Mathematics of 3D Reconstruction

Critical Two-View Configurations via Homaloidal Parametrizations

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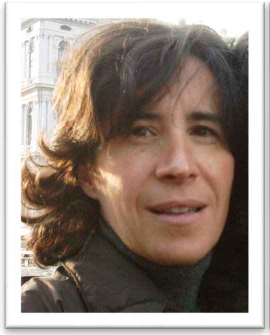
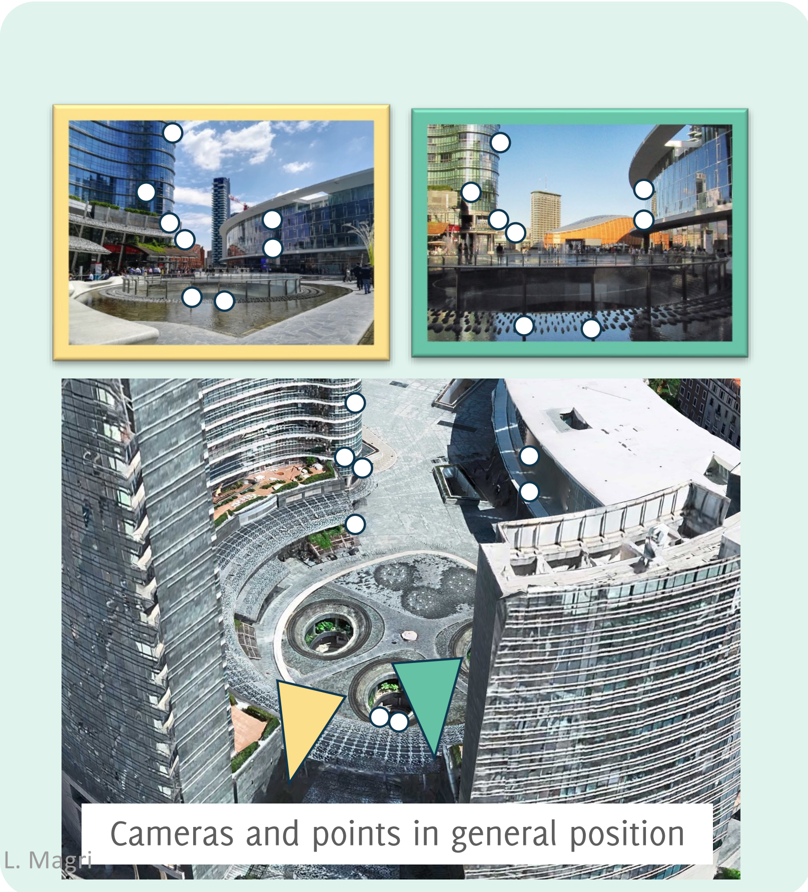
   

Acknowledgments

Joint work with **Marina Bertolini, Cristina Turrini** (Università degli Studi di Milano) **and Rakshit Madhavan, Federica Arrigoni** (Politecnico di Milano).

What is a critical configuration? (idea)

A configuration of *3D points* and *cameras* for **uncalibrated two-view reconstruction** is critical, if from two images alone, the scene cannot be reconstructed uniquely (up to a projectivity).



