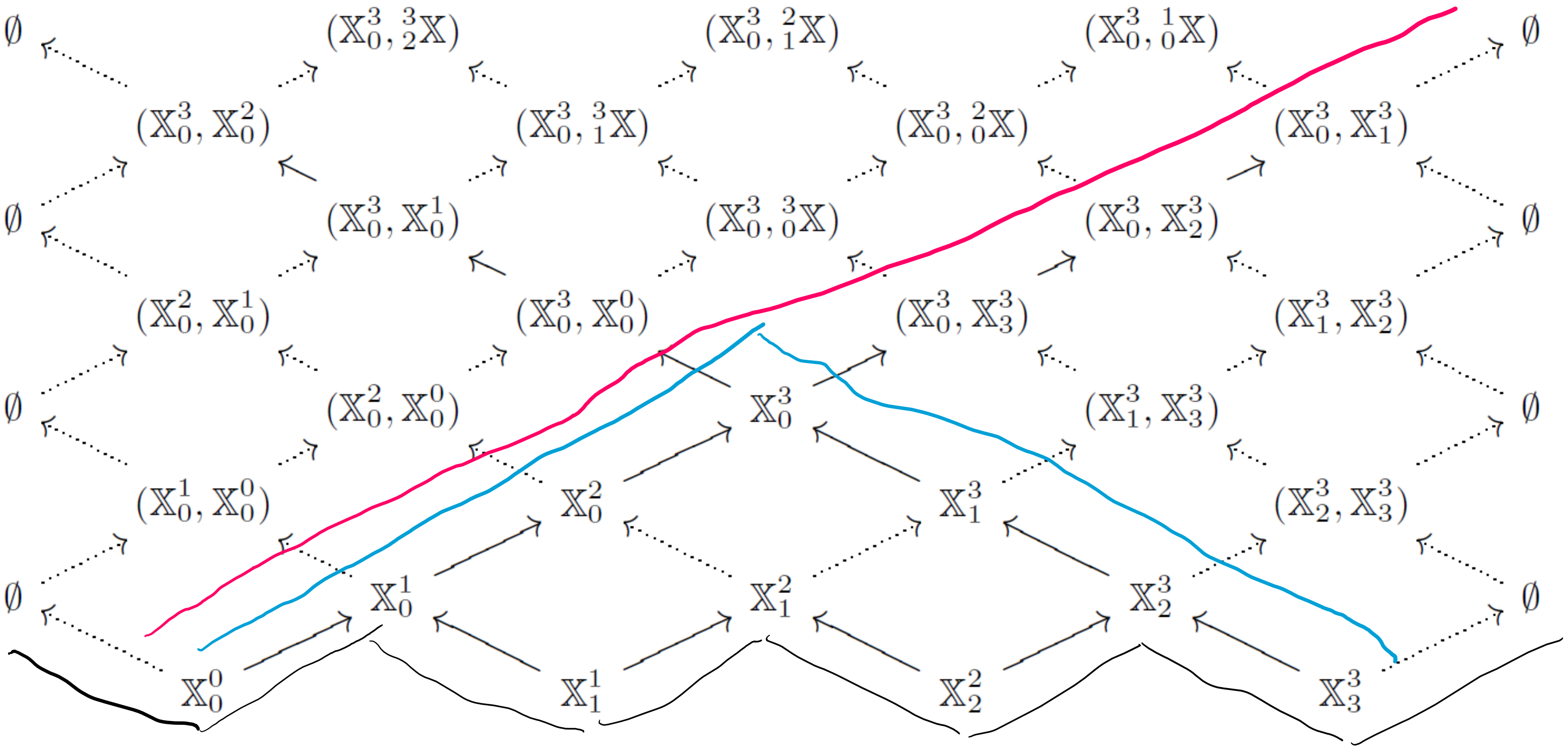
$$\hat{\mathcal{E}} : L_0 \cup \{\omega\} \hookrightarrow \cdots \hookrightarrow L_n \cup \{\omega\} = \hat{K} \cup \omega \cdot L_{2n} \hookrightarrow \hat{K} \cup \omega \cdot L_{2n-1} \hookrightarrow \cdots \hookrightarrow \hat{K} \cup \omega \cdot L_n$$





Gunnar Carlsson, Vin de Silva, and Dmitriy Morozov. **Zigzag persistent homology and real-valued functions**. 2009.

Mayer-Vietoris pyramid



$$\begin{tikzcd}
\emptyset & & (\mathbb{X}_0^3, {}_2^3\mathbb{X}) & & (\mathbb{X}_0^3, {}_1^2\mathbb{X}) & & (\mathbb{X}_0^3, {}_0^1\mathbb{X}) & & \emptyset \\
& (\mathbb{X}_0^3, \mathbb{X}_0^2) \arrow[ul, dotted] \arrow[ur, dotted] & & (\mathbb{X}_0^3, {}_1^3\mathbb{X}) \arrow[ul, dotted] \arrow[ur, dotted] & & (\mathbb{X}_0^3, {}_0^2\mathbb{X}) \arrow[ul, dotted] \arrow[ur, dotted] & & (\mathbb{X}_0^3, \mathbb{X}_1^3) \arrow[ul, dotted] \arrow[ur, dotted] & \\
\emptyset \arrow[ur, dotted] & & (\mathbb{X}_0^3, \mathbb{X}_0^1) \arrow[ul] \arrow[ur, dotted] & & (\mathbb{X}_0^3, {}_0^3\mathbb{X}) \arrow[ul, dotted] \arrow[ur, dotted] & & (\mathbb{X}_0^3, \mathbb{X}_2^3) \arrow[ul, dotted] \arrow[ur] & & \emptyset \arrow[ul, dotted] \\
& (\mathbb{X}_0^2, \mathbb{X}_0^1) \arrow[ul, dotted] \arrow[ur, dotted] & & (\mathbb{X}_0^3, \mathbb{X}_0^0) \arrow[ul] \arrow[ur, dotted] & & (\mathbb{X}_0^3, \mathbb{X}_3^3) \arrow[ul, dotted] \arrow[ur] & & (\mathbb{X}_1^3, \mathbb{X}_2^3) \arrow[ul, dotted] \arrow[ur, dotted] & \\
\emptyset \arrow[ur, dotted] & & (\mathbb{X}_0^2, \mathbb{X}_0^0) \arrow[ul, dotted] \arrow[ur, dotted] & & \mathbb{X}_0^3 \arrow[ul] \arrow[ur] & & (\mathbb{X}_1^3, \mathbb{X}_3^3) \arrow[ul, dotted] \arrow[ur, dotted] & & \emptyset \arrow[ul, dotted] \\
& (\mathbb{X}_0^1, \mathbb{X}_0^0) \arrow[ul, dotted] \arrow[ur, dotted] & & \mathbb{X}_0^2 \arrow[ul, dotted] \arrow[ur] & & \mathbb{X}_1^3 \arrow[ul] \arrow[ur, dotted] & & (\mathbb{X}_2^3, \mathbb{X}_3^3) \arrow[ul, dotted] \arrow[ur, dotted] & \\
\emptyset \arrow[ur, dotted] & & \mathbb{X}_0^1 \arrow[ul, dotted] \arrow[ur] & & \mathbb{X}_1^2 \arrow[ul, dotted] \arrow[ur, dotted] & & \mathbb{X}_2^3 \arrow[ul] \arrow[ur, dotted] & & \emptyset \arrow[ul, dotted] \\
& \mathbb{X}_0^0 \arrow[ul, dotted] \arrow[ur] & & \mathbb{X}_1^1 \arrow[ul] \arrow[ur] & & \mathbb{X}_2^2 \arrow[ul] \arrow[ur] & & \mathbb{X}_3^3 \arrow[ul] \arrow[ur, dotted] &
\end{tikzcd}$$

Gunnar Carlsson, Vin de Silva, and Dmitriy Morozov. Zigzag persistent homology and real-valued functions. 2009.

Multiparameter persistence

Multiparameter persistence

- **Algebraic: A functor $M\colon P \to Vec_F, M(p) = V_p, M(p \le q) = v_{pq}$; $P = \mathbb{Z}^d$ or $\mathbb{N}^d$**
- **Combinatorial: Multi-filtrations (inducing persistence module)**

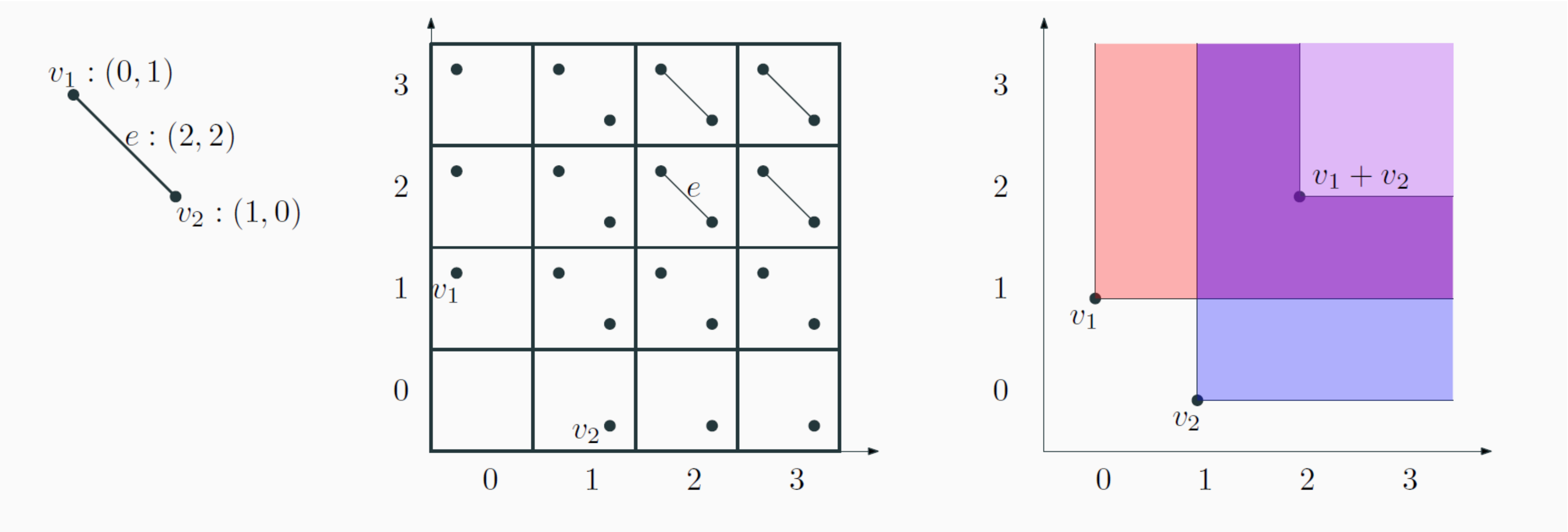
$$\begin{tikzcd}
V_{0,n} \arrow[r] & V_{1,n} \arrow[r] & \cdots \arrow[r] & V_{n,n} \\
\vdots \arrow[u] & \vdots \arrow[u, "v_{ij}"] & & \vdots \arrow[u] \\
V_{0,1} \arrow[r] \arrow[u] & V_{1,1} \arrow[r] \arrow[u] & \cdots \arrow[r] & V_{n,1} \arrow[u] \\
V_{0,0} \arrow[r] \arrow[u] & V_{1,0} \arrow[r] \arrow[u] & \cdots \arrow[r] & V_{n,0} \arrow[u]
\end{tikzcd}$$

$$\xleftarrow{V_{i,j} = H(K_{i,j})}$$

$$\begin{tikzcd}
K_{0,n} \arrow[r, phantom, "\subseteq"] & K_{1,n} \arrow[r, phantom, "\subseteq"] & \cdots \arrow[r, phantom, "\subseteq"] & K_{n,n} \\
\vdots \arrow[u, phantom, sloped, "\subseteq"] & \vdots \arrow[u, phantom, sloped, "\subseteq"] \arrow[u, phantom, "{\sigma_{ij}^{1}, \cdots, \sigma_{ij}^{k}}"'] & & \vdots \arrow[u, phantom, sloped, "\subseteq"] \\
K_{0,1} \arrow[r, phantom, "\subseteq"] \arrow[u, phantom, sloped, "\subseteq"] & K_{1,1} \arrow[r, phantom, "\subseteq"] \arrow[u, phantom, sloped, "\subseteq"] & \cdots \arrow[r, phantom, "\subseteq"] & K_{n,1} \arrow[u, phantom, sloped, "\subseteq"] \\
K_{0,0} \arrow[r, phantom, "\subseteq"] \arrow[u, phantom, sloped, "\subseteq"] & K_{1,0} \arrow[r, phantom, "\subseteq"] \arrow[u, phantom, sloped, "\subseteq"] & \cdots \arrow[r, phantom, "\subseteq"] & K_{n,0} \arrow[u, phantom, sloped, "\subseteq"]
\end{tikzcd}$$

Input as chain complex

- $C_{p-1} \xleftarrow{\partial_p} C_p \xleftarrow{\partial_{p+1}} C_{p+1}$ $\partial_p \circ \partial_{p+1} = 0,$ $H_p = Ker\ \partial_p / \ Im\ \partial_{p+1}$

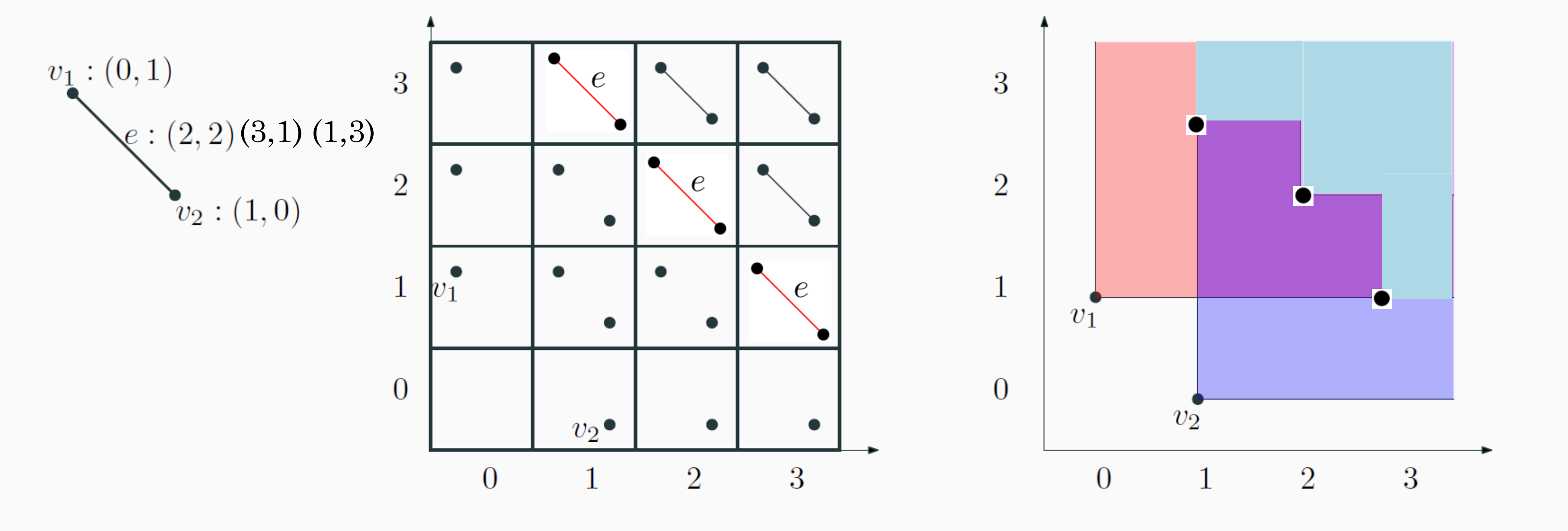


$H_0 - module$: Generators: v_1 (0,1), v_2 (1,0), Relation: $v_1 + v_2 = 0$ (2,2)

🟥 dim 1   🟦 dim 1   🟪 dim 2   ◻ dim 1

Input as chain complex (modules not Free)

- $C_{p-1} \xleftarrow{\partial_p} C_p \xleftarrow{\partial_{p+1}} C_{p+1}$ $\partial_p \circ \partial_{p+1} = 0,$ $H_p = Ker\ \partial_p / \ Im\ \partial_{p+1}$



