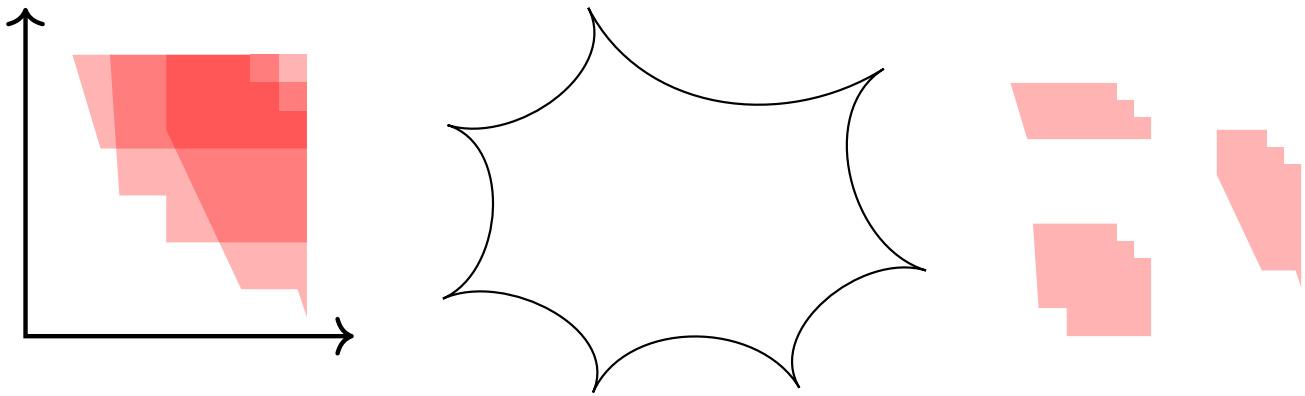
UNIVERSITY AT ALBANY
State University of New York

Computing approximate decompositions of multiparameter modules

Håvard Bakke Bjerkevik, Jan Jendrysiak and Fabian Lenzen

University at Albany, SUNY

May 19, 2026



Joining forces

- Stabilizing Decomposition of Multiparameter Persistence Modules (B.)

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- Current project (B., Jan Jendrysiak, Fabian Lenzen)

The persistence barcode

In one-parameter persistence, we can decompose modules algebraically to get a barcode/persistence diagram.

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We would like a multiparameter barcode!

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Theorem (Bauer, Scoccola (2022))

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Do these problems show up in practice?

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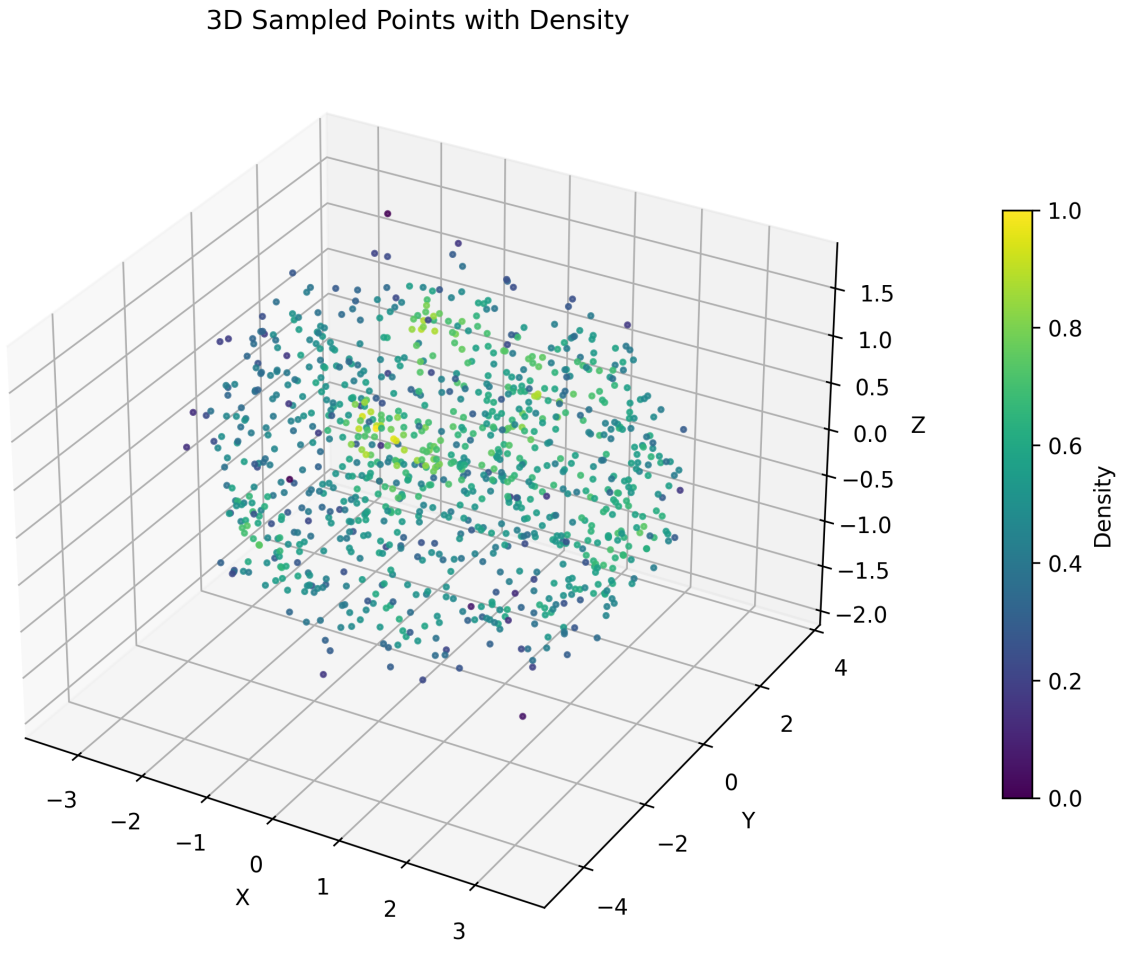

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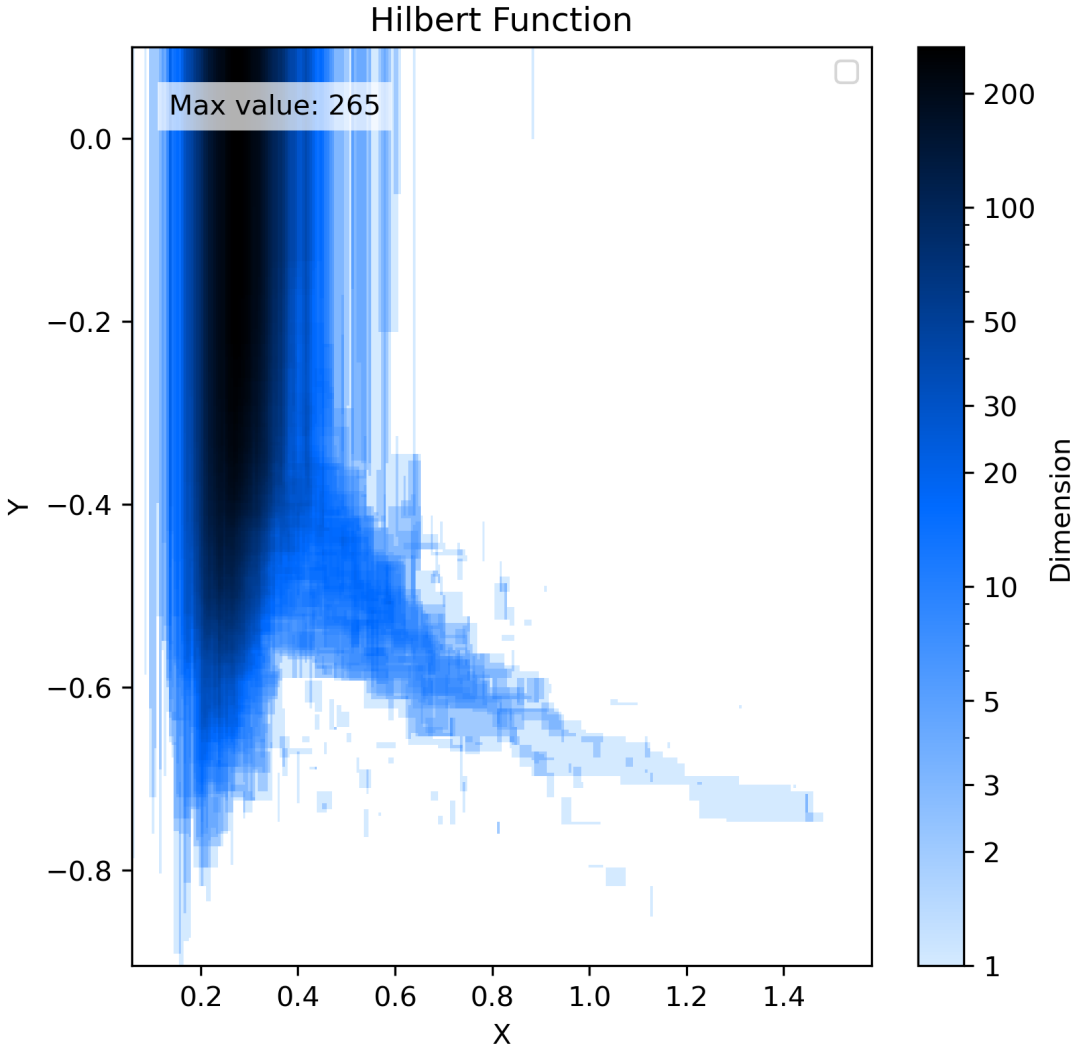