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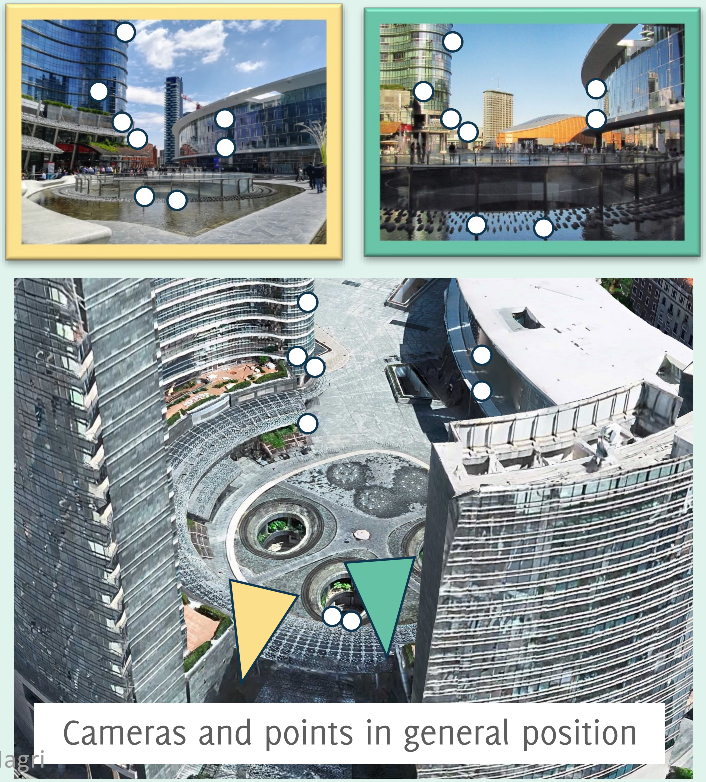
Not-critical

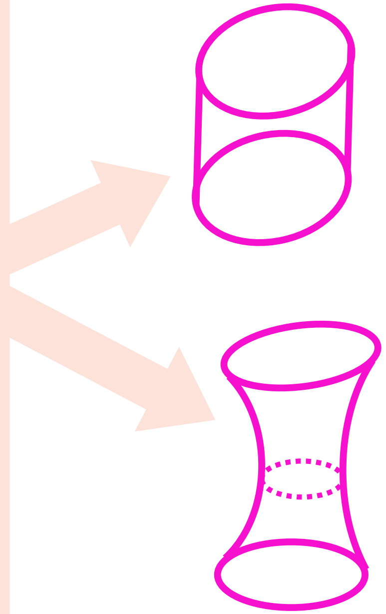


Cameras and points in general position

Critical

Cameras and points on a cylinder

Two alternative reconstructions



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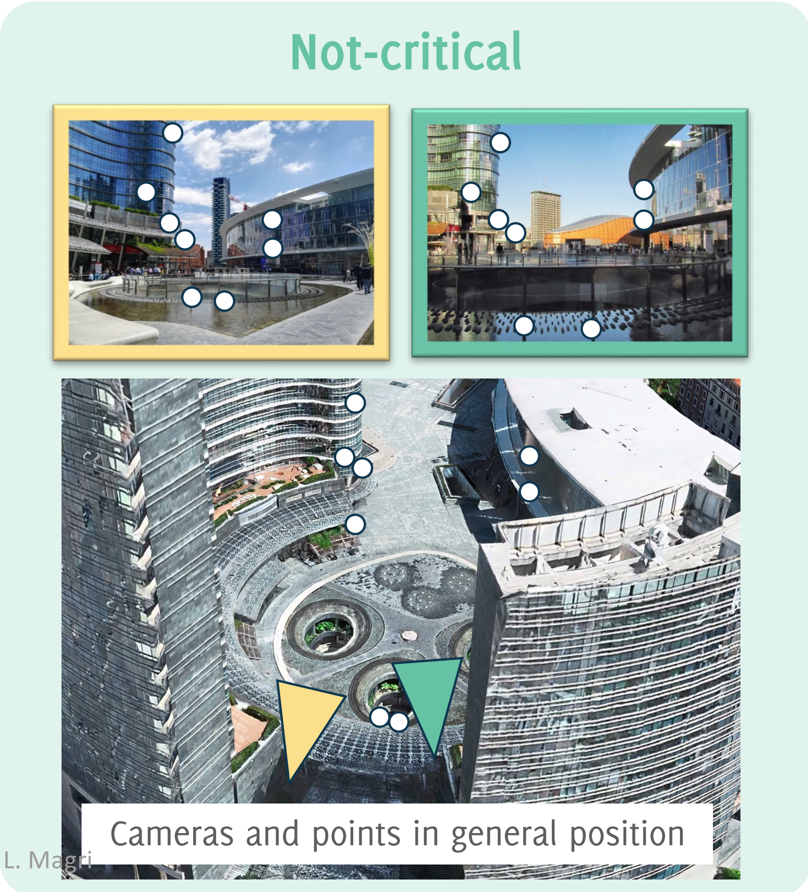