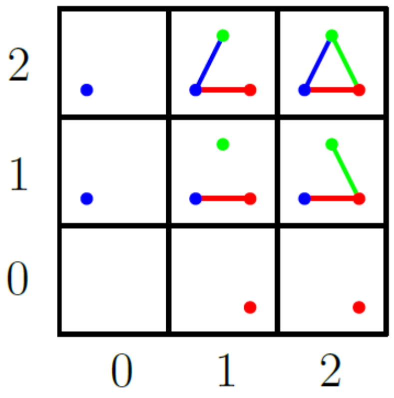
$H_0 - module$: Generators: v_1 (0,1), v_2 (1,0), Relation: $v_1 + v_2 = 0$ (2,2) (3,1)(1,3)

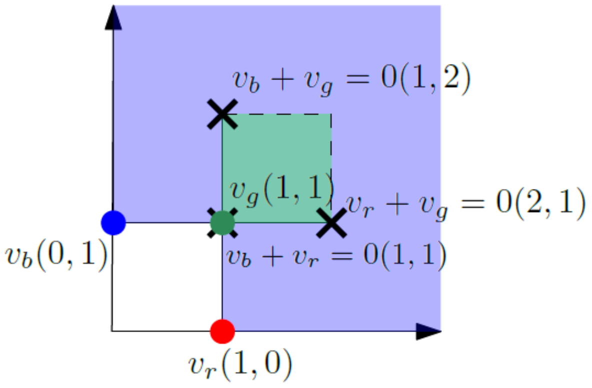
🟥 dim 1 🟦 dim 1 🟪 dim 2 ▨ dim 1

Goal: Compute Presentation of Homology

1. Input Chain complex : $\boldsymbol{C}$: $\boldsymbol{C_{p-1} \leftarrow C_p \leftarrow C_{p+1}}$
2. Compute Free chain complex: $\boldsymbol{D}$: $\boldsymbol{D_{p-1} \leftarrow D_p \leftarrow D_{p+1}}$ *s.t.* $\boldsymbol{H_p(C) \cong H_p(D)}$ [CSV17]
3. Compute an exact sequence: $\boldsymbol{0 \leftarrow H_p(D) \leftarrow P_0 \leftarrow P_1}$

$\boldsymbol{P_0}$ **and** $\boldsymbol{P_1}$ **are free (no relations)**, $\boldsymbol{A}$: $\boldsymbol{P_0 \leftarrow P_1}$ is called presentation, can be expressed as a matrix.





$$\begin{array}{cccc} [\partial_1] & e_r^{(1,1)} & e_b^{(1,2)} & e_g^{(2,1)} \\ v_b^{(0,1)} & \mathbf{t}^{(1,0)} & \mathbf{t}^{(1,1)} & 0 \\ v_r^{(1,0)} & \mathbf{t}^{(0,1)} & 0 & \mathbf{t}^{(1,1)} \\ v_g^{(1,1)} & 0 & \mathbf{t}^{(0,1)} & \mathbf{t}^{(1,0)} \end{array} \longrightarrow \begin{array}{cccc} \mathbf{A} & c_1^{(1,1)} & c_2^{(1,2)} & c_3^{(2,1)} \\ r_1^{(0,1)} & 1 & 1 & 0 \\ r_2^{(1,0)} & 1 & 0 & 1 \\ r_3^{(1,1)} & 0 & 1 & 1 \end{array}$$

2. **Minimal presentation from a bi-filtration can be computed in** $\boldsymbol{O(n^3)}$ ***time***

[LW22]

W. Chakolski, M. Scolamiero, F. Vaccarino. Combinatorial presentation of multidimensional persistent homology. Journal of Pure and Applied Algebra,221(5), 1055-1075, 2017.

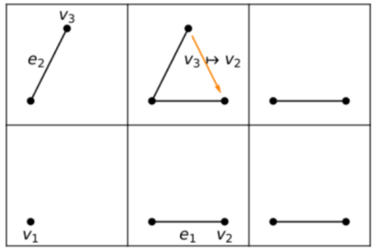
M. Lesnick, M. Wright. Computing minimal presentation and bigraded betti numbers of 2-parameter persistent homology. SIAM J. Appl. Algebra Geom., 6(2), 267-298, 2022.

Two new results

1. Input Chain complex (filtration): $\boldsymbol{C}$: $\boldsymbol{C_{p-1} \leftarrow C_p \leftarrow C_{p+1} \leftarrow C_{p+2} \leftarrow \cdots}$
2. Compute Free chain resolution: $\boldsymbol{D}$: $\boldsymbol{D_{p-1} \leftarrow D_p \leftarrow D_{p+1} \leftarrow D_{p+2} \leftarrow \cdots}$

   $\boldsymbol{s.t.\ H_k(C) \cong H_k(D)\ \forall k \geq 0}$

O(n $\log^2$n) algo in 2-parameter:[non-minimal resolution better, uses graphs] **[BDKRS26]**



$$\begin{tikzcd}
\mathbb{k} \arrow[r, "\begin{pmatrix}0\\1\end{pmatrix}"] & \mathbb{k}^2 \arrow[r, "\begin{pmatrix}1 & 1\end{pmatrix}"] & \mathbb{k} \\
0 \arrow[u] \arrow[r] & \mathbb{k} \arrow[u, "\begin{pmatrix}1\\0\end{pmatrix}"] \arrow[r, "1"] & \mathbb{k} \arrow[u, "1"']
\end{tikzcd}$$

$C_1(K)$

1. Input Chain complex of towers : $\boldsymbol{T}$: $\boldsymbol{T_{p-1} \leftarrow T_p \leftarrow T_{p+1}}$

2. Compute Free chain complex: $\boldsymbol{D}$: $\boldsymbol{A \overset{g}{\leftarrow} B \overset{h}{\leftarrow} C\ s.t.\ H_p(T) \cong ker\ g / \ im\ h}$

   Assume indexing set is a finite poset P, |P|=t

   An O(nt^2) algo : minimal resolution, uses graphs to compute R and RR [DR26]

U. Bauer, T. Dey, M. Kerber, F. Russo, M. Sols. Fast free resolution of bifiltered chain complexes. Proc. 42nd SoCG, 2026. arXiv.org/abs/2512.08652.

T. Dey, F. Russo. Computing Projective Implicit Representation from Poset Towers. arXiv.org/abs/2505.08755 (2025)

Generalized Rank and Its Computation

Invariants

1. **A property R for a category of persistence modules is invariant iff $\boldsymbol{R(M) = R(N)}$ whenever $\boldsymbol{M \cong N}$; $\boldsymbol{R}$** is incomplete when `only if' is dropped

Barcode for 1-parameter modules is a complete invariant

There is no complete discrete invariant for multiparameter modules **[CZ09]**

2. **Some discrete invariants: (stability, algorithms not discussed here)**

Hilbert function: $dim: P \to \mathbb{Z}$, $\mathrm{p} \mapsto \dim(M(p))$

Rank function: $rank: P \times P \to \mathbb{Z}$, $p \times q_{\{p \leq q\}} \mapsto rank(M(p \leq q))$ **[CZ09]**

Generalized rank: defined by rank of limit to colimit maps **[KM21]**

Betti numbers : #generators of free modules in a (minimal) free resolution **[OS24]**

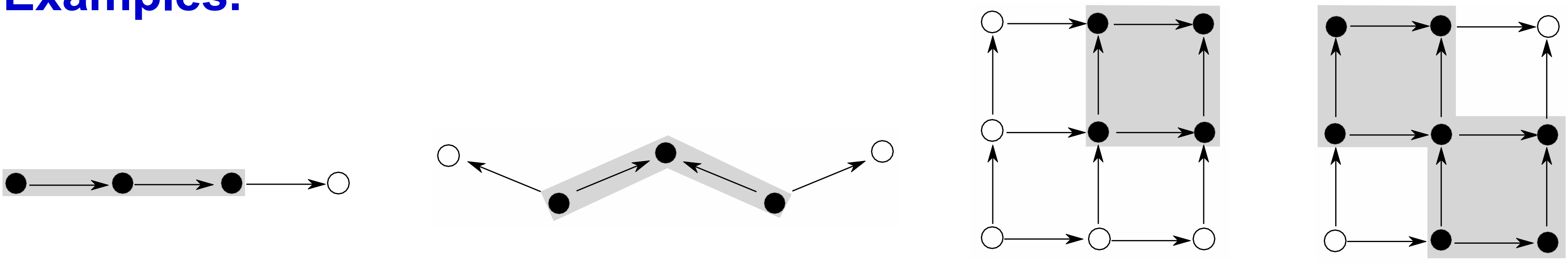
Epsilon-pruning : stabilizing decompositions **[B25]**

G. Carlsson, A. Zomorodian. The theory of multidimensional persistence. Discrete & Comput. Geom, 42(1):72–93, 2009.
W. Kim, F. Memoli. Generalized persistence diagrams for persistence modules over posets. J. Applied Comput. Topology, 5(4), 533—581, 2021.
S. Oudot, L. Soccola. On the stability of multigraded Betti numbers and Hilbert functions. SIAM Appl. Algebra Geom. 8(1), 54-88, 2024
H. Bjerkevik. Stabilizing decomposition of multiparameter persistence modules. arXiv 2305.15550 (2025).

Intervals in posets

- $I \subset \mathrm{P}$ is called an **interval** if I is **connected** and **order-convex**, i.e. $[p, q \in I$ and $p \leq r \leq q] \Rightarrow r \in I$.

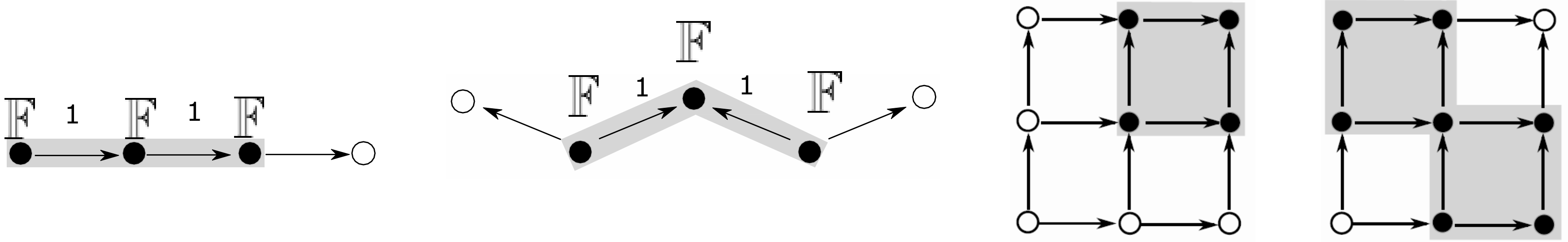
Examples.



Intervals in posets

- $I \subset \mathrm{P}$ is called an **interval** if I is **connected** and **order-convex**, i.e. $[p, q \in I \text{ and } p \leq r \leq q] \Rightarrow r \in I$.

An **interval module** $\mathrm{I}^{I} : \mathrm{P} \to \mathbf{vec}$ is an indicator module over I.



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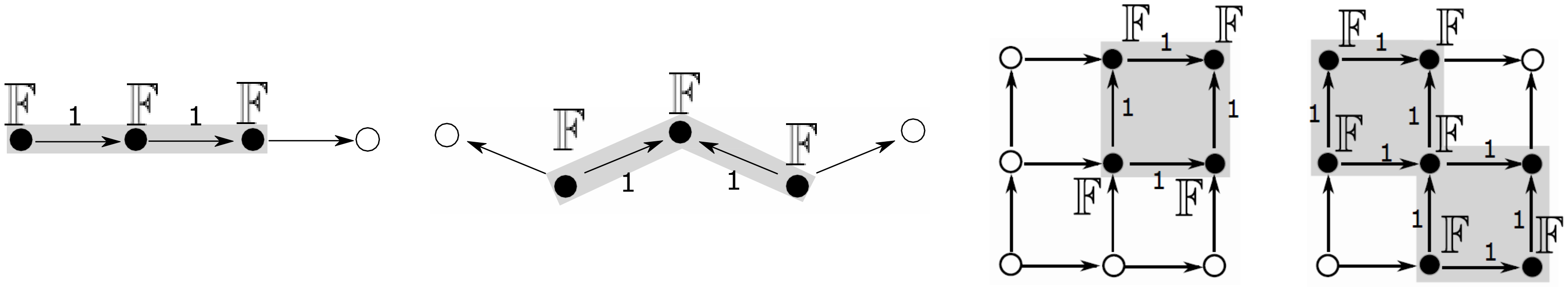
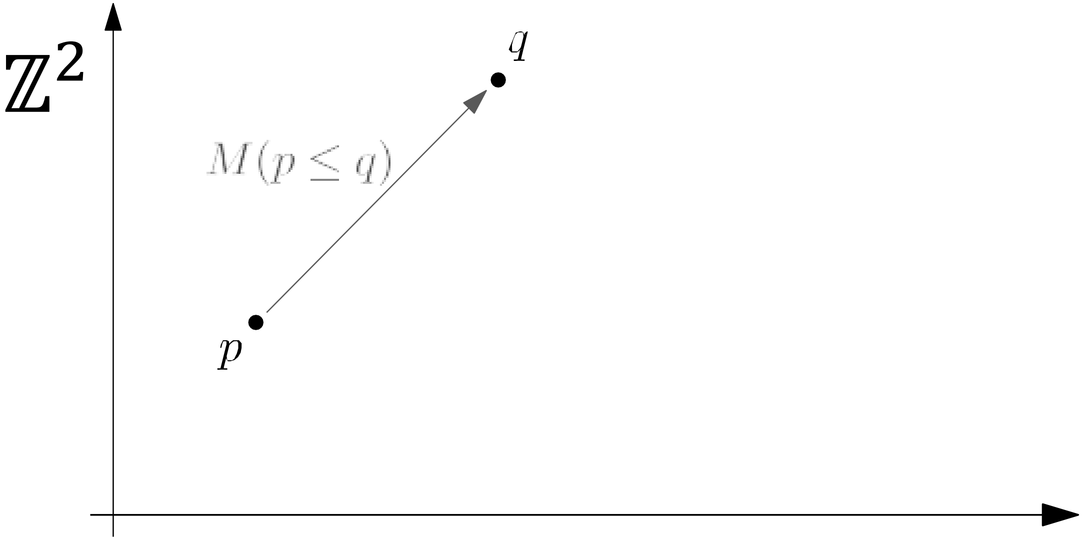


Interval modules are **indecomposable**.

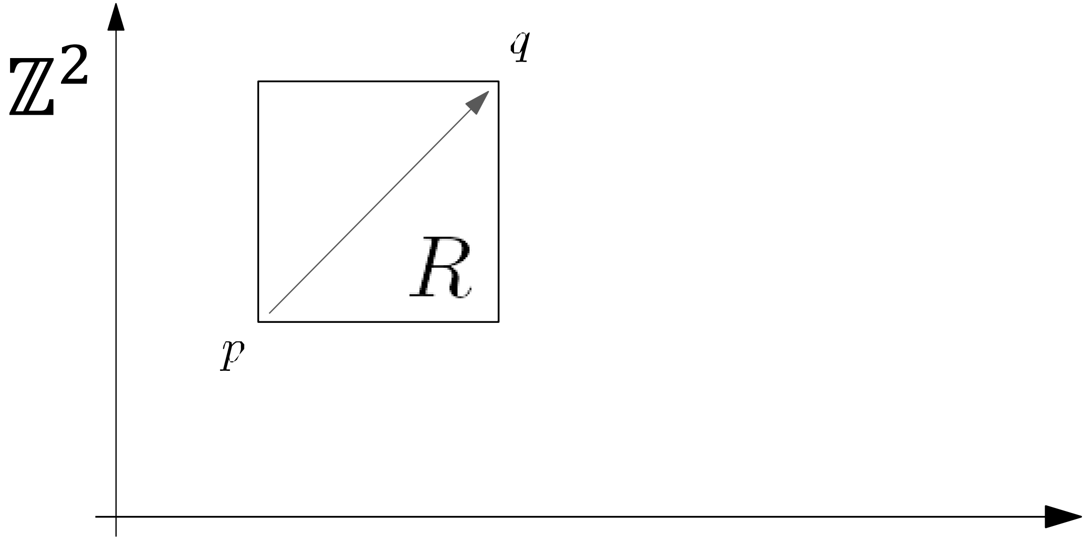
$M \cong \bigoplus I^{I_i}$ is called interval decomposable and $\{I_i\}$ is called the barcode of M.