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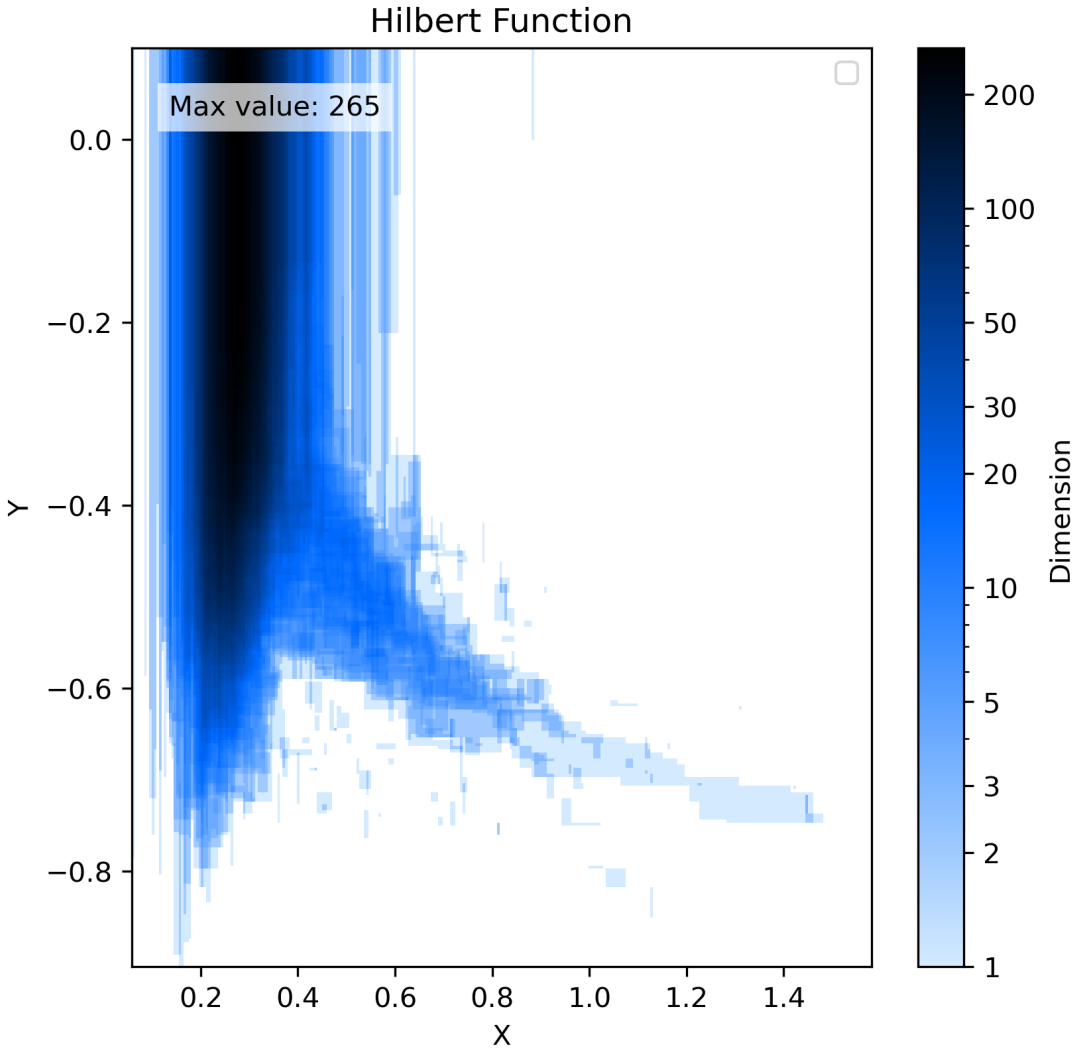
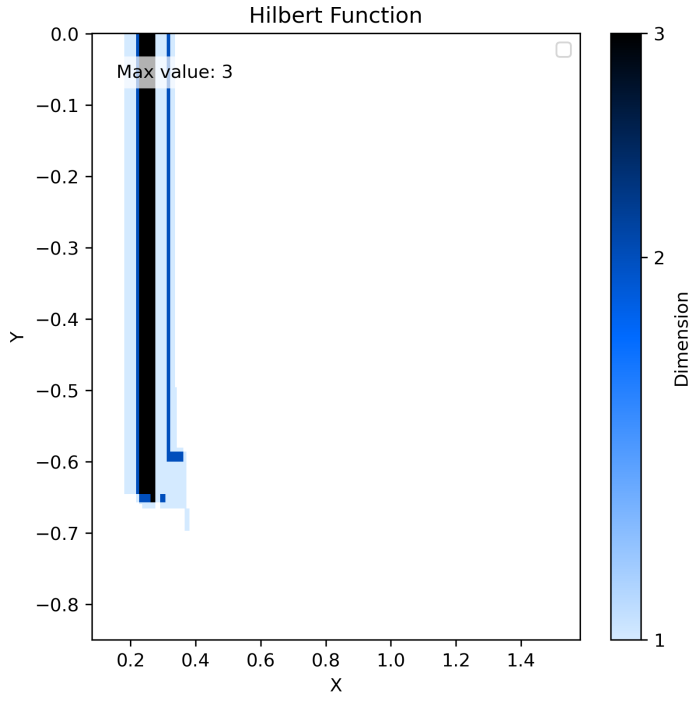
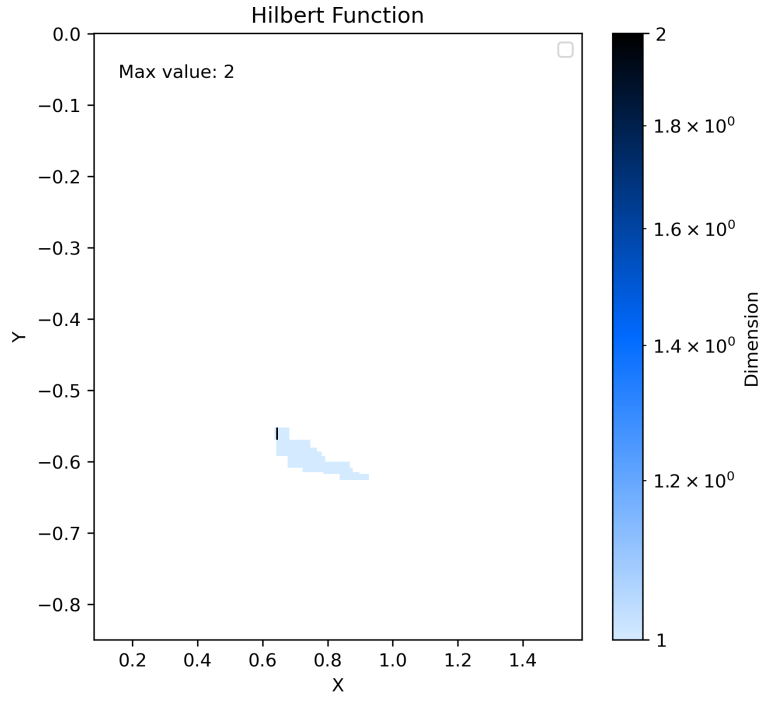
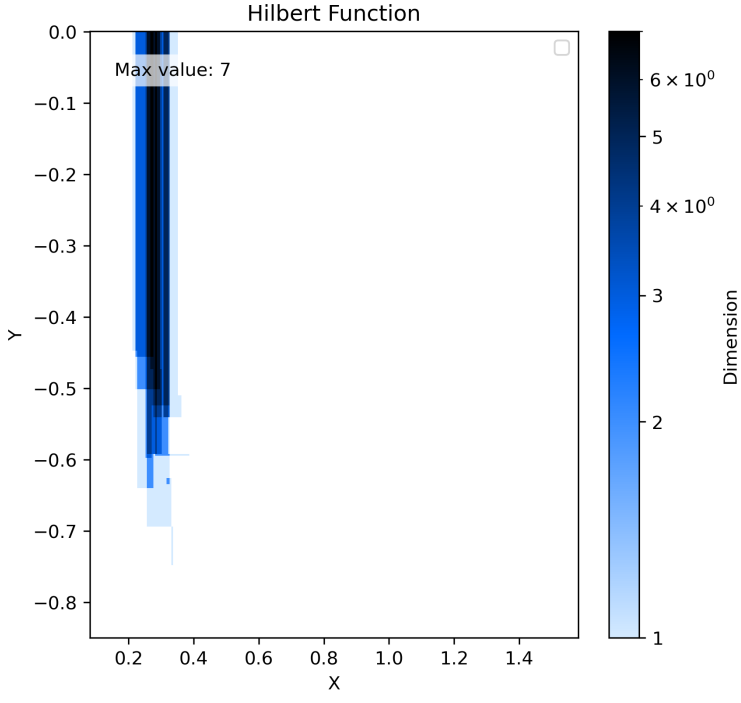
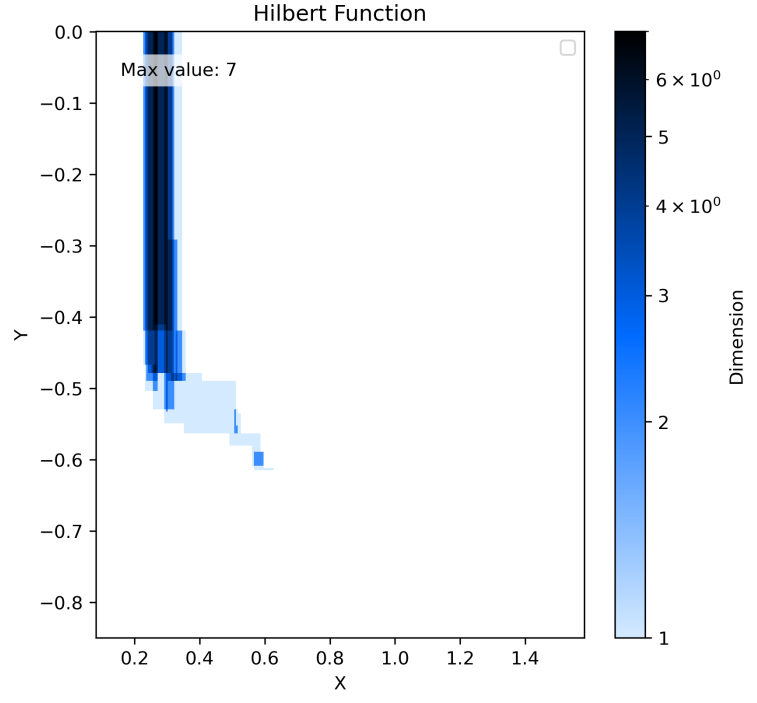