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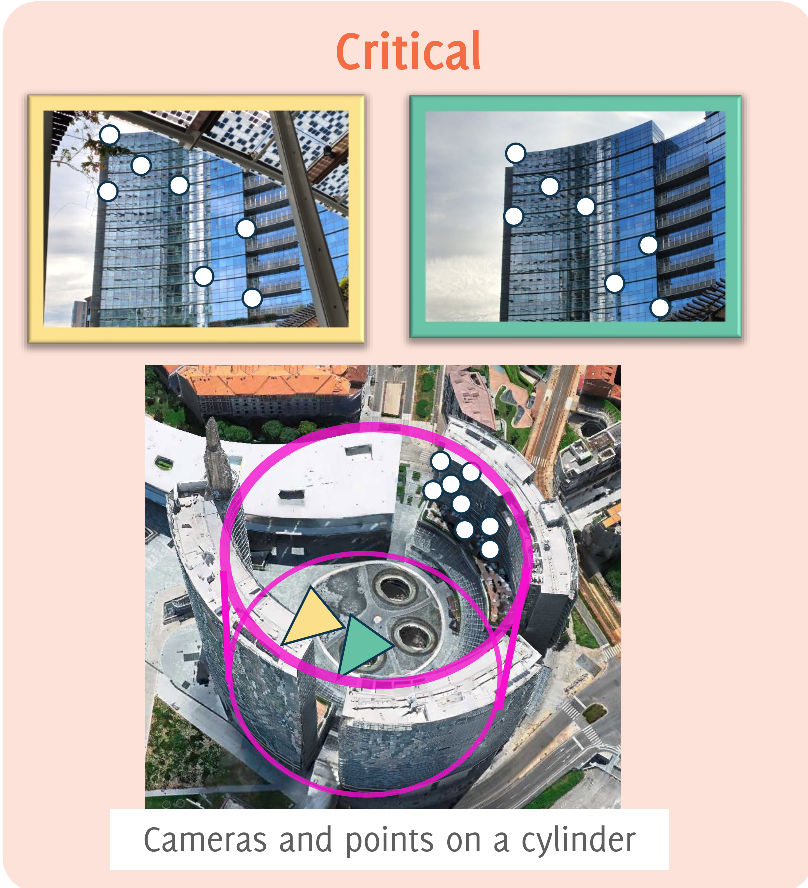

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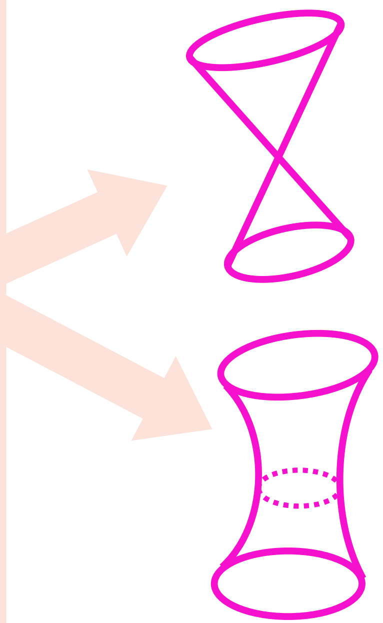
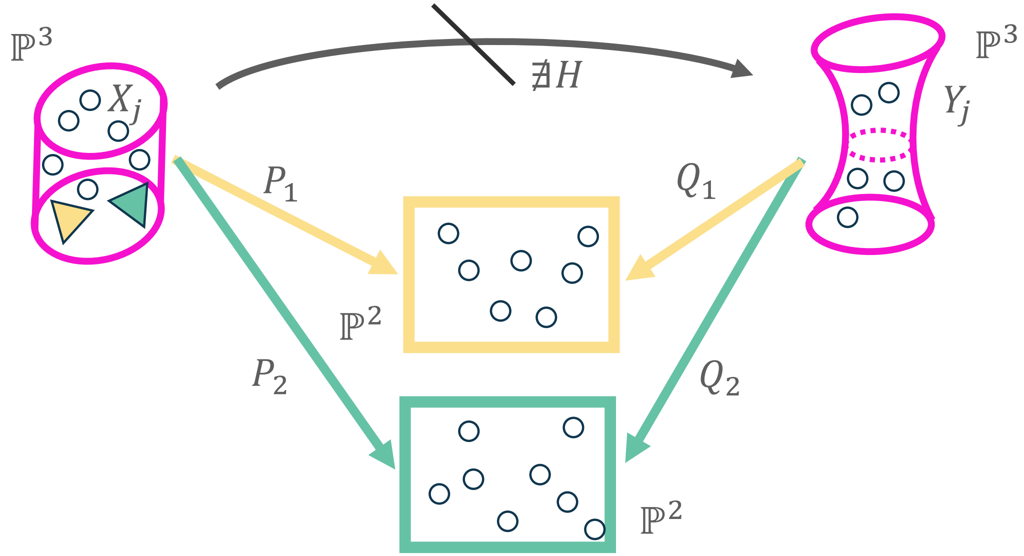