2-parameter persistence and rank invariant



$M : \mathbb{Z}^2 \rightarrow$ **vec**

2-parameter persistence and rank invariant



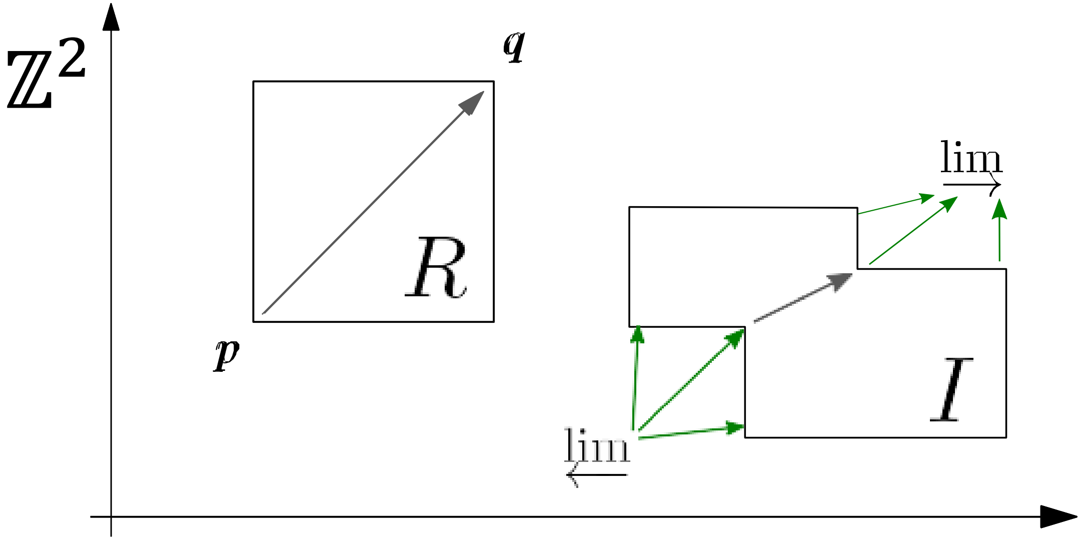
$M : \mathbb{Z}^2 \to$ **vec**

Rank invariant [Carlsson, Zomordian]

$(p \leq q) \to \mathbf{rk}_M(p \leq q)$, $R \to \mathbf{rk}_M(R)$

G. Carlsson, A. Zomorodian. The theory of multidimensional persistence. Discrete & Comput. Geom, 42(1):72–93, 2009.

2-parameter persistence and rank invariant



$M : \mathbb{Z}^2 \to \textbf{vec}$

Rank invariant [Carlsson, Zomordian]

$(p \leq q) \to \mathbf{rk}_M(p \leq q)$, $R \to \mathbf{rk}_M(R)$

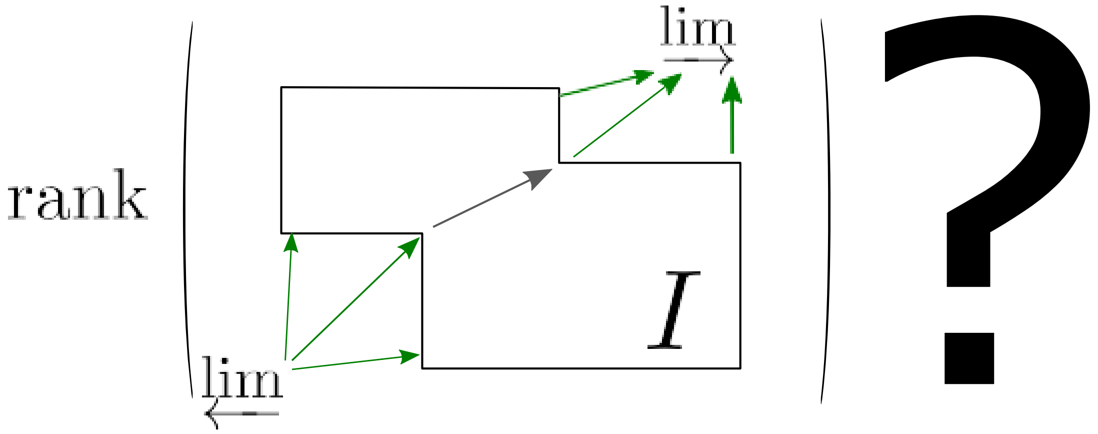
Generalized Rank invariant [Kim,Memoli] [Patel]

$I \to \mathbf{rk}_M(I)$ A complete stable invariant for interval decomposable modules

W. Kim, F. Memoli. Generalized persistence diagrams for persistence modules over posets. J. Applied Comput. Topology, 5(4), 533—581, 2021.

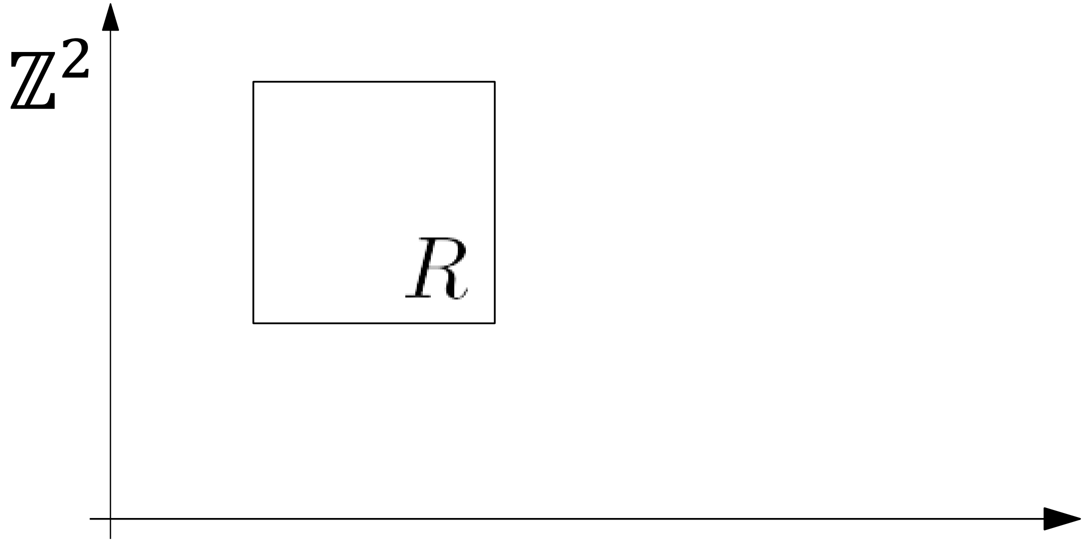
How to compute the generalized rank invariant?

How to compute



Generalized Rank to Zigzag

$M : \mathbb{Z}^2 \to \textbf{vec}$

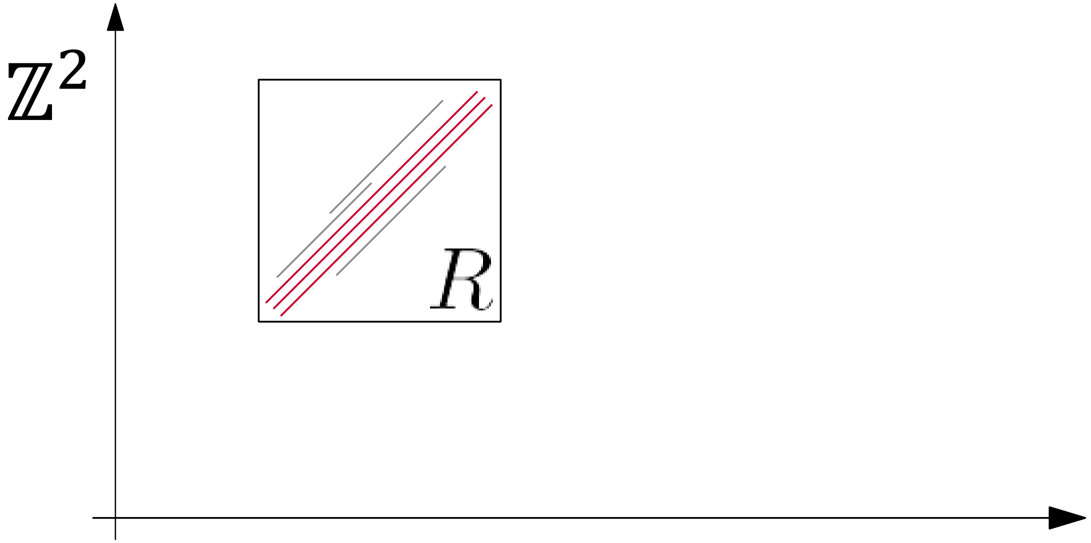


Rank invariant

$R \to \mathbf{rk}_M(R)$

Generalized Rank to Zigzag

$M : \mathbb{Z}^2 \to$ **vec**



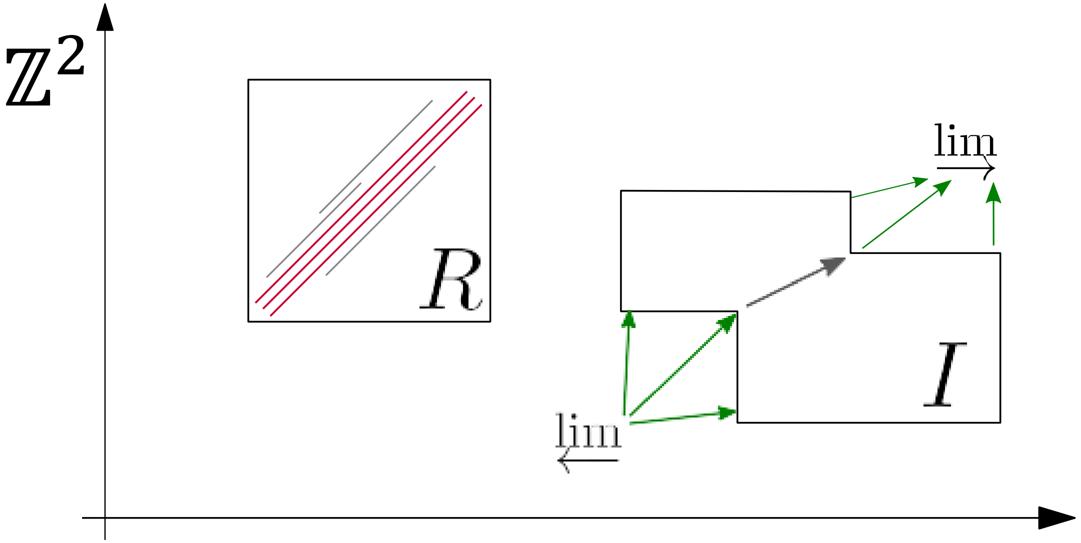
Rank invariant (known)

$R \to \mathbf{rk}_M(R)$

$= \#($"Full" bars$)$

Generalized Rank to Zigzag

$M : \mathbb{Z}^2 \to \textbf{vec}$



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Generalized Rank to Zigzag

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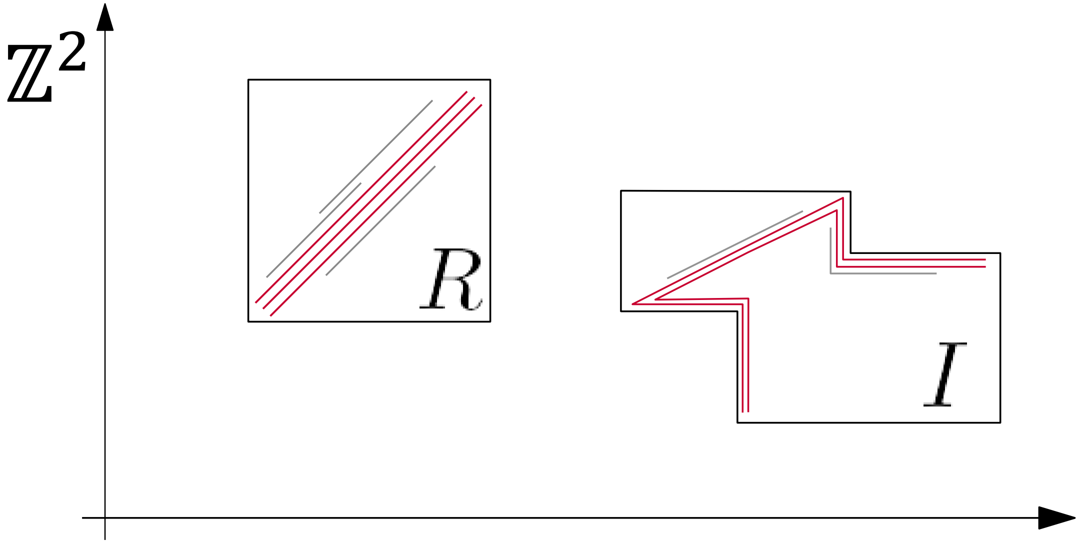

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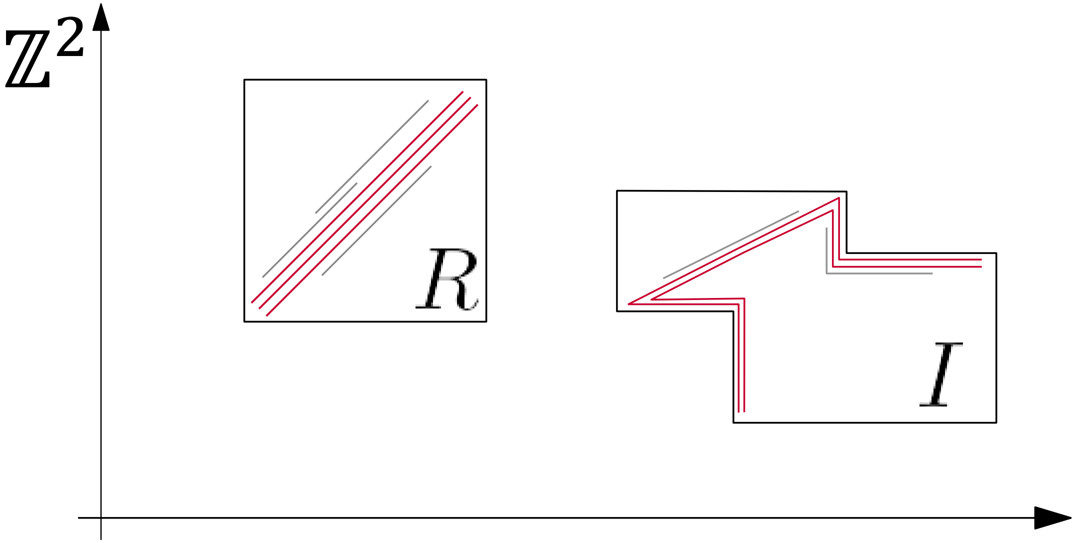
T. Dey, W. Kim, F. Memoli. Computing generalized rank invariant for 2-parameter persistence modules via zigzag persistence and its applications. Discrete & Comput. Geometry, 71(1), 67—94, 2024.