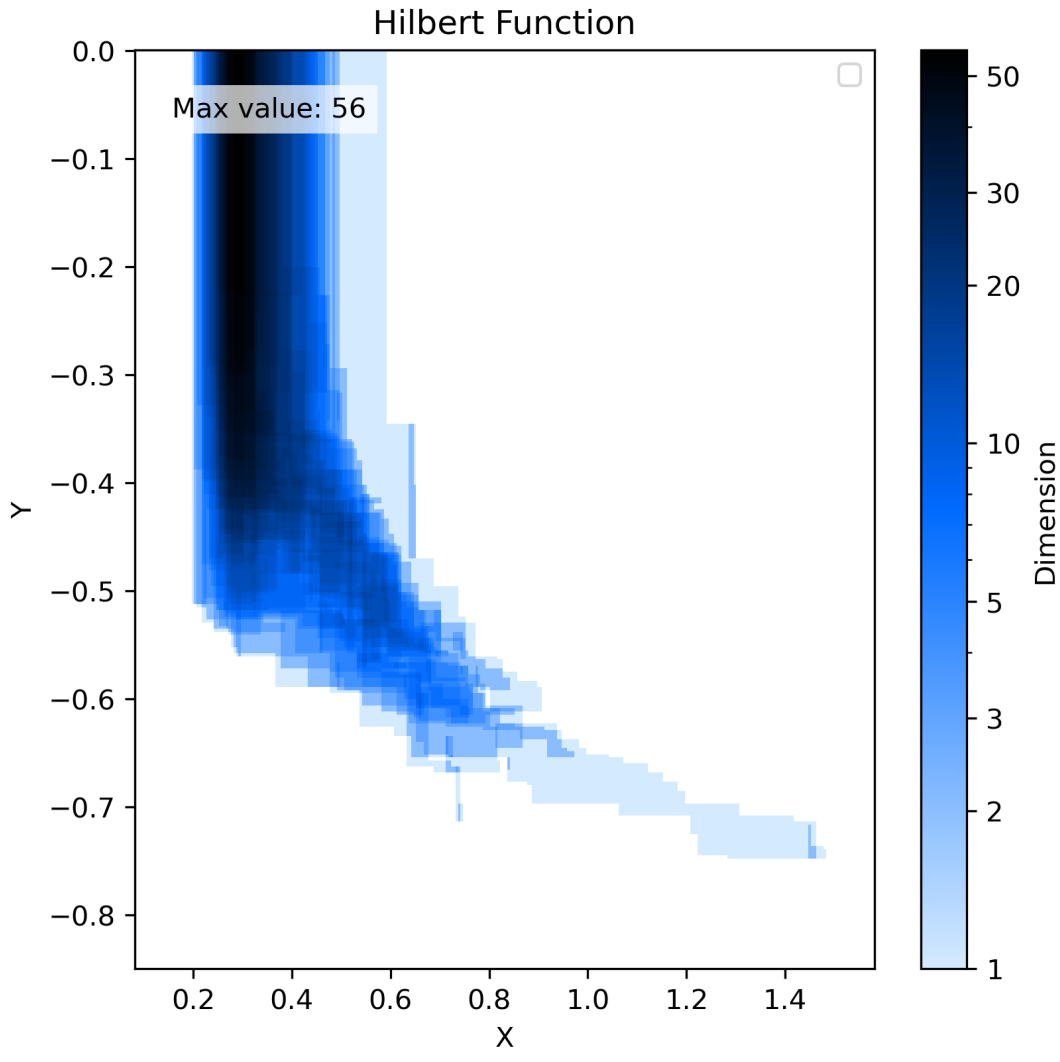
2742 components.

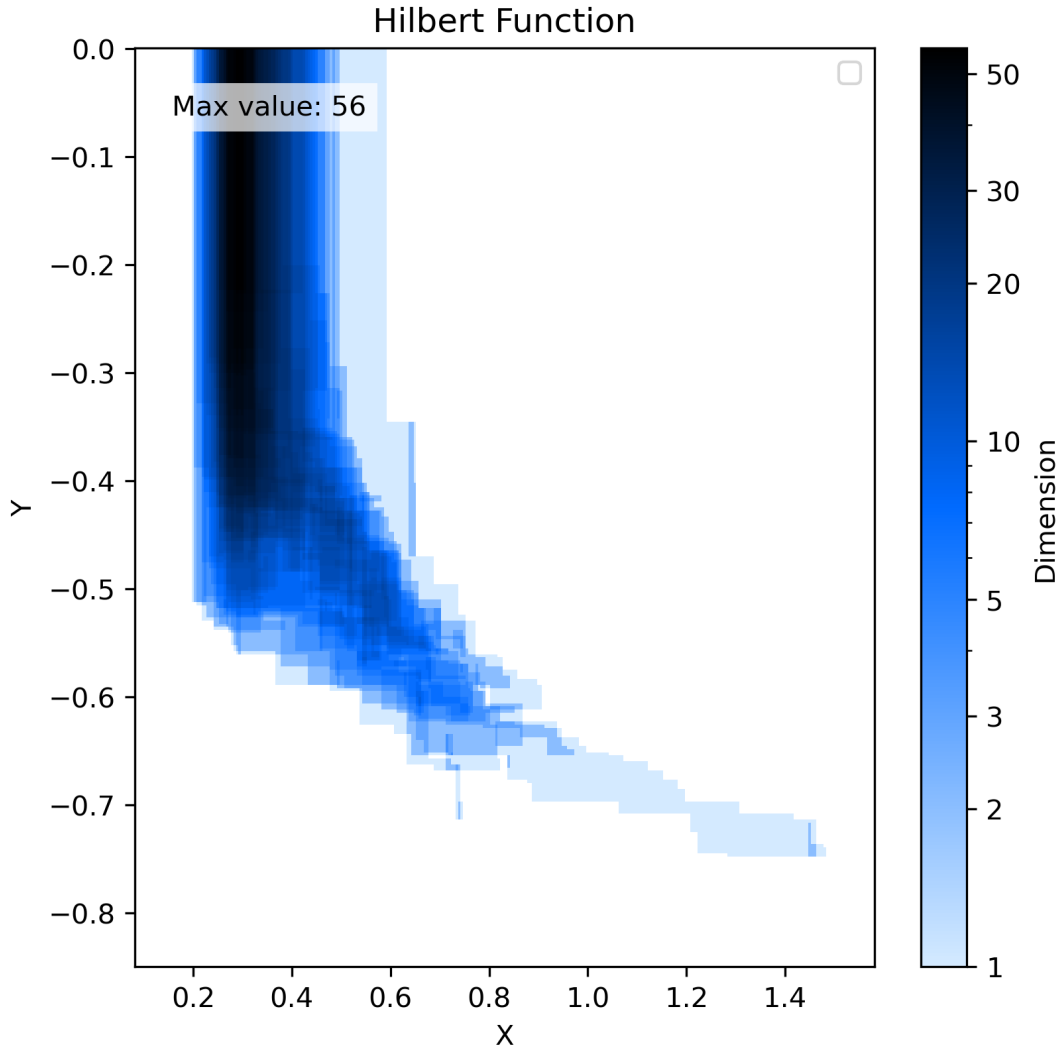

Hilbert Function
Max value: 3
Dimension
X
Y


Hilbert Function
Max value: 2
Dimension
X
Y


Hilbert Function
Max value: 7
Dimension
X
Y


Hilbert Function
Max value: 7
Dimension
X
Y


Hilbert Function
Max value: 56
0.0
−0.1
−0.2
−0.3
−0.4
−0.5
−0.6
−0.7
−0.8
Y
0.2
0.4
0.6
0.8
1.0
1.2
1.4
X
50
30
20
10
5
3
2
1
Dimension



There is a “giant component”.

Pruning

Idea: Even if a module is indecomposable, it might be *almost* decomposable.

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Definition (B. (2024))

Let M be a module. Let I be the largest submodule of M such that for any morphism $f\colon M \to M(2\epsilon)$, $f(I) \subseteq M_{0\to 2\epsilon}(I)$. Let K be the smallest submodule of I such that for any morphism $f\colon M \to M(2\epsilon)$, $I^{-1}_{0\to 2\epsilon}(f(K)) \subseteq K$. We define the ϵ**-pruning pair** of M to be (I, K), and the ϵ**-pruning** of M to be $\mathrm{Pru}_\epsilon(M) := (I/K)(-\epsilon)$.

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Theorem (B. (2024))

Let $\epsilon \geq 0$, and let M and N be ϵ-interleaved pfd modules, and let $r = \operatorname{supdim} M < \infty$. Then $\operatorname{Pru}_\epsilon(M)$ is a $2r\epsilon$-refinement of N.