What is a critical configuration?

A pair of cameras (P_1, P_2) with 3D points $\{X_j\}$ is critical if there exists a conjugate configuration $(Q_1, Q_2, \{Y_j\})$, not projectively equivalent to it, producing the same images:

i. $P_i X_j \cong Q_i Y_j$

ii. $\nexists H \in \mathbb{P}GL(4)\ s.t.\ HX_j \cong Y_j$



Equivalently: the configuration is degenerate when the fundamental matrix compatible with the correspondences is not unique.

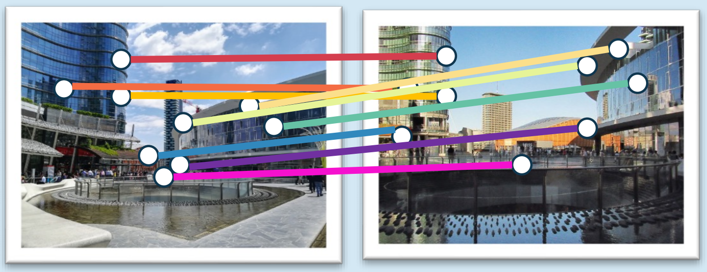
Non-critical two-view reconstruction

Given $N \geq 8$ correspondences $\{\boldsymbol{x}_j\ \boldsymbol{x}'_j\}_{j=1,\ldots N}$,
the fundamental matrix F exists and is the **unique** rank 2 matrix satisfying

$$\left(\boldsymbol{x}'^{\top}_j \otimes \boldsymbol{x}^{\top}_j\right)\mathrm{vec}(F) = \mathbf{0}$$

To get a reconstruction

Input:
matches$\{\boldsymbol{x}_i, \boldsymbol{x}'_i\}_{i=1,\ldots N}$

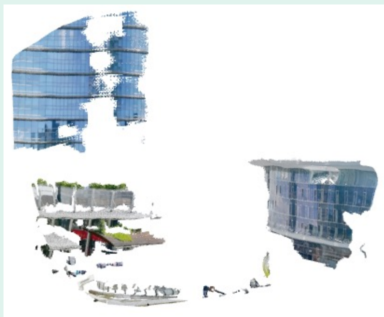


Solve
$\left(\boldsymbol{x}'^{\top}_i \otimes \boldsymbol{x}^{\top}_i\right)\ \mathrm{vec}(F) = \mathbf{0}$