Generalized Rank to Zigzag

$M : \mathbb{Z}^2 \to \textbf{vec}$



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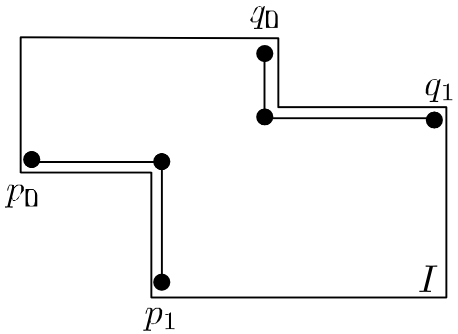
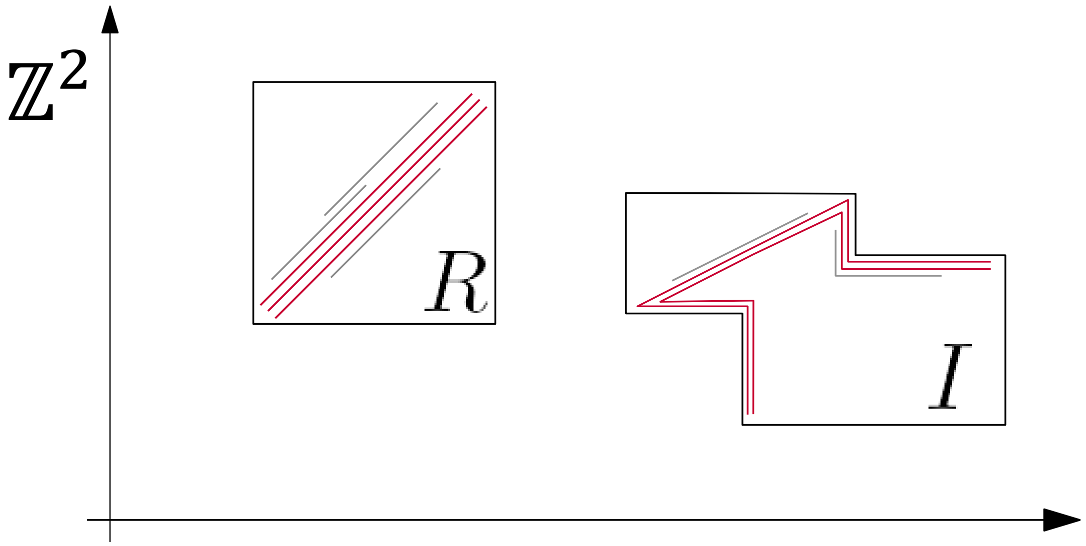


Generalized Rank to Zigzag on Boundary Cap

$M : \mathbb{Z}^2 \to$ **vec**



Rank invariant (known)

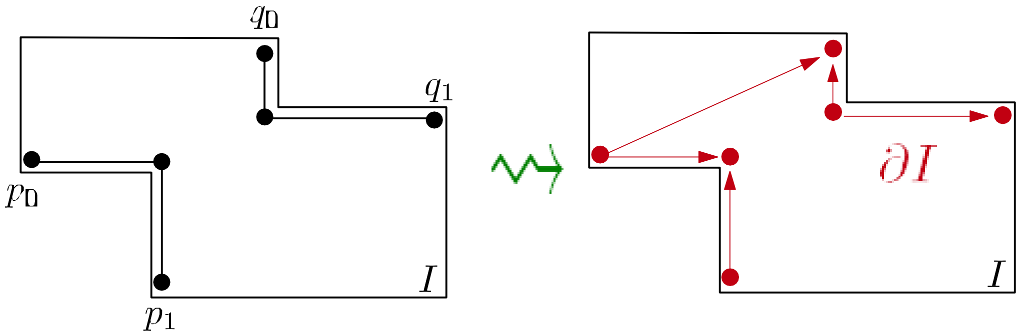
$R \to \mathbf{rk}_M(R)$

= #("Full" bars)

Theorem 1 [D., Kim, Me'moli]

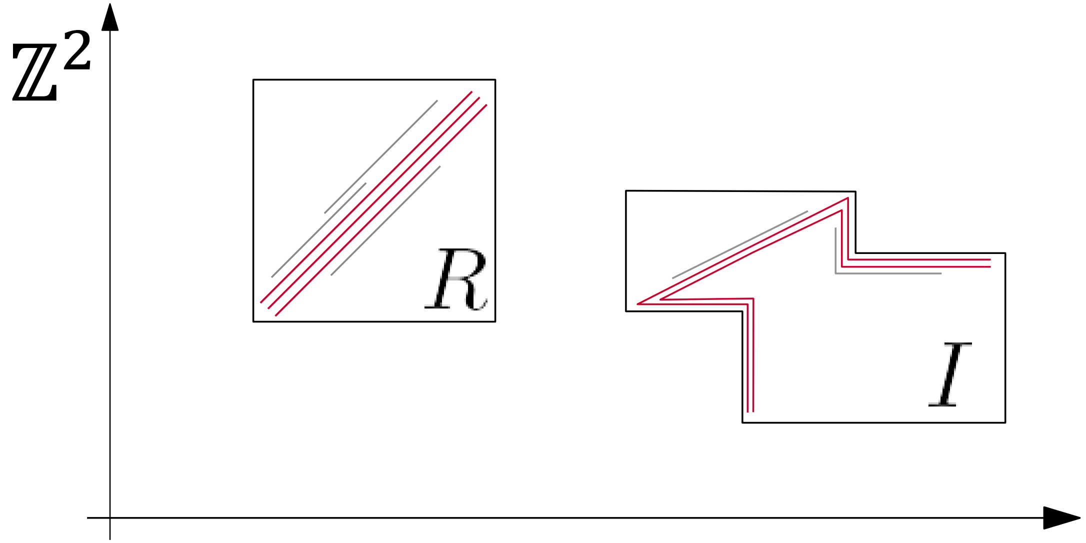
$I \to \mathbf{rk}_M(I)$

= #("Full" bars) on bnd cap zigzag



Generalized Rank to Zigzag on Boundary Cap

$M : \mathbb{Z}^2 \to$ **vec**



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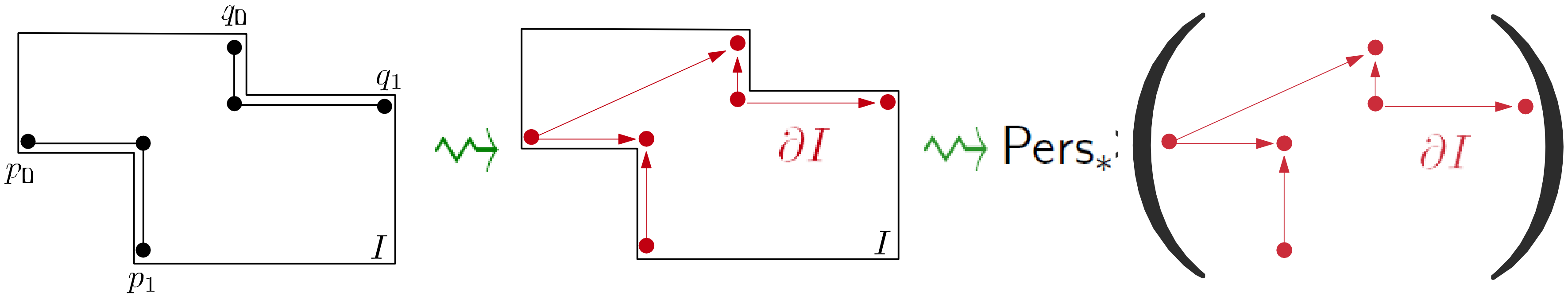


New results for (co)limits for d-parameter modules [DL26]

- $M: Q \subseteq \mathbb{N}^d \to Vec$ *given by presentation* $M \leftarrow F_0 \leftarrow F_1$ *where* F_i *has rank* $\leq r$
- *Q has n minima and maxima*

1. **(co)lim** *M can be computed in*

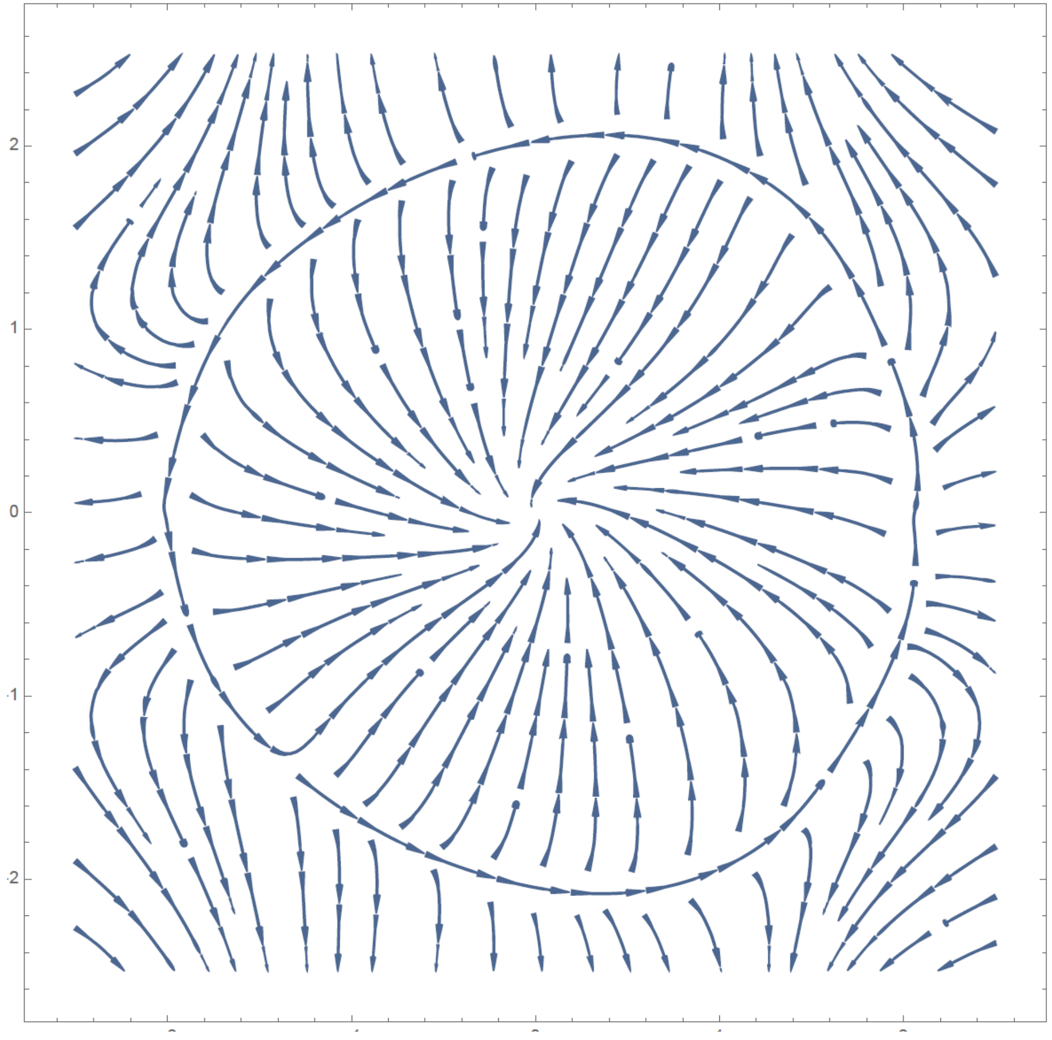
   (i) $O(n\log n + r^3)$ time for $d = 2$
   (ii) $O((nr)^\omega)$ time for $d=3$
   (iii) $O(n^4 + n^{\omega+1} r^\omega)$ *time for* $d > 3$

2. *Generalized rank of M can be computed with the same time complexity.*

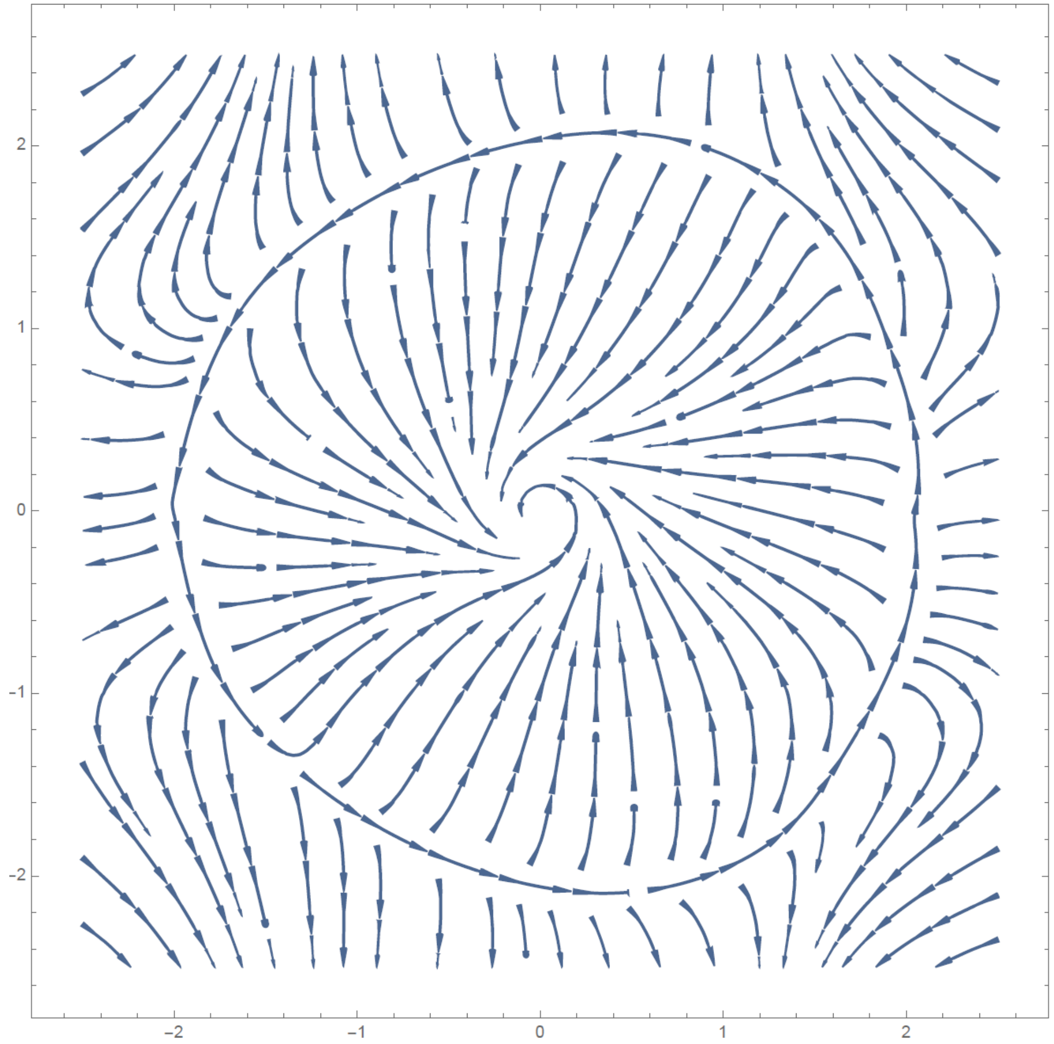
T. Dey, M. Lesnick. Limit computation over posets via minimal initial functors. arXiv.org/abs/2601.00209 (2026)