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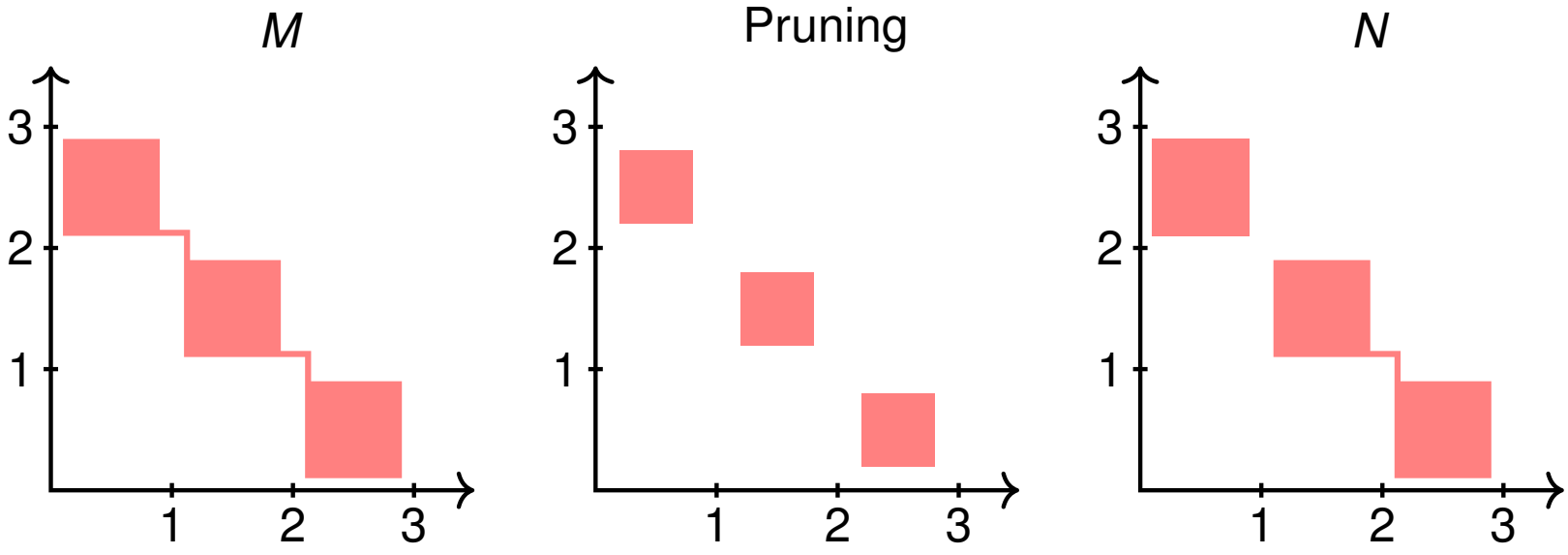

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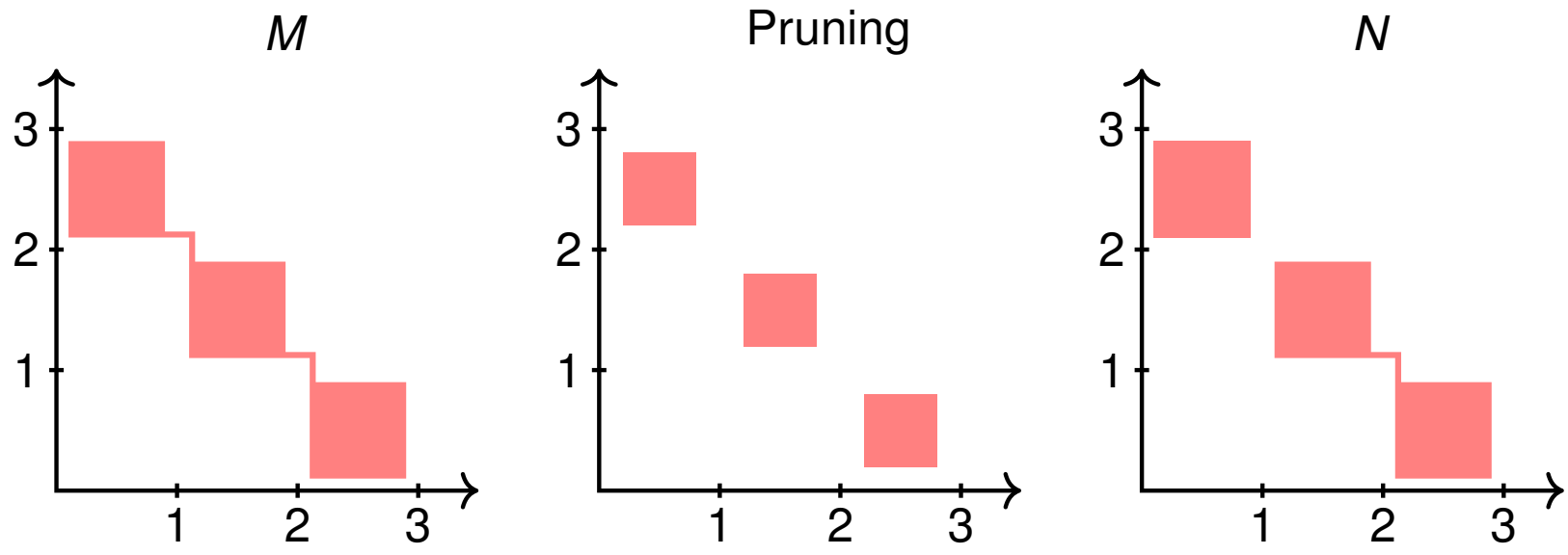

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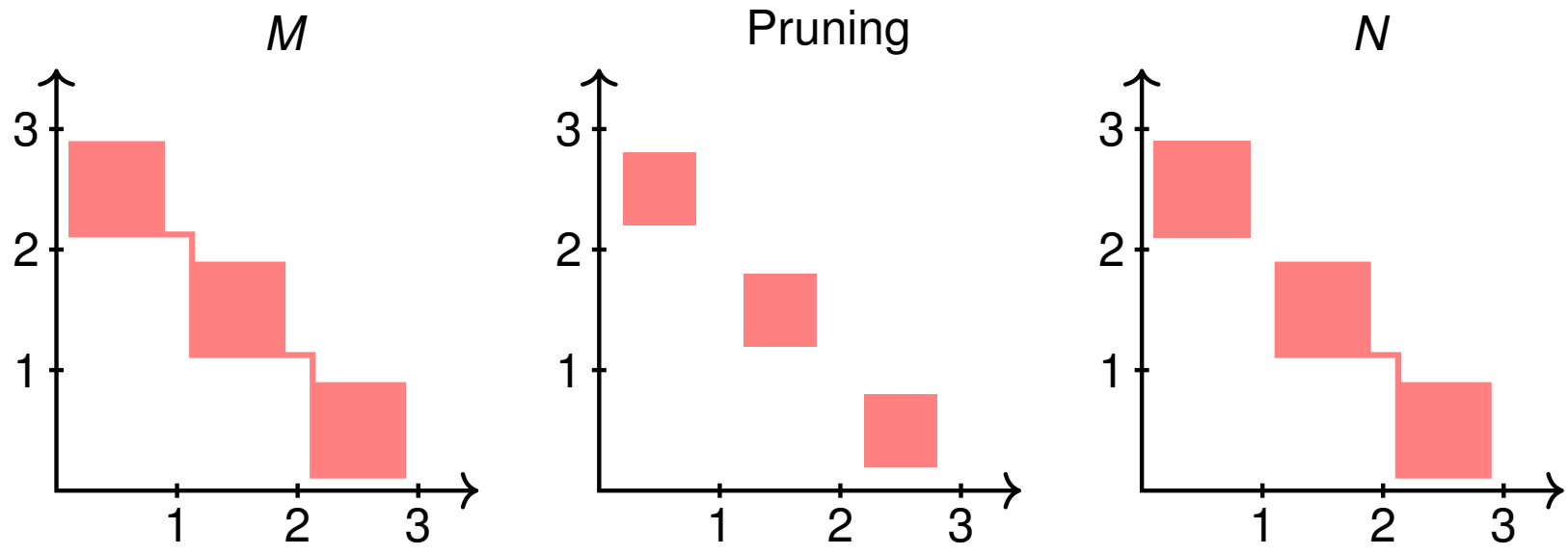

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Drawback: The pruning shrinks M. In the worst case, it shrinks it by a lot.

Let's compute the pruning! But how?

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A more practical description

Lemma

Let (I, K) be the ϵ-pruning pair of M. Let $\Gamma = \mathrm{Hom}(M, M(2\epsilon))$. Then $I = \bigcap_{i=0}^{\infty} I_i$, where we define $I_0 = M$, and for all $i \geq 1$,

- $I_i = \bigcap_{f \in \Gamma} f^{-1}(M_{0 \to 2\epsilon}(I_{i-1}))$.

$K = \bigcup_{i=0}^{\infty} K_i$, where we define $K_0 = 0$, and for all $i \geq 1$,

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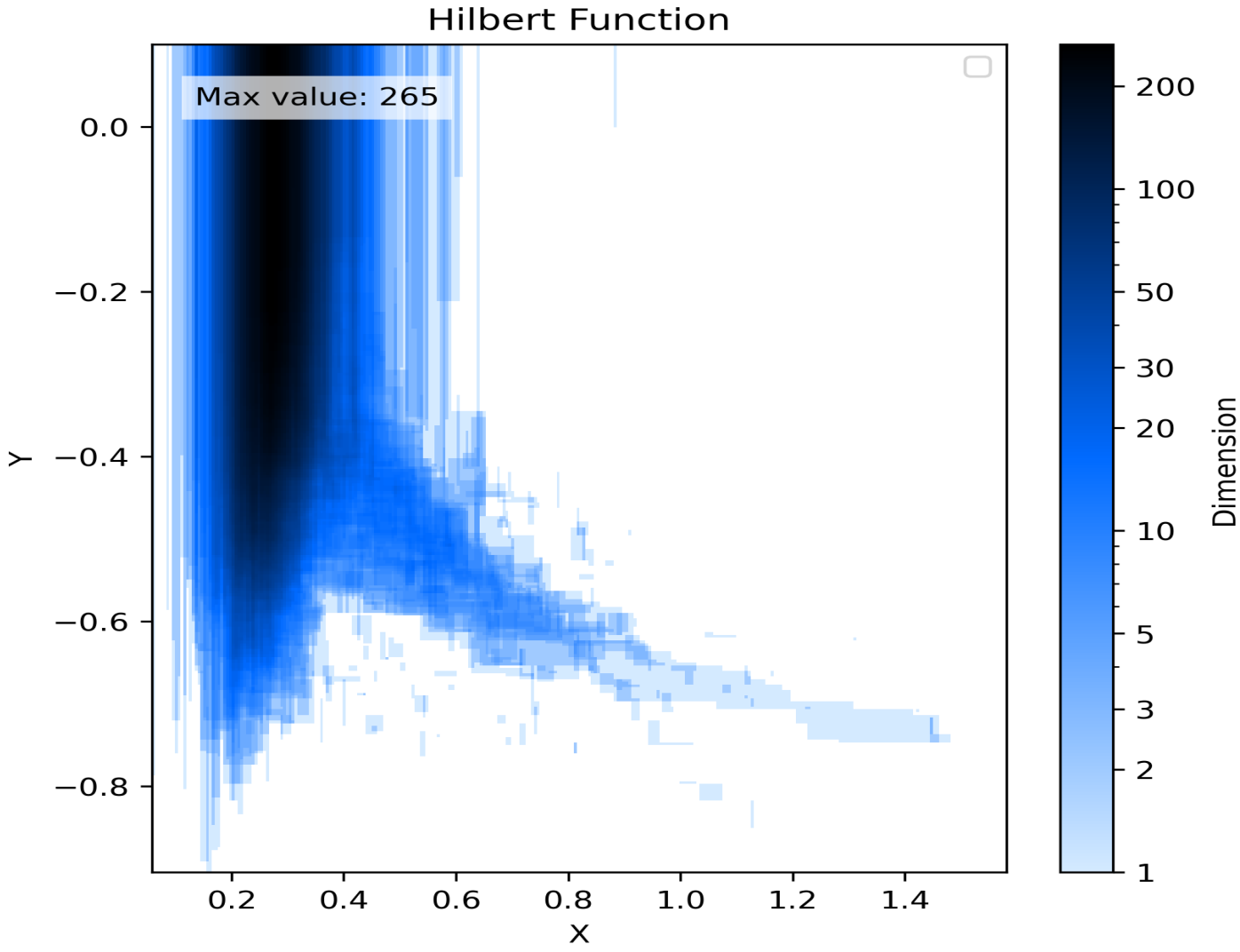
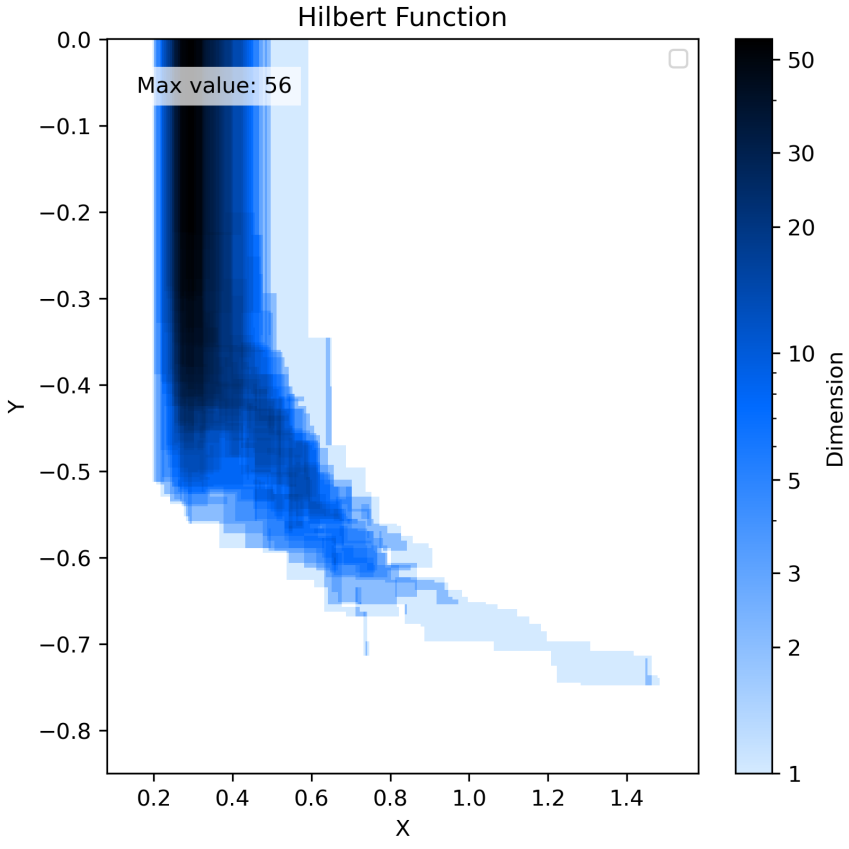
Plus further simplifications. Some implemented, some not.


Hilbert Function
Max value: 265
0.0
−0.2
−0.4
−0.6
−0.8
Y
0.2
0.4
0.6
0.8
1.0
1.2
1.4
X
200
100
50
30
20
10
5
3
2
1
Dimension

Results

Its largest component and the 0.02-pruning of the largest component:



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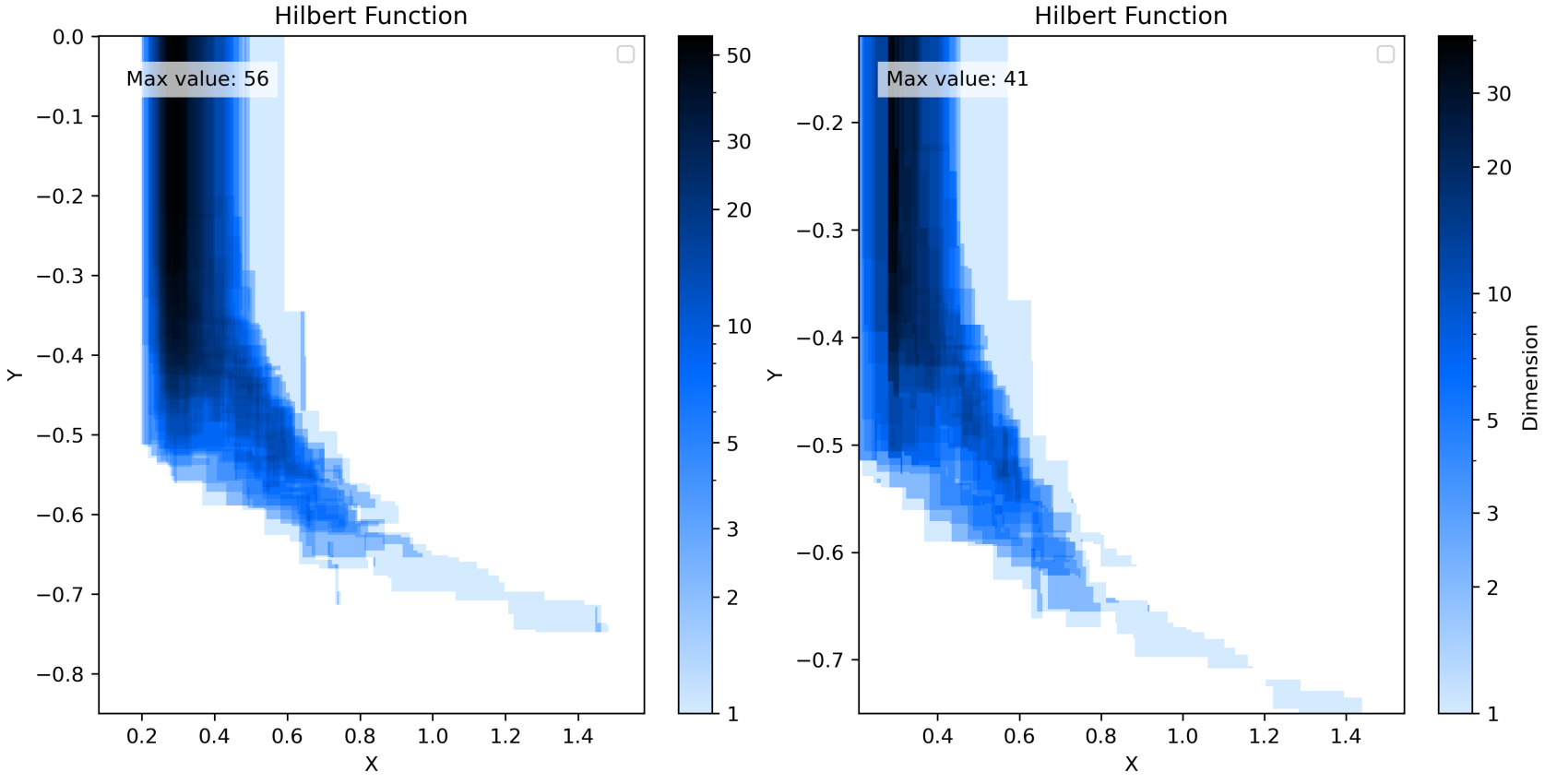