Combinatorial Dynamical System

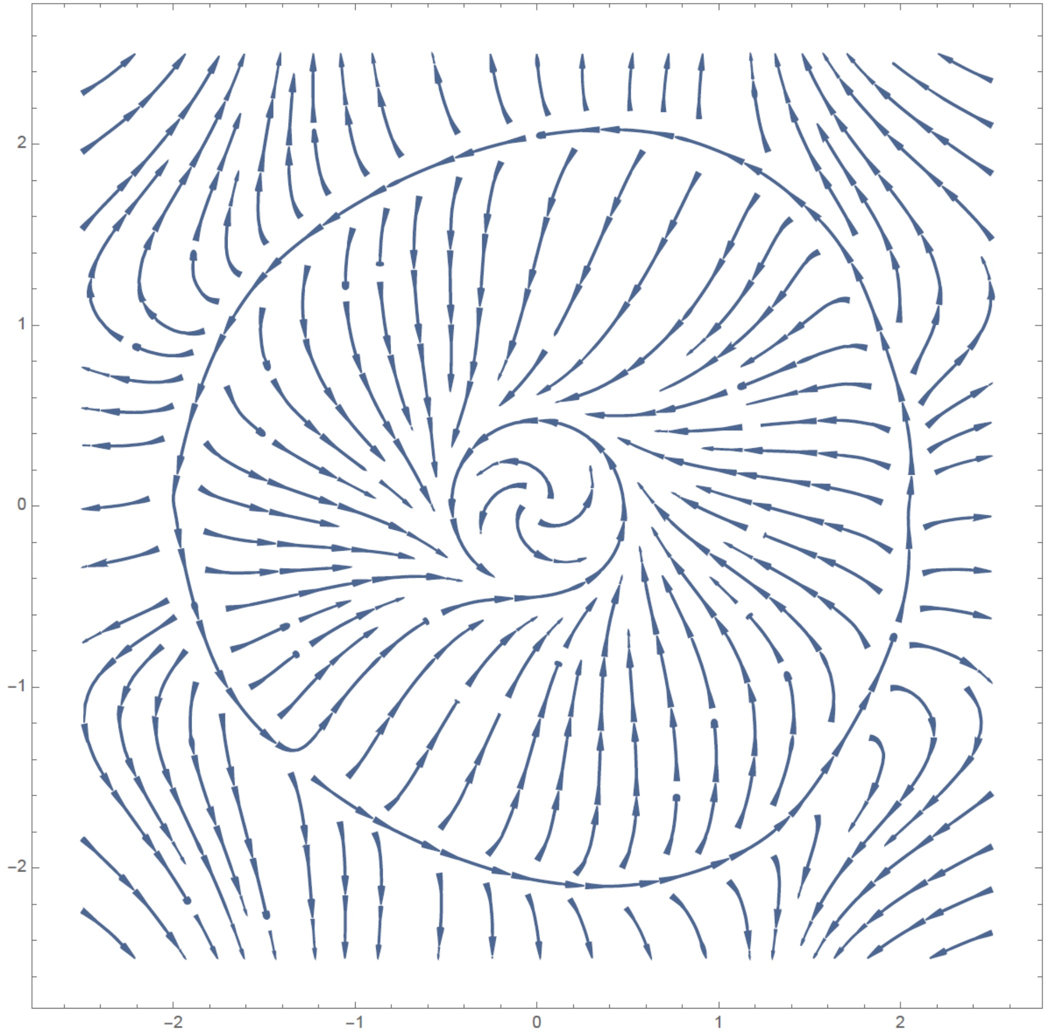
Continuous vector field



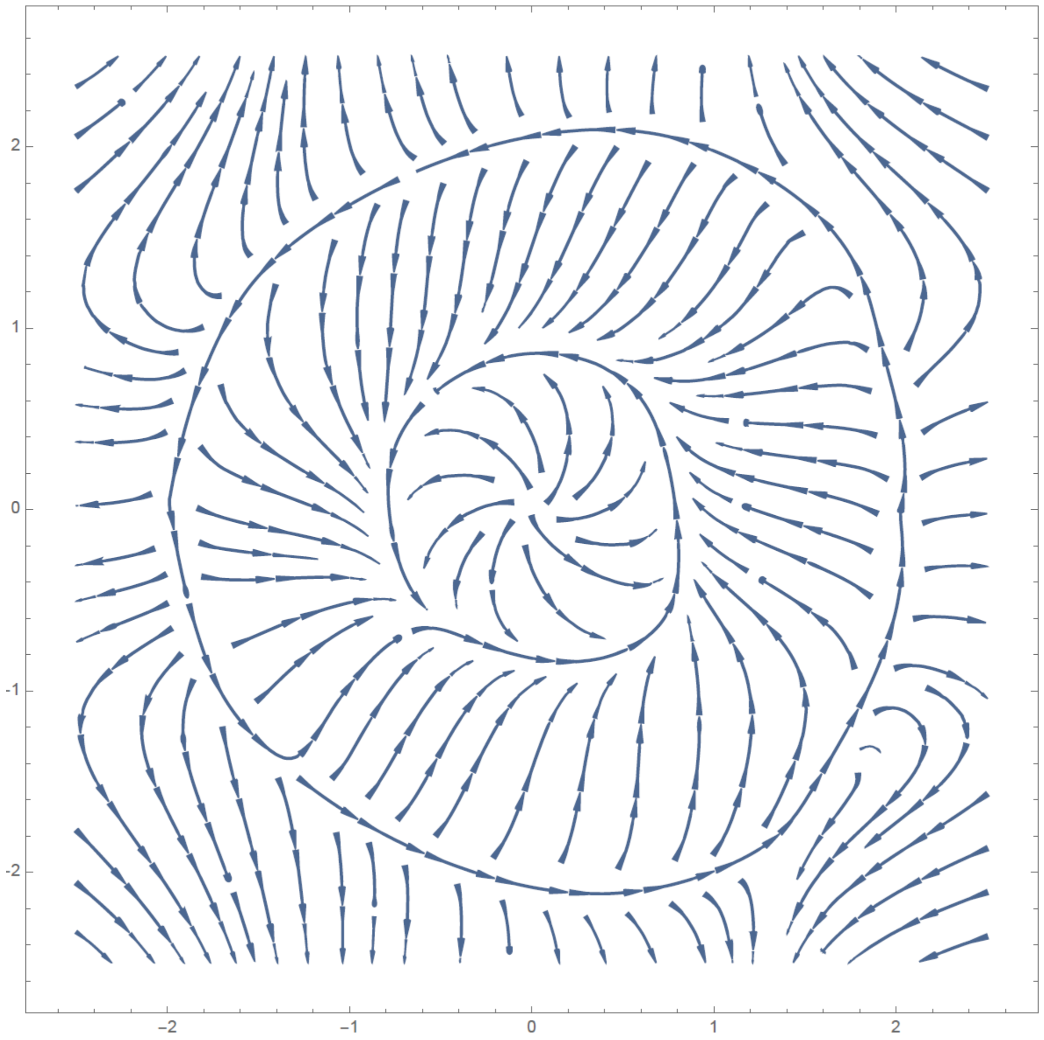
Continuous vector field



Continuous vector field



Continuous vector field



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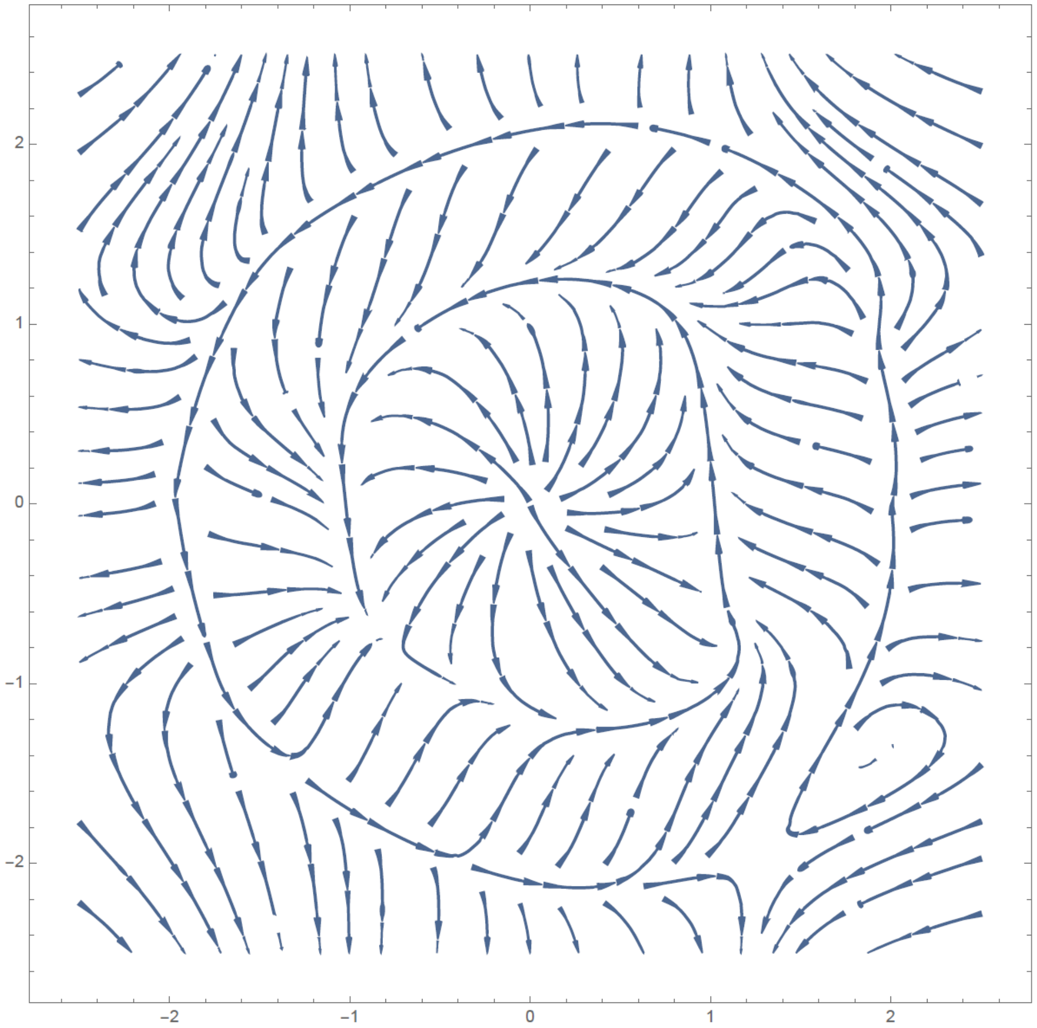

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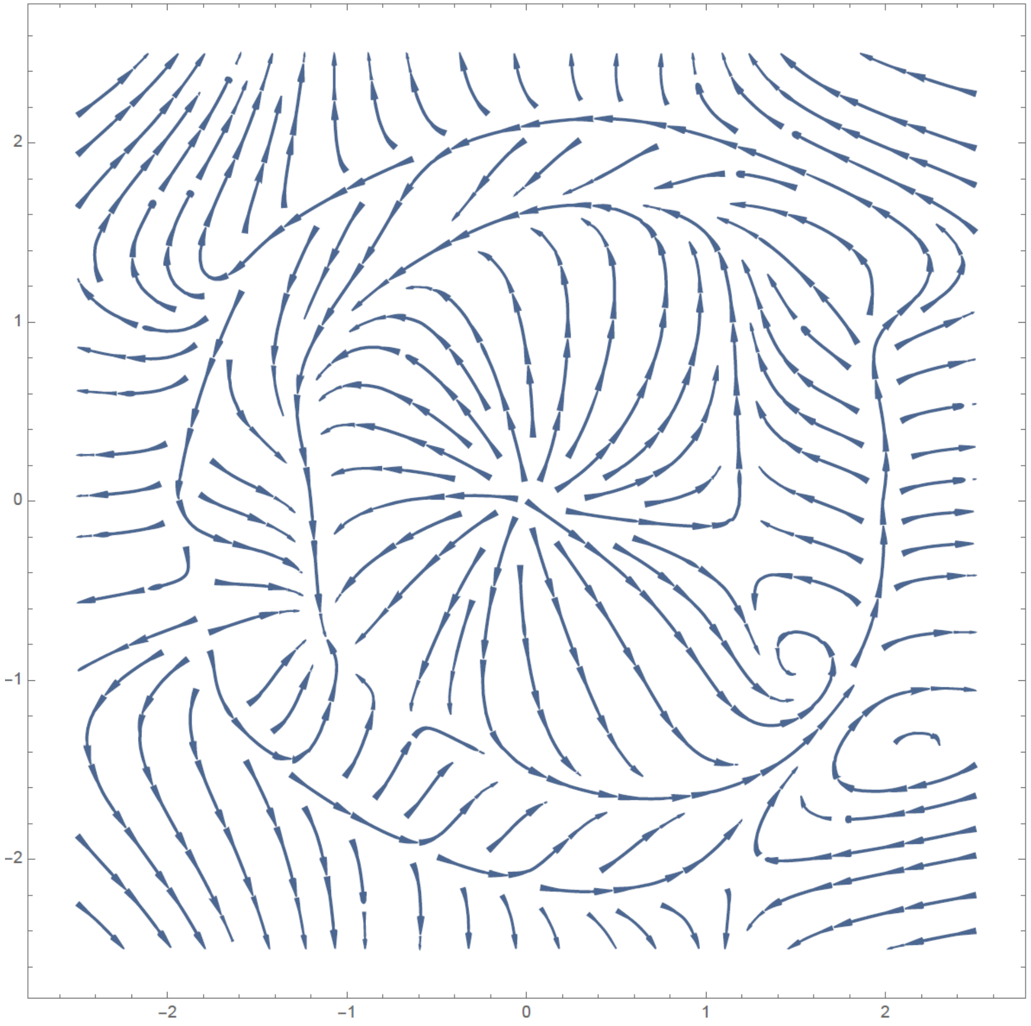
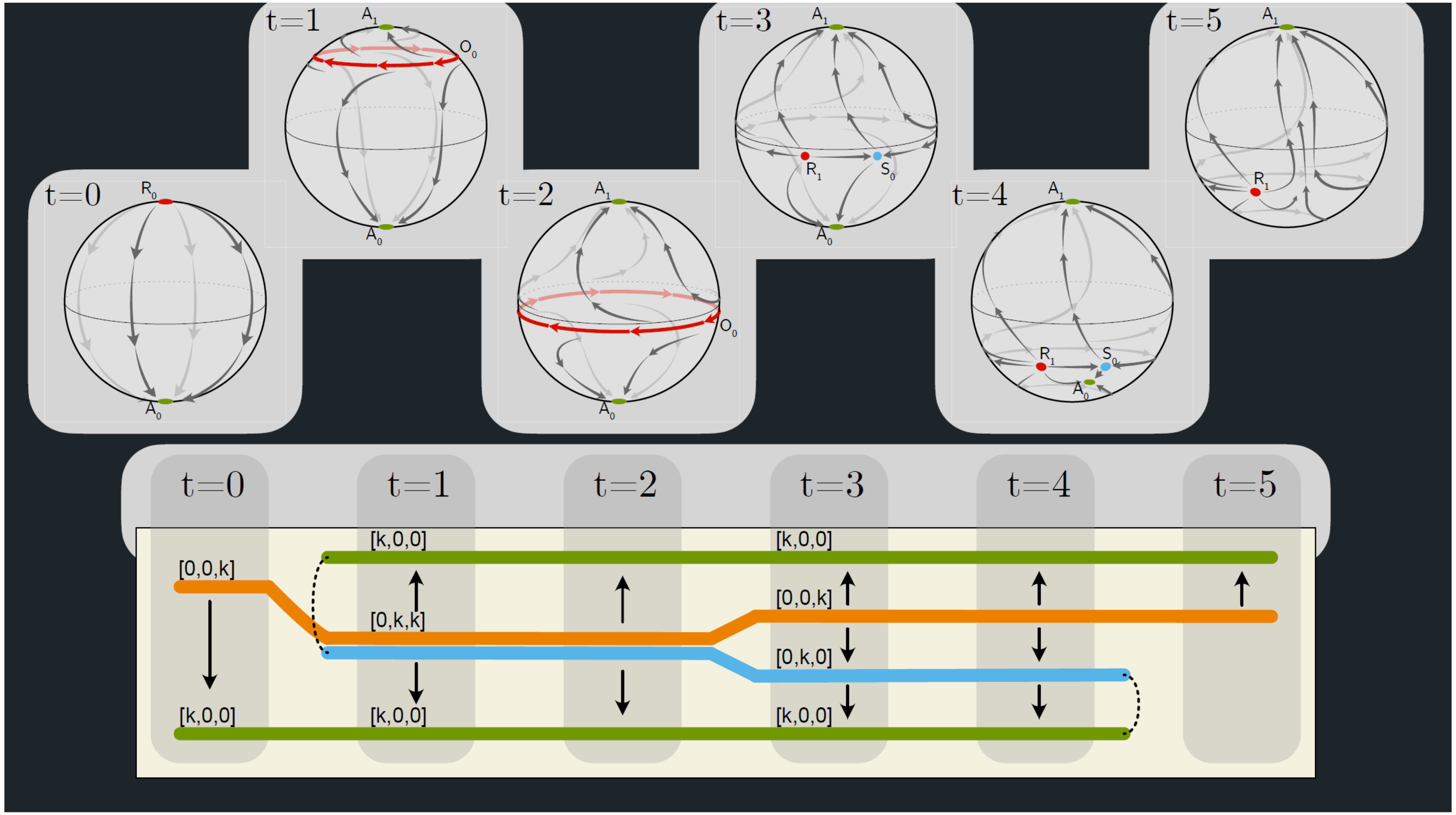
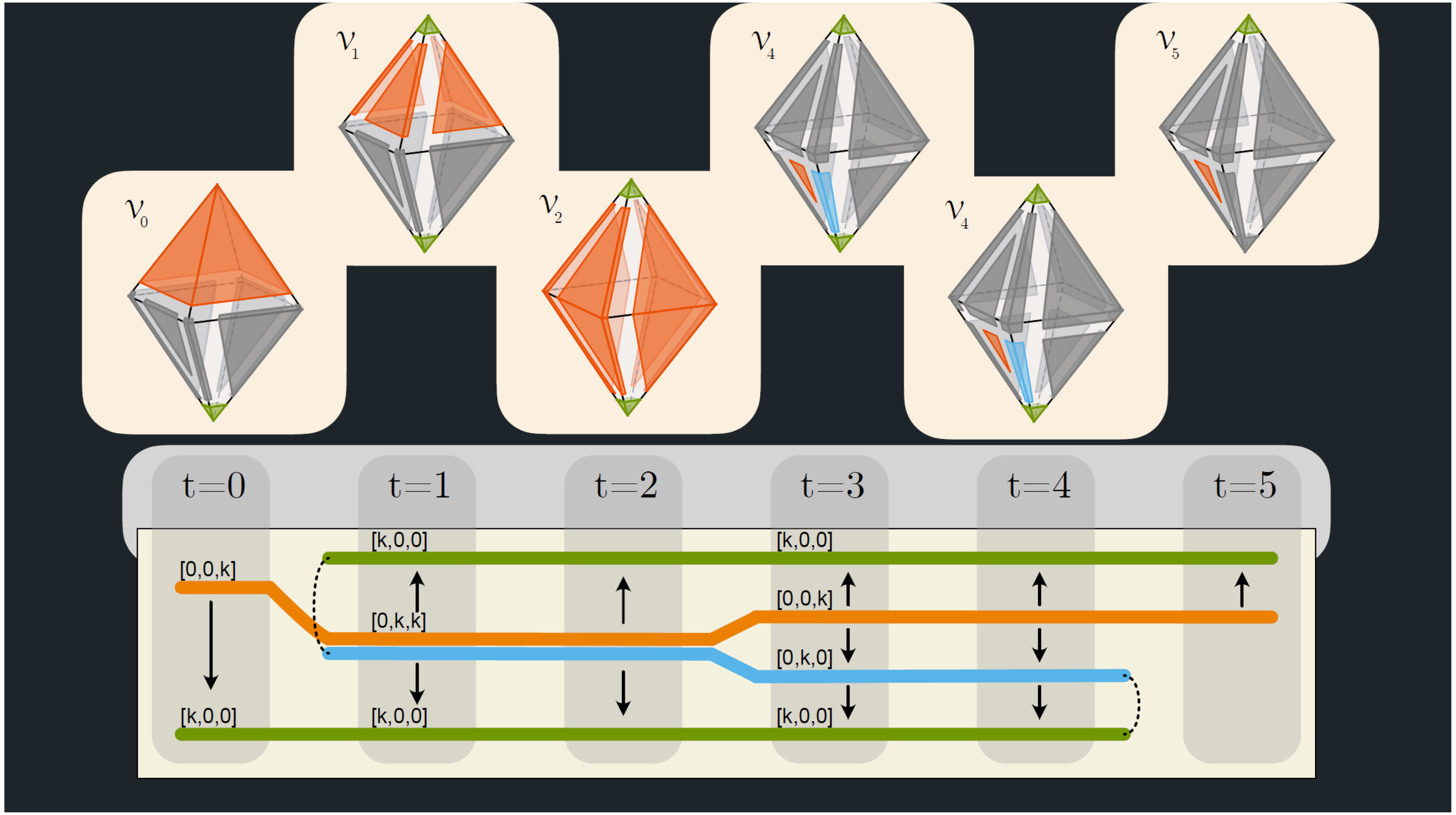


Conely-Morse barcode [DLS26]



T. K. Dey, M. Lipinsky, M. Soriano-Trigueros. Conely-Morse persistence barcode: A homological signature of a combinatorial bifurcation.Foundations of Computational Mathematics (FoCM), to appear; arXiv.org/abs/2504.17105 (2026).