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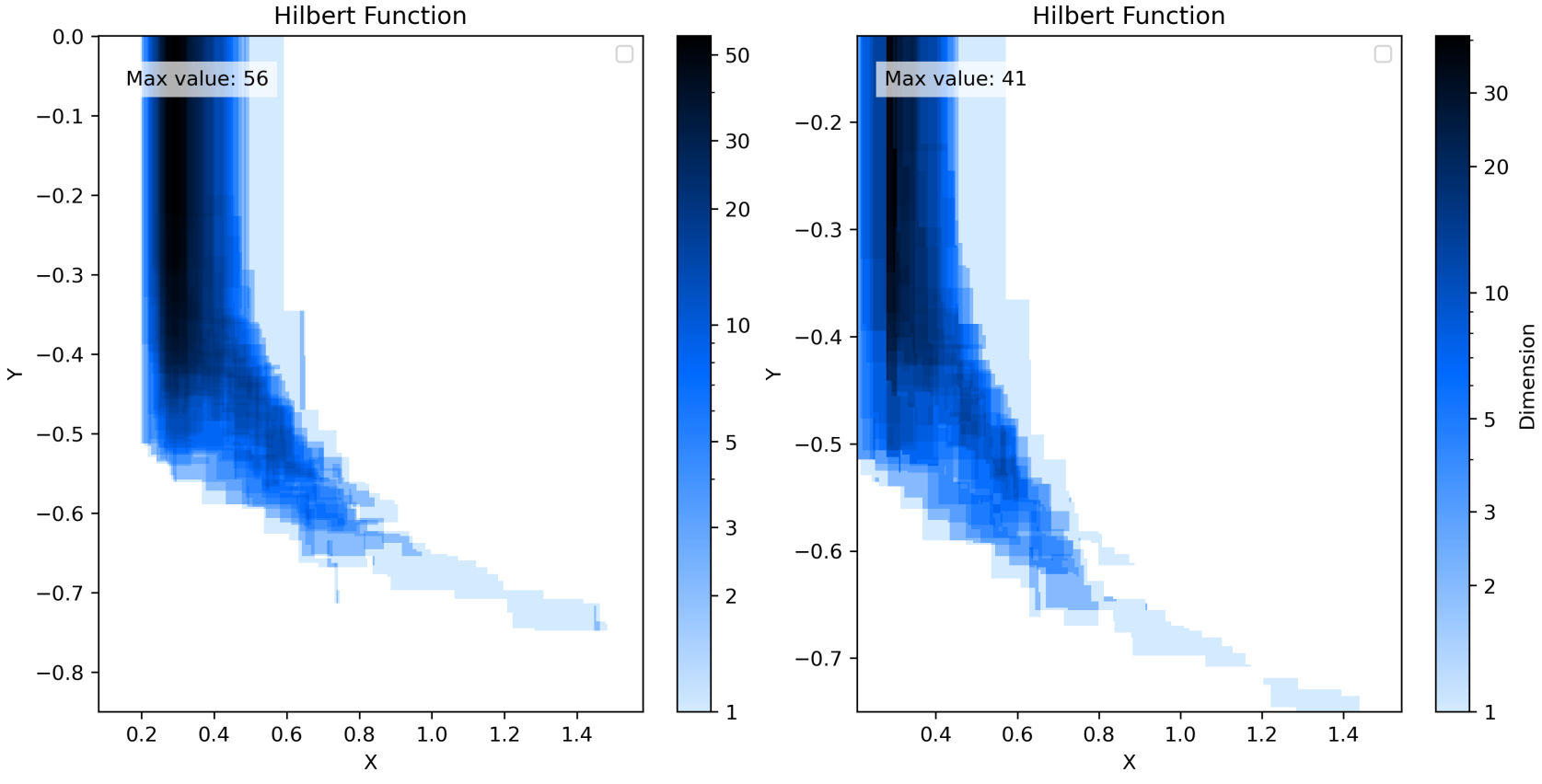
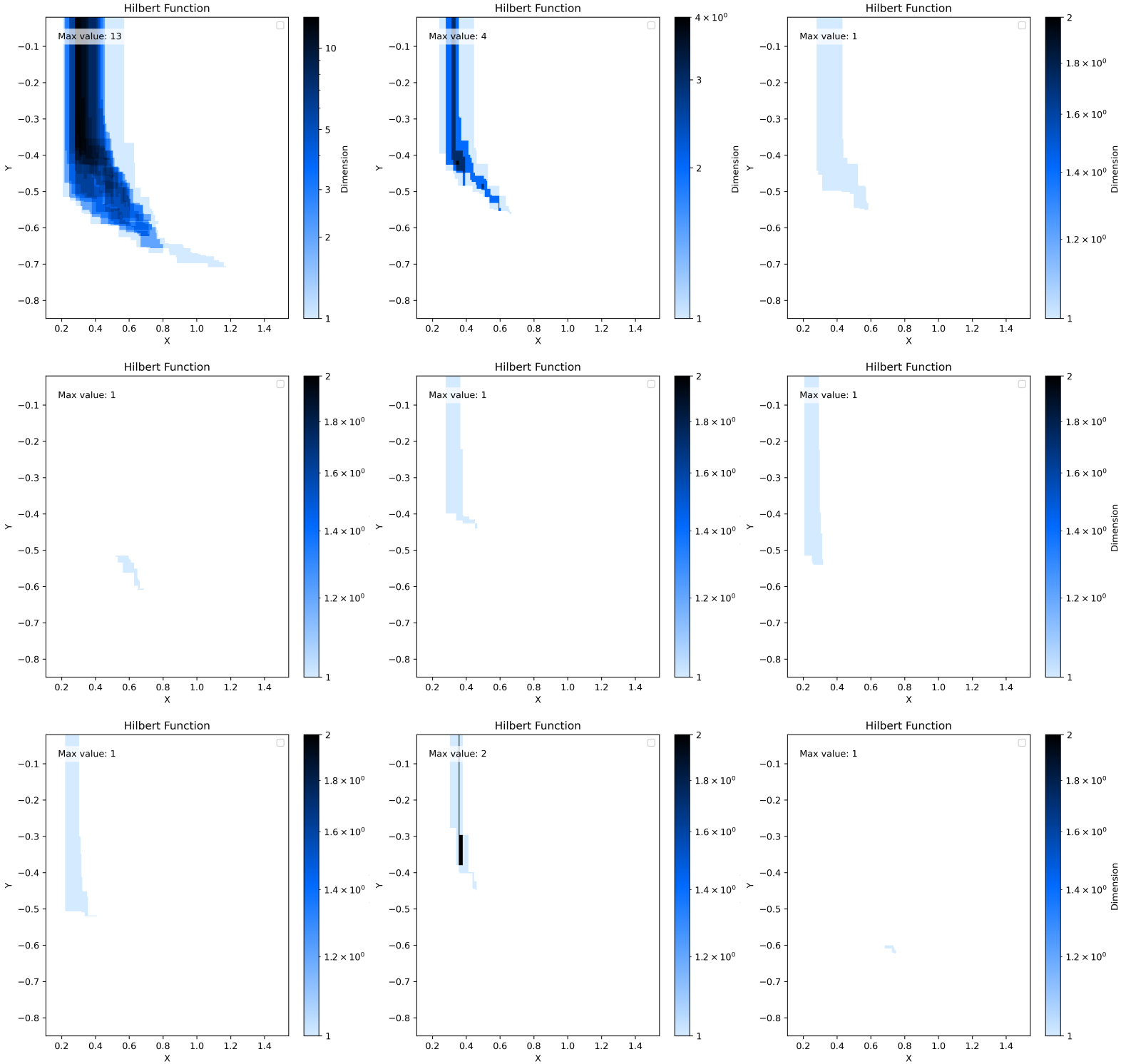
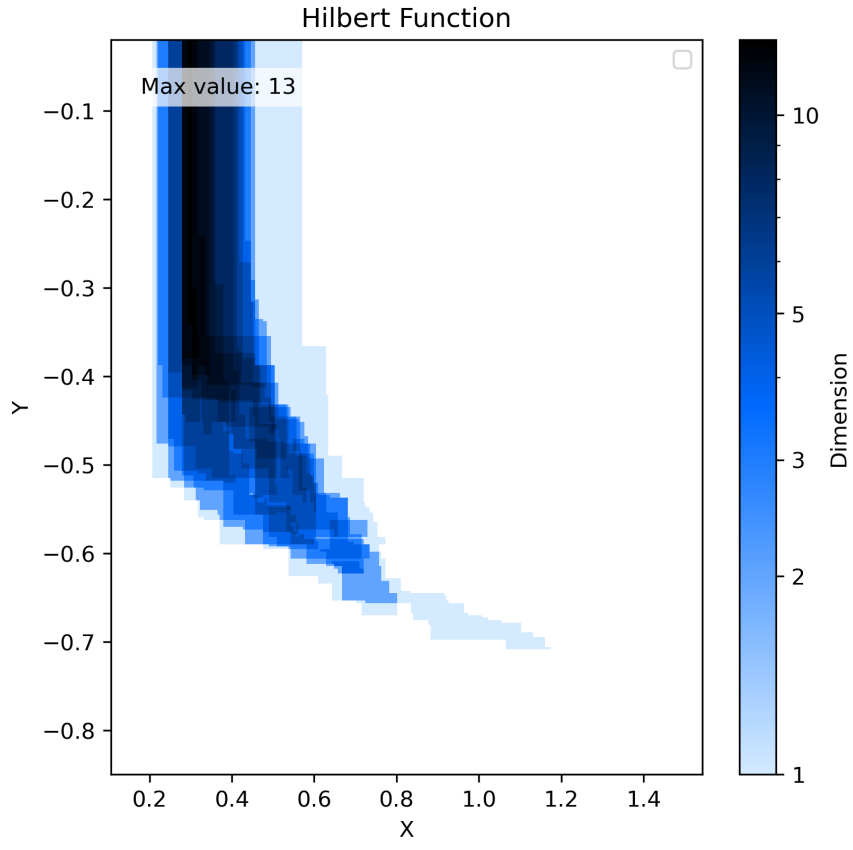


86 components


Hilbert Function
Max value: 13
Hilbert Function
Max value: 4
Hilbert Function
Max value: 1
Hilbert Function
Max value: 1
Hilbert Function
Max value: 1
Hilbert Function
Max value: 1
Hilbert Function
Max value: 1
Hilbert Function
Max value: 2
Hilbert Function
Max value: 1
X
Y
Dimension

The largest component of the 0.02-pruning and its 0.03-pruning:



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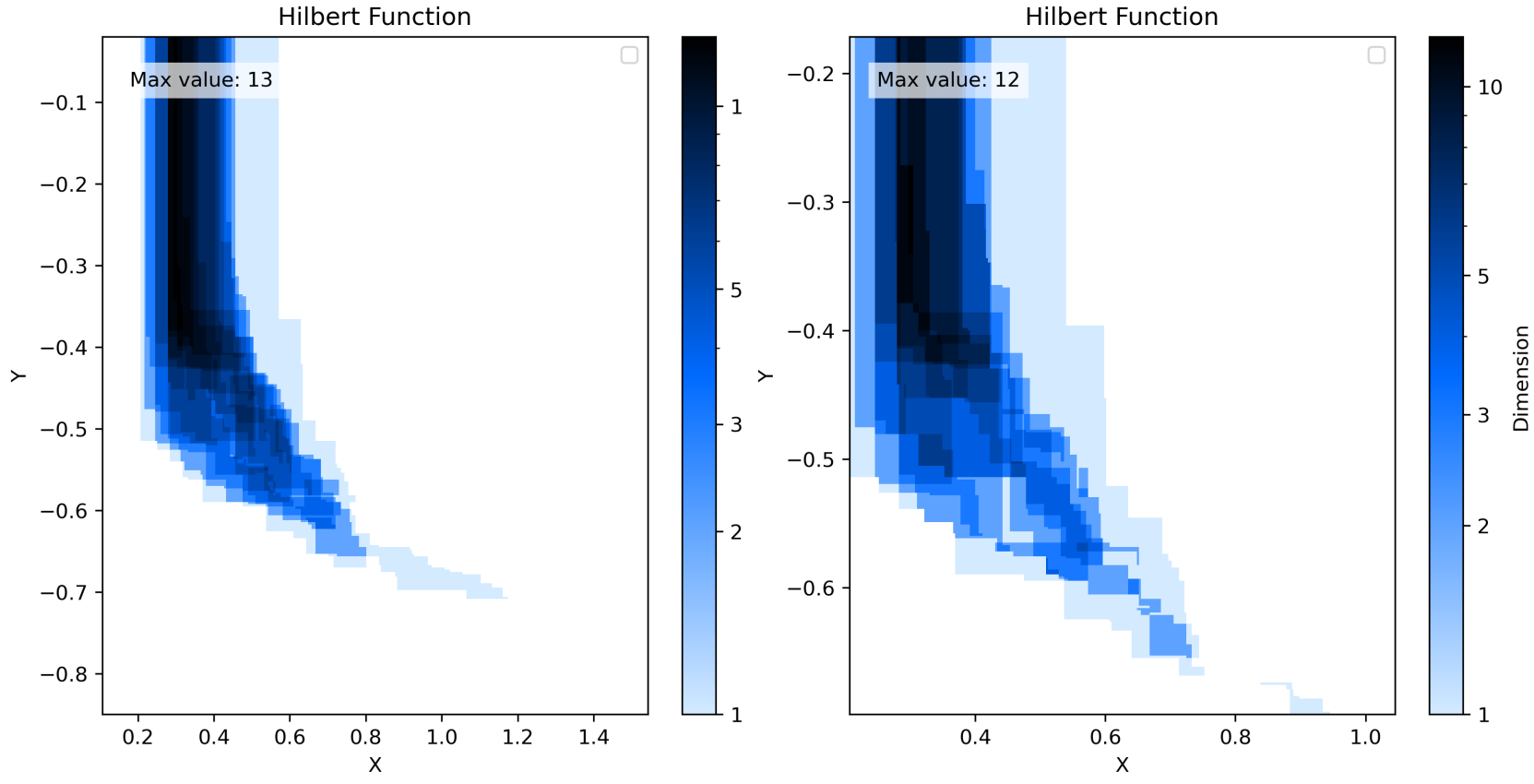