Conely-Morse barcode

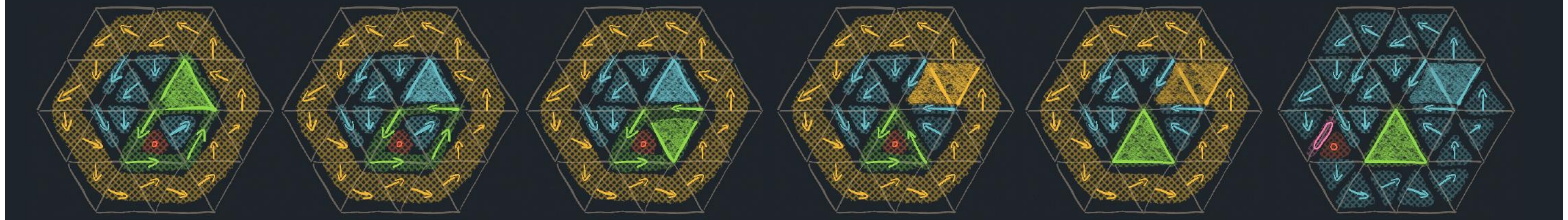


- Multivector: $v \subset K\ convex,\ if\ \sigma, \tau \in v$ $with\ \sigma \leq \mu \leq \tau, then\ \mu \in K$
- Multivector field: V a partition of K into multivectors **[Forman98,Mrozek17]**

Robin Forman, Morse theory for cell complexes, Advances in Mathematics 134 (1998), no. 1, 90–145.

Marian Mrozek, Conley–Morse–Forman theory for combinatorial multivector fields on Lefschetz complexes, Foundations of Computational Mathematics 17 (2017), no. 6, 1585–1633.

Combinatorial continuation



parameterized multivector field:

$$\mathscr{V}_0 \sqsupseteq \mathscr{V}_1 \sqsubseteq \mathscr{V}_2 \sqsupseteq \mathscr{V}_3 \sqsubseteq \mathscr{V}_4 \sqsupseteq \mathscr{V}_5$$

corresponding sequence of block decompositions :

$$\mathscr{N}_0 \sqsupseteq \mathscr{N}_1 \sqsubseteq \mathscr{N}_2 \sqsupseteq \mathscr{N}_3 \sqsubseteq \mathscr{N}_4 \sqsupseteq \mathscr{N}_5$$

T. K. Dey, M. Lipinsky, M. Soriano-Trigueros. Conely-Morse persistence barcode: A homological signature of a combinatorial bifurcation.Foundations of Computational Mathematics (FoCM), to appear; arXiv.org/abs/2504.17105 (2026).

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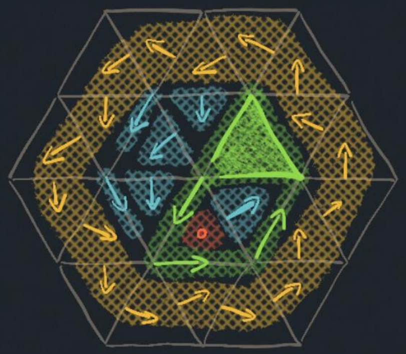

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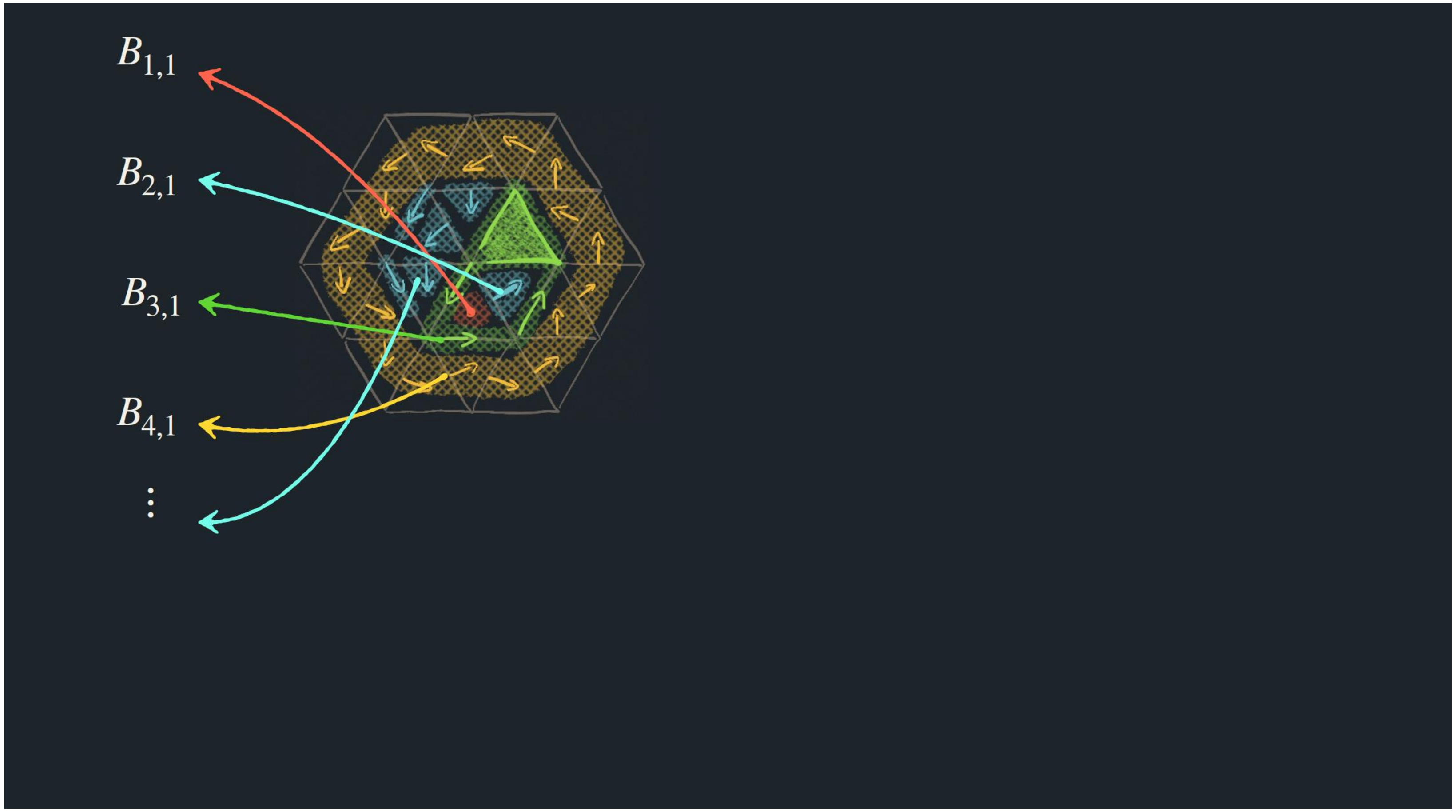

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Conely-Morse barcode

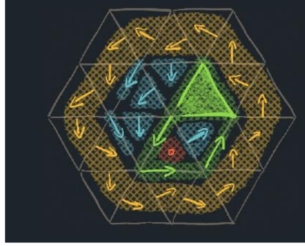
$B_{1,1}$

$B_{2,1}$

$B_{3,1}$

$B_{4,1}$

$\vdots$



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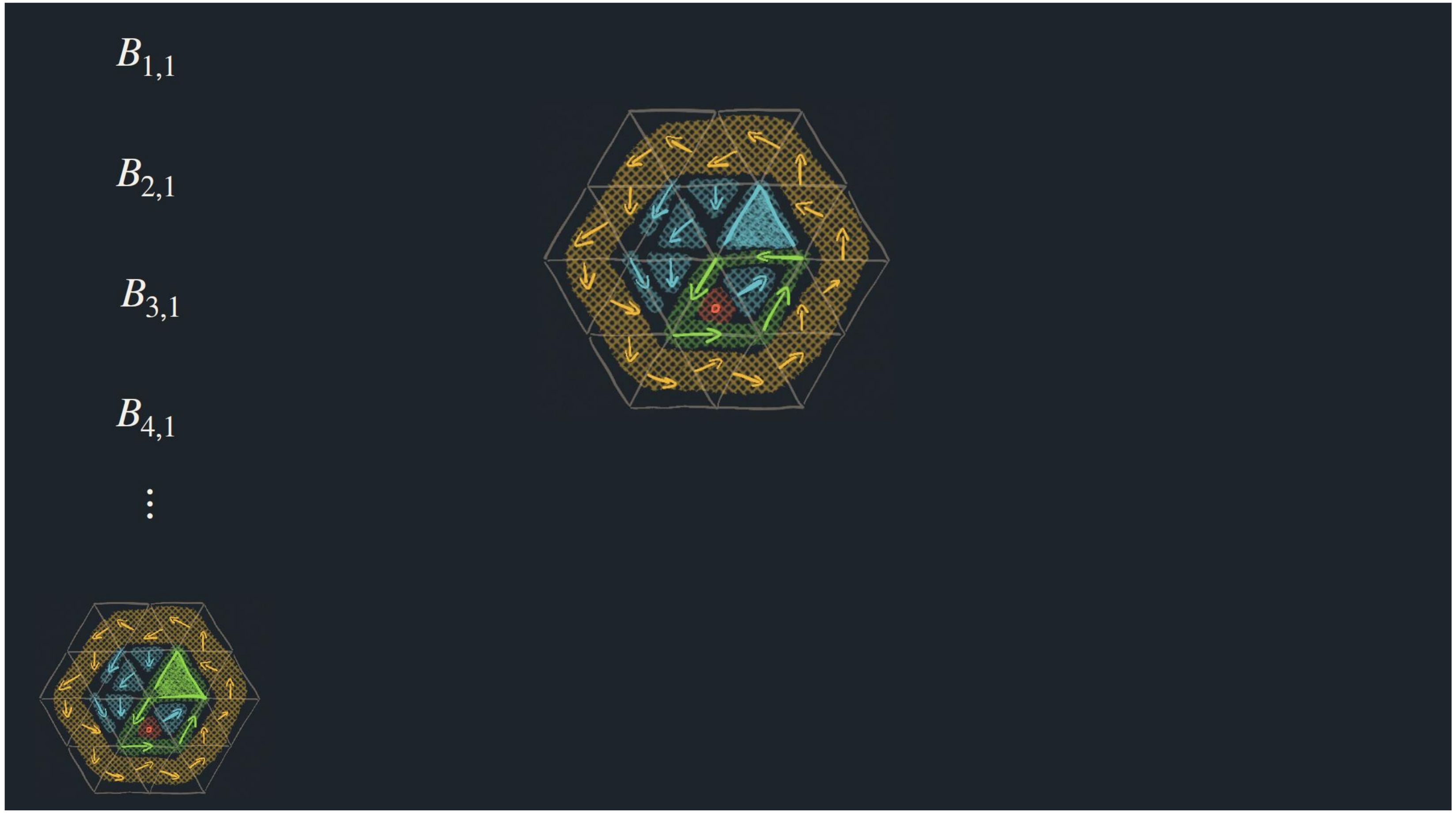

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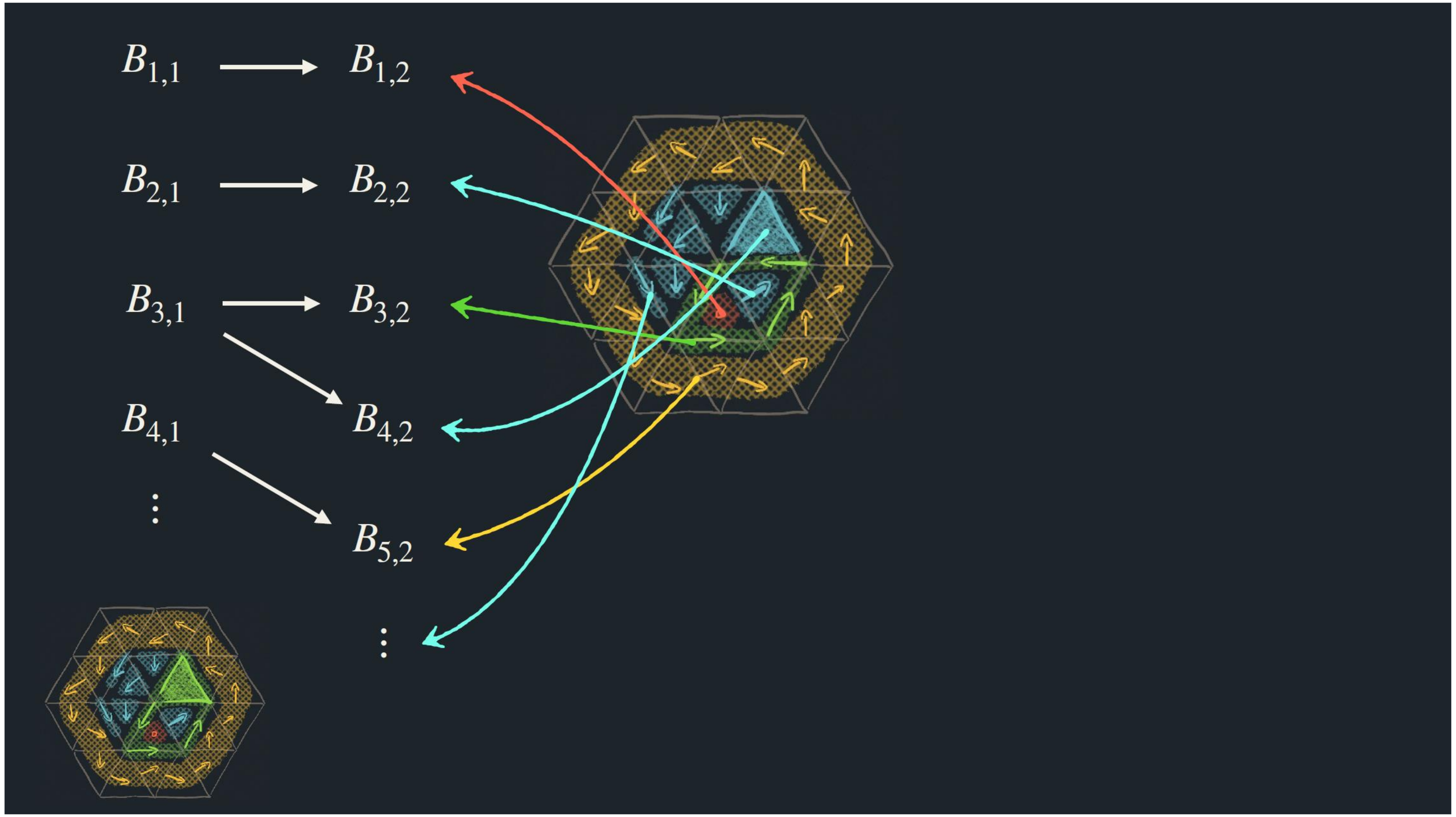