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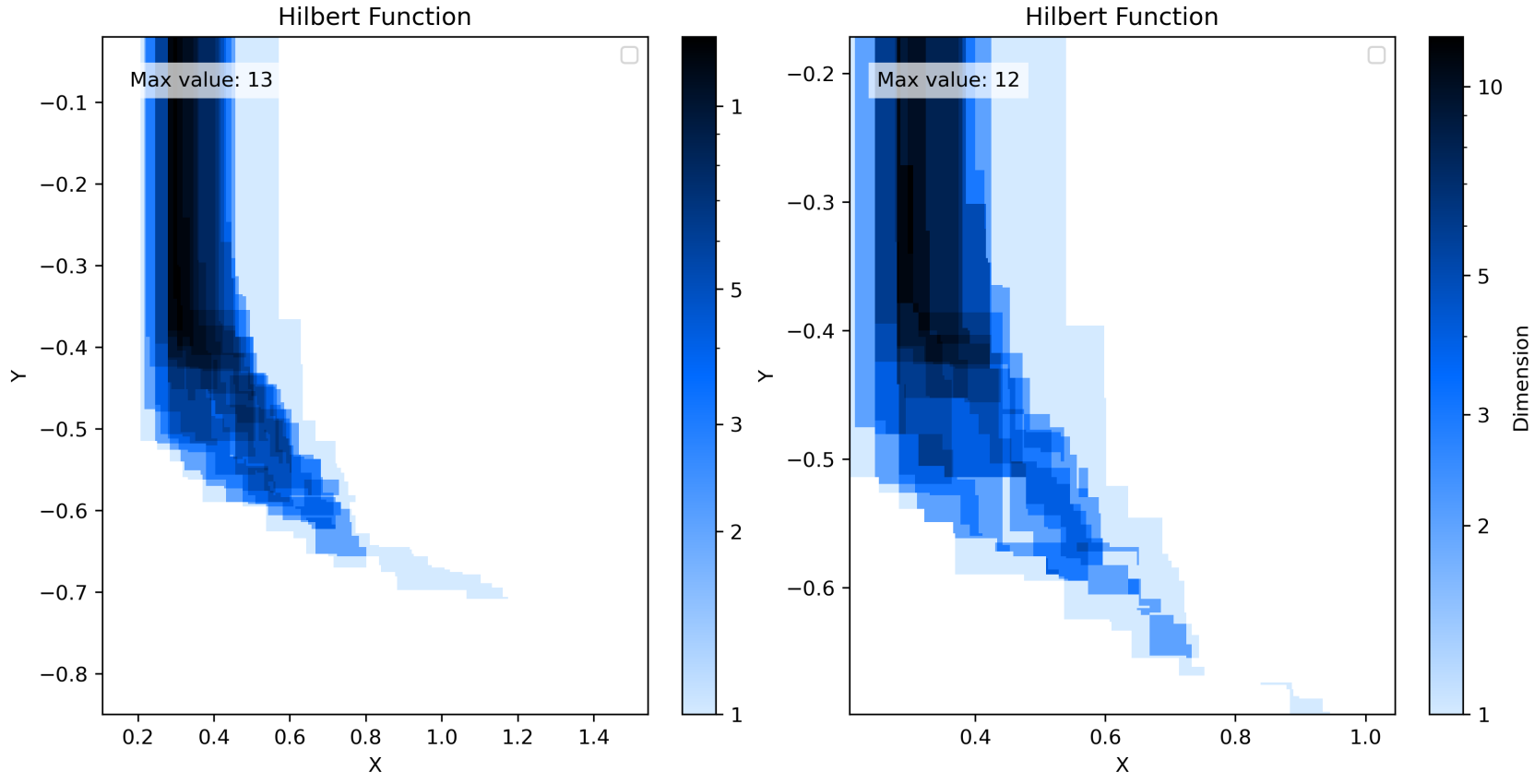
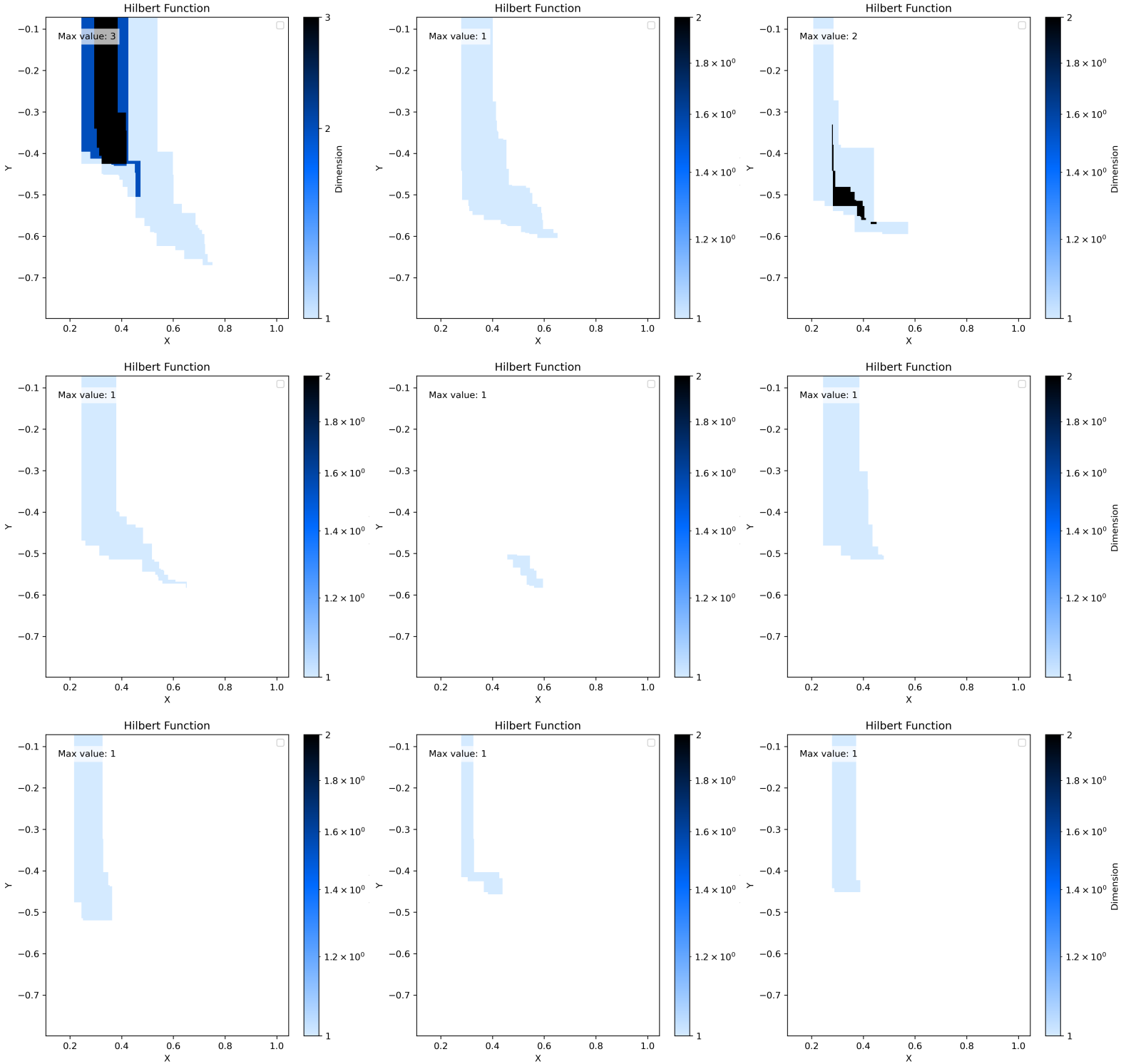
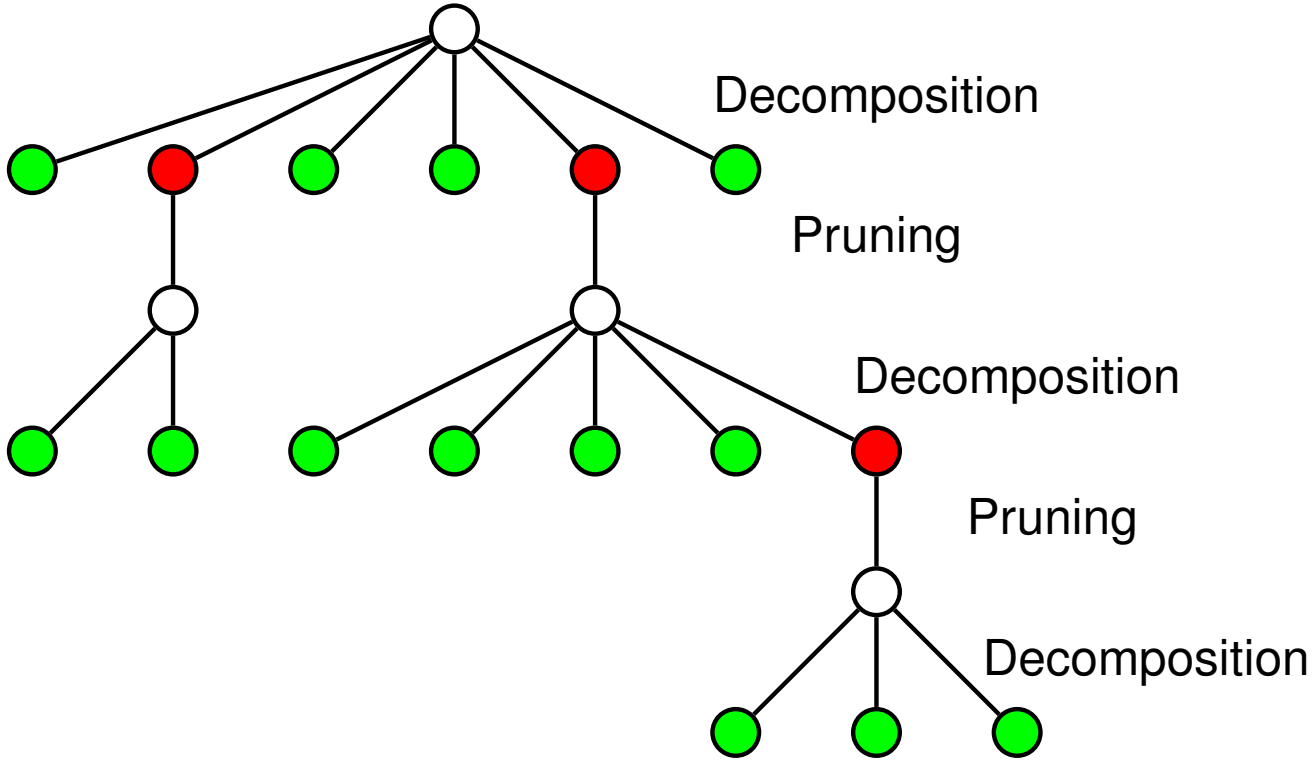


20 components


Hilbert Function
Max value: 3
Hilbert Function
Max value: 1
Hilbert Function
Max value: 2
Hilbert Function
Max value: 1
Hilbert Function
Max value: 1
Hilbert Function
Max value: 1
Hilbert Function
Max value: 1
Hilbert Function
Max value: 1
Hilbert Function
Max value: 1
X
Y
Dimension

Tree of decompositions? (Barkcode?)

One could do one big pruning or a sequence of smaller prunings to prune everything into simple pieces.



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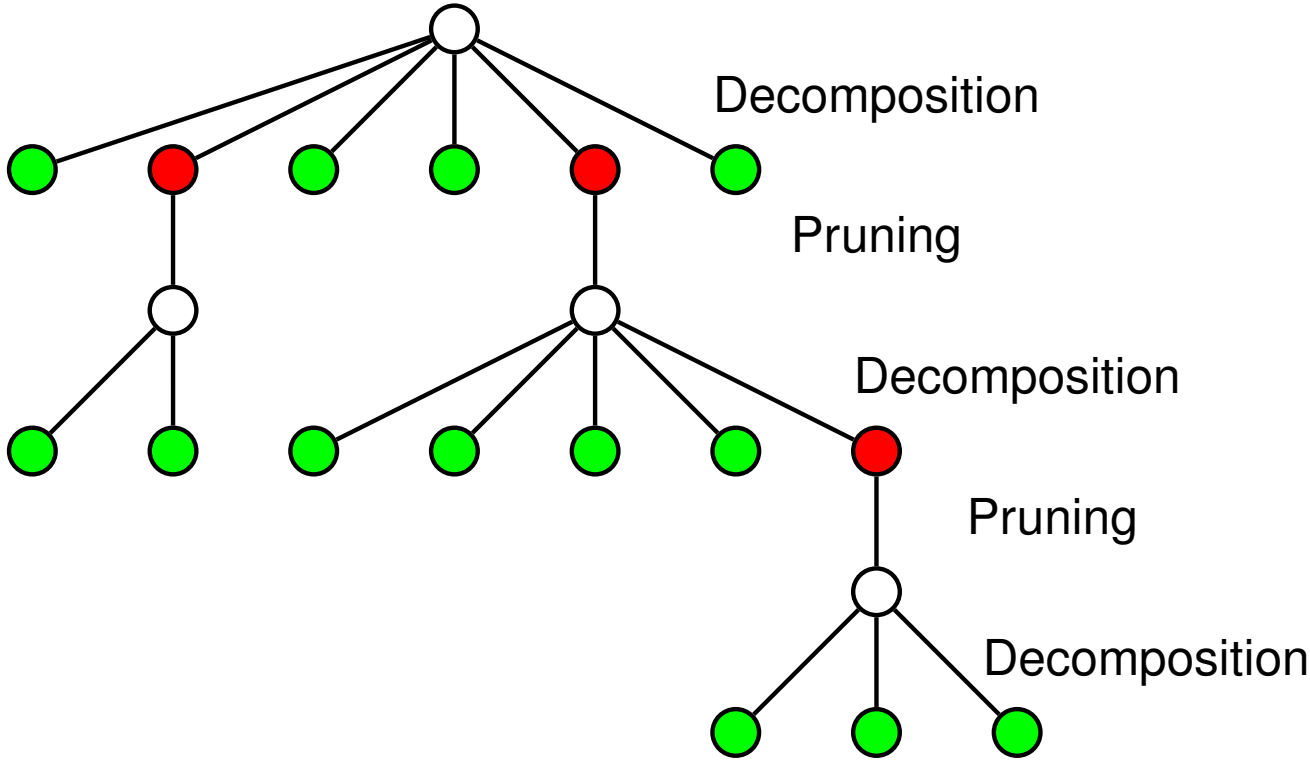

Resulting barcode = {leaf nodes}.

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