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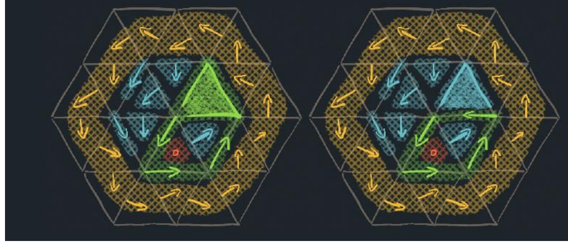
Conely-Morse barcode

$$\begin{tikzcd}
B_{1,1} \arrow[r] & B_{1,2} \\
B_{2,1} \arrow[r] & B_{2,2} \\
B_{3,1} \arrow[r] \arrow[dr] & B_{3,2} \\
B_{4,1} \arrow[dr] & B_{4,2} \\
\vdots & B_{5,2}
\end{tikzcd}$$



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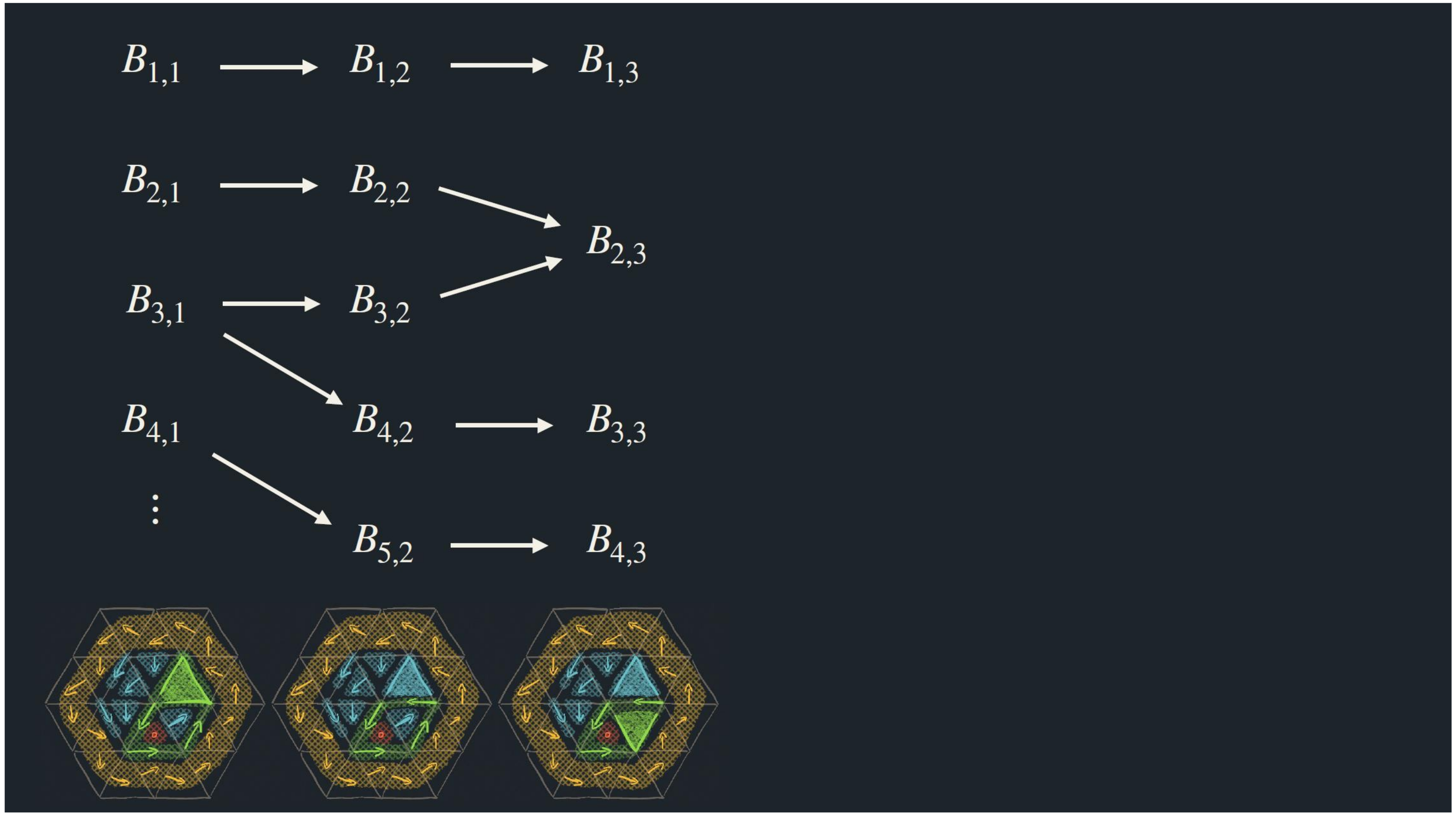

$$\begin{tikzcd}
B_{1,1} \arrow[r] & B_{1,2} \arrow[r] & B_{1,3} \\
B_{2,1} \arrow[r] & B_{2,2} \arrow[dr] & \\
 & & B_{2,3} \\
B_{3,1} \arrow[r] \arrow[dr] & B_{3,2} \arrow[ur] & \\
B_{4,1} \arrow[dr] & B_{4,2} \arrow[r] & B_{3,3} \\
\vdots & B_{5,2} \arrow[r] & B_{4,3}
\end{tikzcd}$$

T. K. Dey, M. Lipinsky, M. Soriano-Trigueros. Conely-Morse persistence barcode: A homological signature of a combinatorial bifurcation.Foundations of Computational Mathematics (FoCM), to appear; arXiv.org/abs/2504.17105 (2026).

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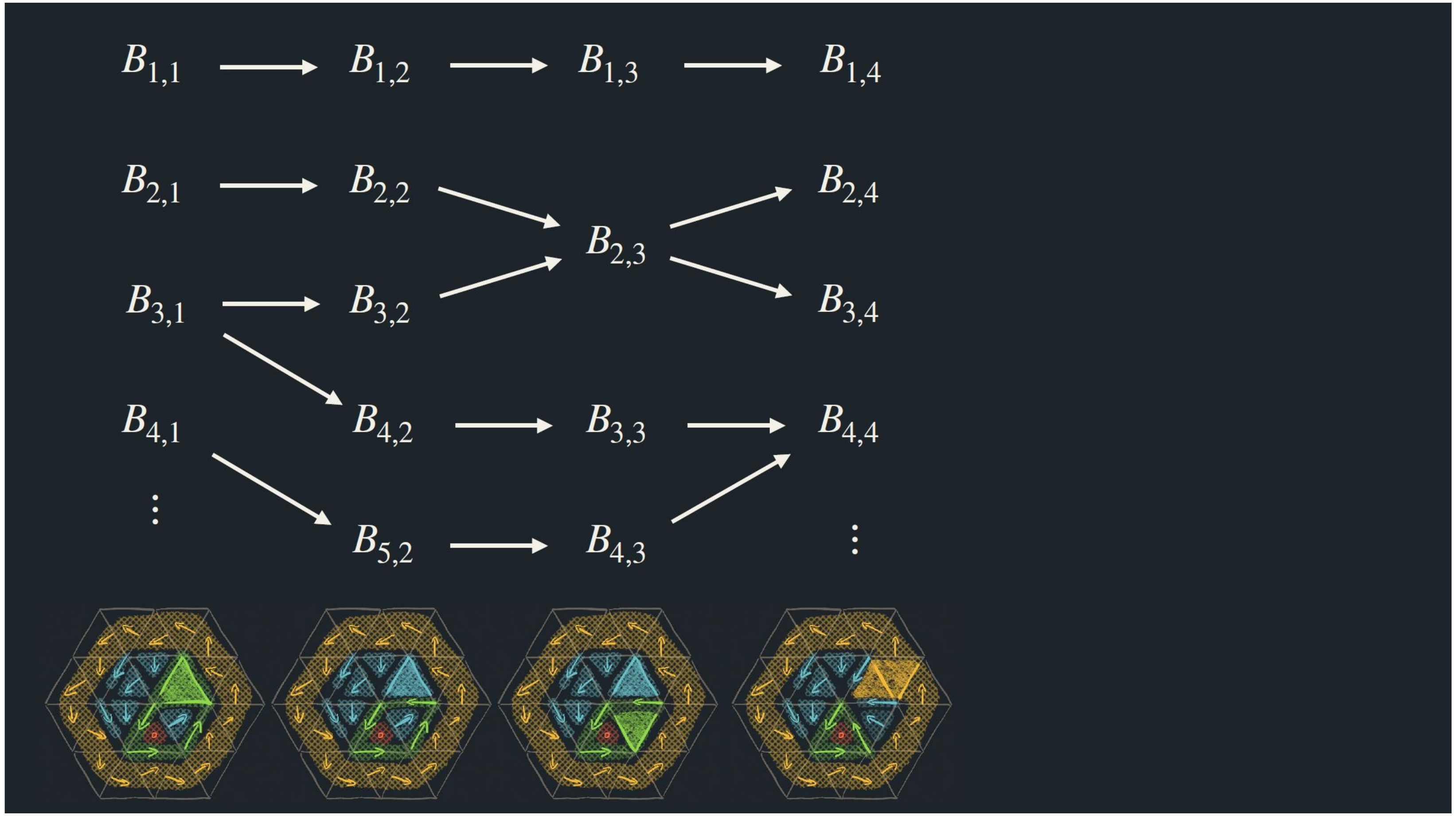

$$\begin{tikzcd}
B_{1,1} \arrow[r] & B_{1,2} \arrow[r] & B_{1,3} \arrow[r] & B_{1,4} \\
B_{2,1} \arrow[r] & B_{2,2} \arrow[dr] & & B_{2,4} \\
& & B_{2,3} \arrow[ur] \arrow[dr] & \\
B_{3,1} \arrow[r] \arrow[dr] & B_{3,2} \arrow[ur] & & B_{3,4} \\
B_{4,1} \arrow[dr] & B_{4,2} \arrow[r] & B_{3,3} \arrow[r] & B_{4,4} \\
\vdots & B_{5,2} \arrow[r] & B_{4,3} \arrow[ur] & \vdots
\end{tikzcd}$$

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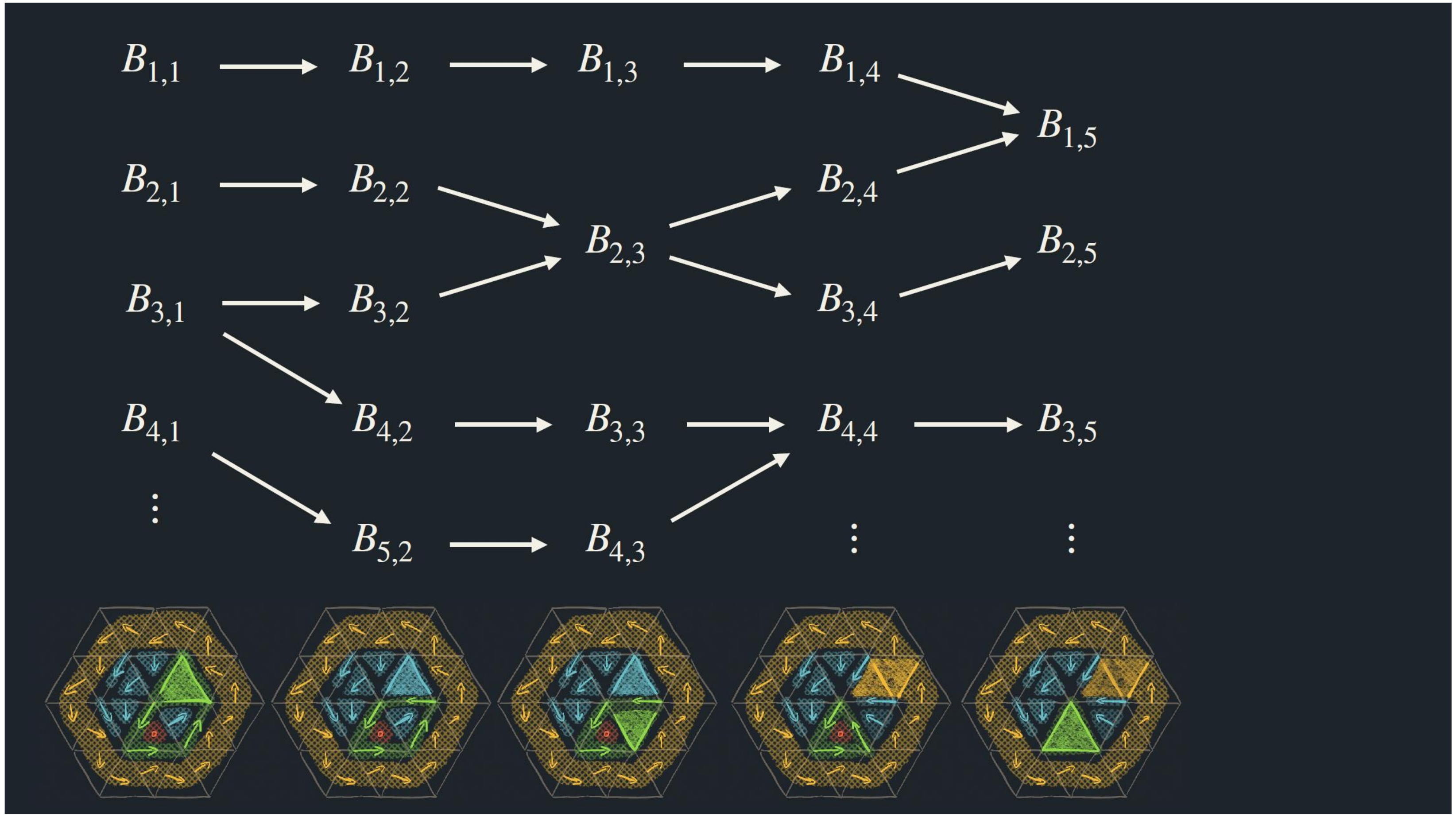

$$\begin{tikzcd}
B_{1,1} \arrow[r] & B_{1,2} \arrow[r] & B_{1,3} \arrow[r] & B_{1,4} \arrow[dr] & \\
 & & & & B_{1,5} \\
B_{2,1} \arrow[r] & B_{2,2} \arrow[dr] & & B_{2,4} \arrow[ur] & \\
 & & B_{2,3} \arrow[ur] \arrow[dr] & & B_{2,5} \\
B_{3,1} \arrow[r] \arrow[dr] & B_{3,2} \arrow[ur] & & B_{3,4} \arrow[ur] & \\
B_{4,1} \arrow[dr] & B_{4,2} \arrow[r] & B_{3,3} \arrow[r] & B_{4,4} \arrow[r] & B_{3,5} \\
\vdots & B_{5,2} \arrow[r] & B_{4,3} \arrow[ur] & \vdots & \vdots
\end{tikzcd}$$

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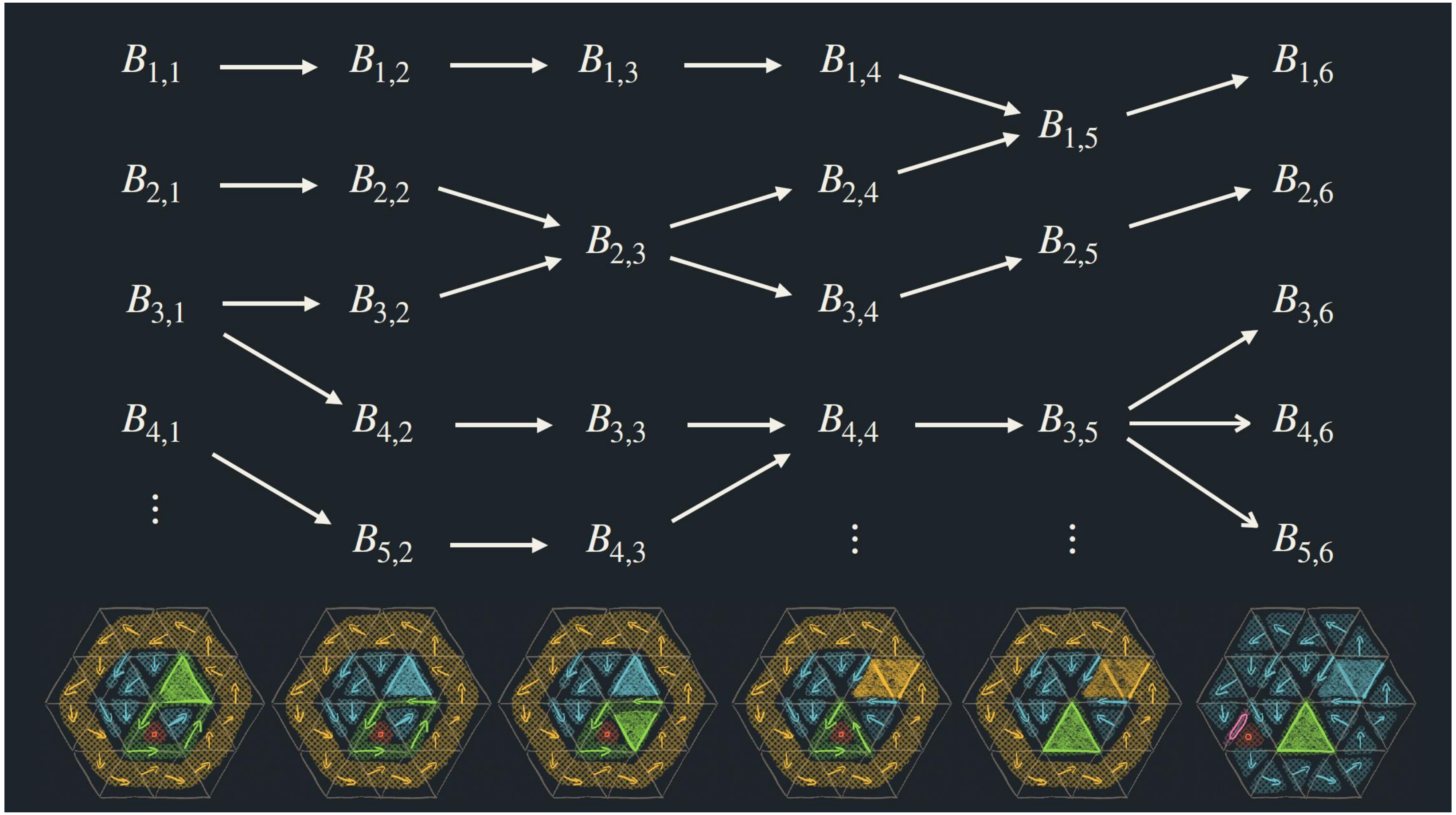
Conely-Morse barcode



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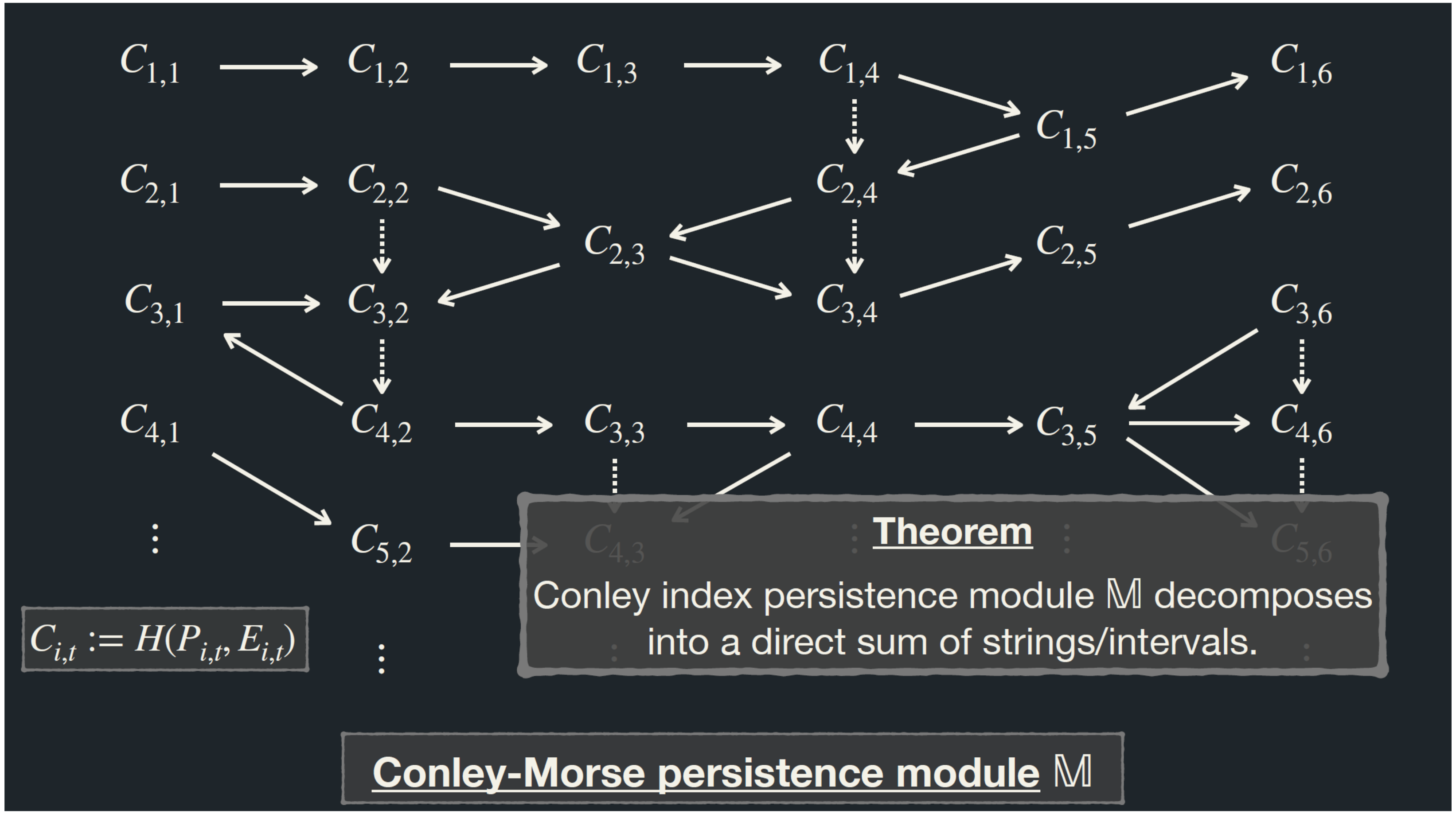
Conely-Morse barcode



$$\begin{tikzcd}
C_{1,1} \arrow[r] & C_{1,2} \arrow[r] & C_{1,3} \arrow[r] & C_{1,4} \arrow[dr] \arrow[dd, dashed] & & C_{1,6} \\
& & & & C_{1,5} \arrow[ur] \arrow[dl] & \\
C_{2,1} \arrow[r] & C_{2,2} \arrow[dr] \arrow[dd, dashed] & & C_{2,4} \arrow[dl] \arrow[dd, dashed] & & C_{2,6} \\
& & C_{2,3} \arrow[dl] \arrow[dr] & & C_{2,5} \arrow[ur] & \\
C_{3,1} \arrow[r] & C_{3,2} \arrow[d, dashed] & & C_{3,4} \arrow[ur] & & C_{3,6} \arrow[dl] \arrow[d, dashed] \\
C_{4,1} \arrow[dr] & C_{4,2} \arrow[ul] \arrow[r] & C_{3,3} \arrow[r] \arrow[d, dashed] & C_{4,4} \arrow[r] \arrow[dl] & C_{3,5} \arrow[r] \arrow[dr] & C_{4,6} \arrow[d, dashed] \\
\vdots & C_{5,2} \arrow[r] & C_{4,3} & \vdots & \vdots & C_{5,6} \\
& \vdots & \vdots & & & \vdots
\end{tikzcd}$$

$C_{i,t} := H(P_{i,t}, E_{i,t})$

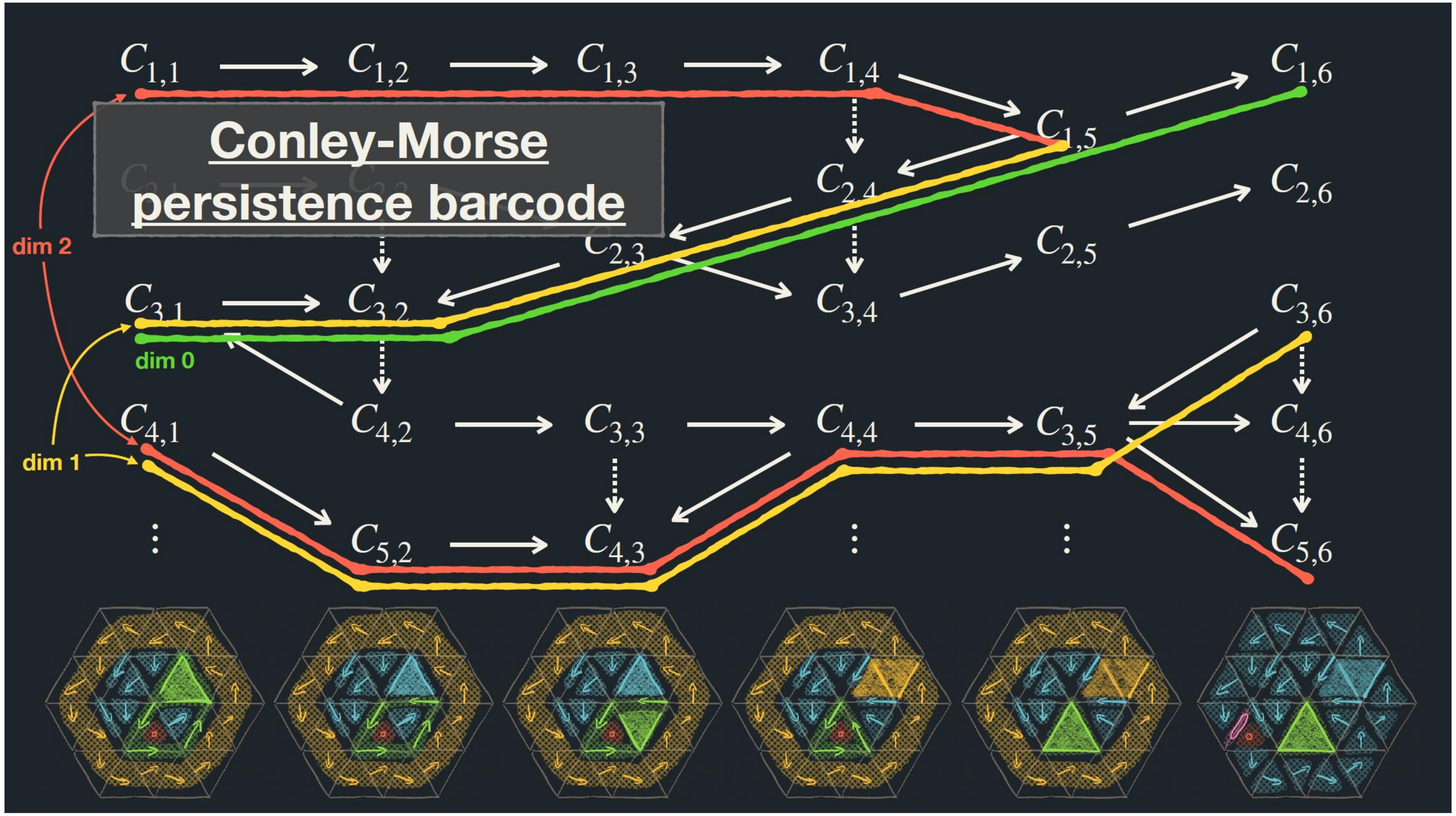
Theorem

Conley index persistence module $\mathbb{M}$ decomposes into a direct sum of strings/intervals.

Conley-Morse persistence module $\mathbb{M}$

T. K. Dey, M. Lipinsky, M. Soriano-Trigueros. Conely-Morse persistence barcode: A homological signature of a combinatorial bifurcation.Foundations of Computational Mathematics (FoCM), to appear; arXiv.org/abs/2504.17105 (2026).

Conely-Morse barcode



T. K. Dey, M. Lipinsky, M. Soriano-Trigueros. Conely-Morse persistence barcode: A homological signature of a combinatorial bifurcation.Foundations of Computational Mathematics (FoCM), to appear; arXiv.org/abs/2504.17105 (2026).

New algorithm for computing Conley-Morse barcode

1. Each block decomposition is presented as a matrix with a notion of homogeneity introduced earlier in the context of connection matrix computation [DHL25]: $\boldsymbol{M_1} \to \boldsymbol{M_2} \to \cdots \to \boldsymbol{M_k}$

2. Compute $\boldsymbol{M_{t+1}}$ from $\boldsymbol{M_t}$ using certain atomic operations while tracking the bars:
   requires reinterpretation of the algebra in transition diagram [DHL26]

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Concluding remarks

- Open questions
    - How to compute presentations (minimal or not) efficiently from a filtration?

      Efficient algorithms known for 2-parameter only [LW22]

    - How to compute a decomposition efficiently?

      Efficient algorithm known for distinctly graded modules [DX22][DJK25]

    - How to compute more discriminating invariants efficiently? [DKM24][MS25]
    - How to compute other invariants from multivector fields?
    - Applications?

- Software
    - RIVET
    - MPFREE
    - ?

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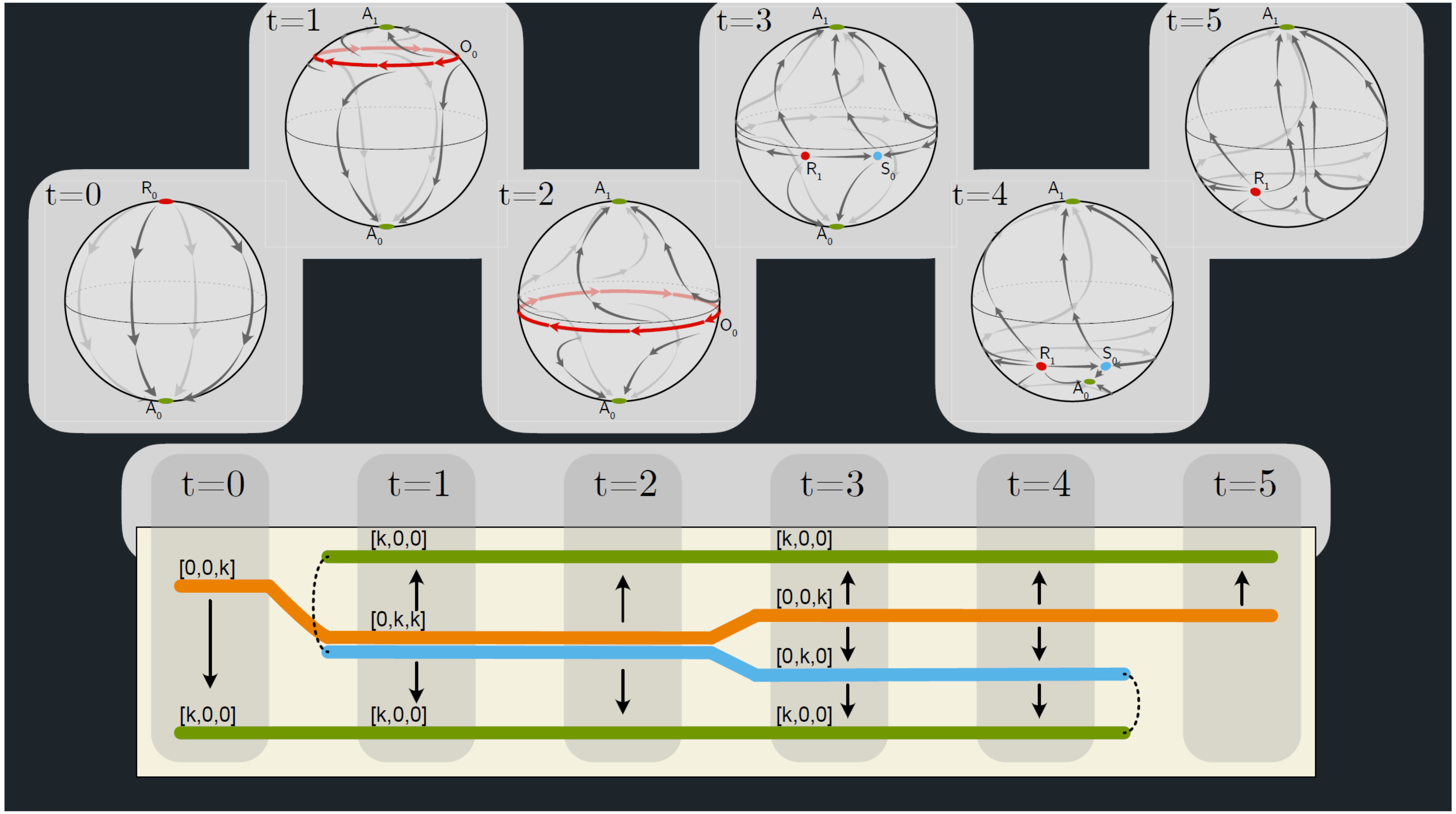
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Thank you!