Get cameras from F

Triangulate points

Output:
3D points X_i & Cameras P_1, P_2

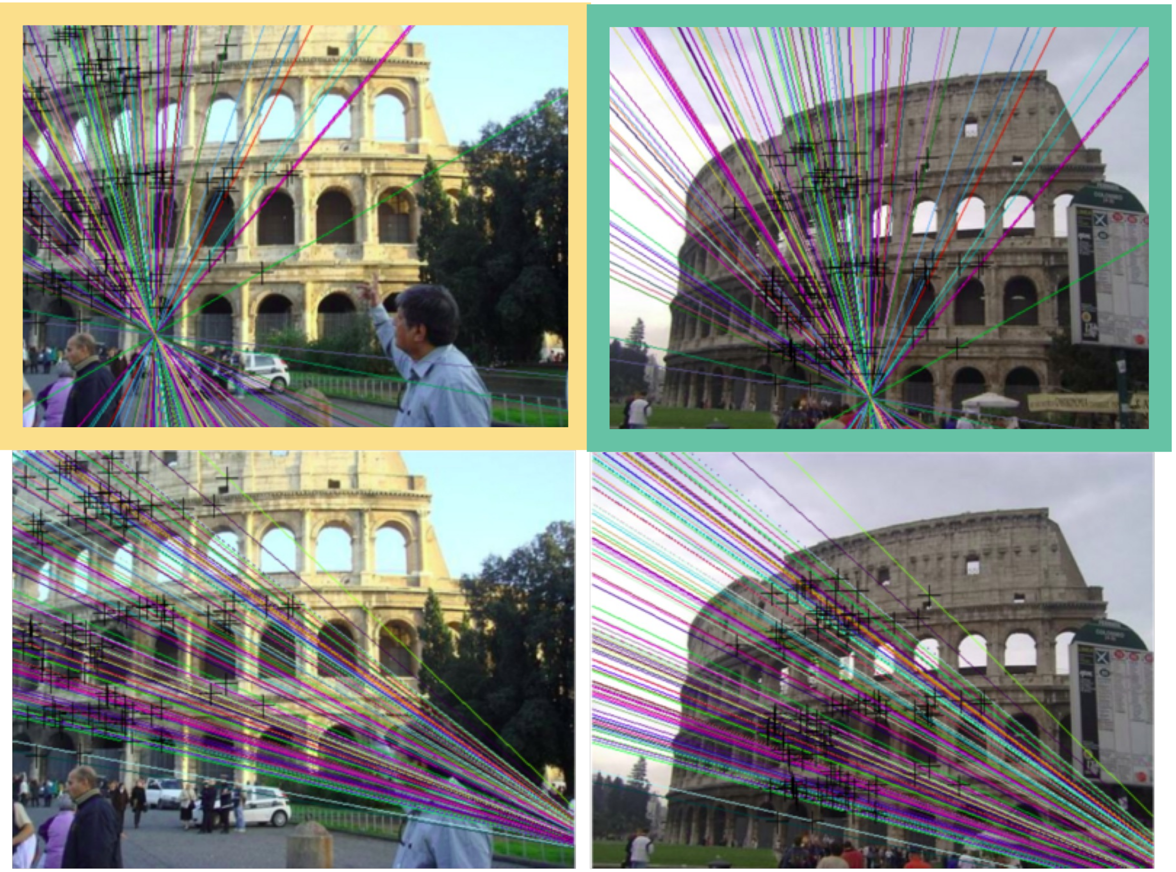


In practice

Even if all the matches are inlier, it is possible to get an erroneous estimate of epipolar geometry.

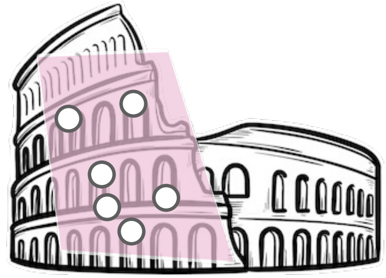
Ground truth

Wrong estimate



Pictures from *Fan, Kileel, Kimia*; CVPR 2022

All matches are inliers ≠ well posed estimation



Why? Ill-posed epipolar geometry

The uniqueness of F, depends on the rank of the null-space N of $\left(\boldsymbol{x}_i'^\top \otimes \boldsymbol{x}_i^\top\right)$:

$$\dim(N) = \begin{cases} 1, & \text{not-critical (unique } F\text{);} \\ 2, & \text{critical configuration, (up to three compatible } F\text{);} \\ 3, & \text{pure rotation/planar scene, } \infty\text{ −family of } F. \end{cases}$$

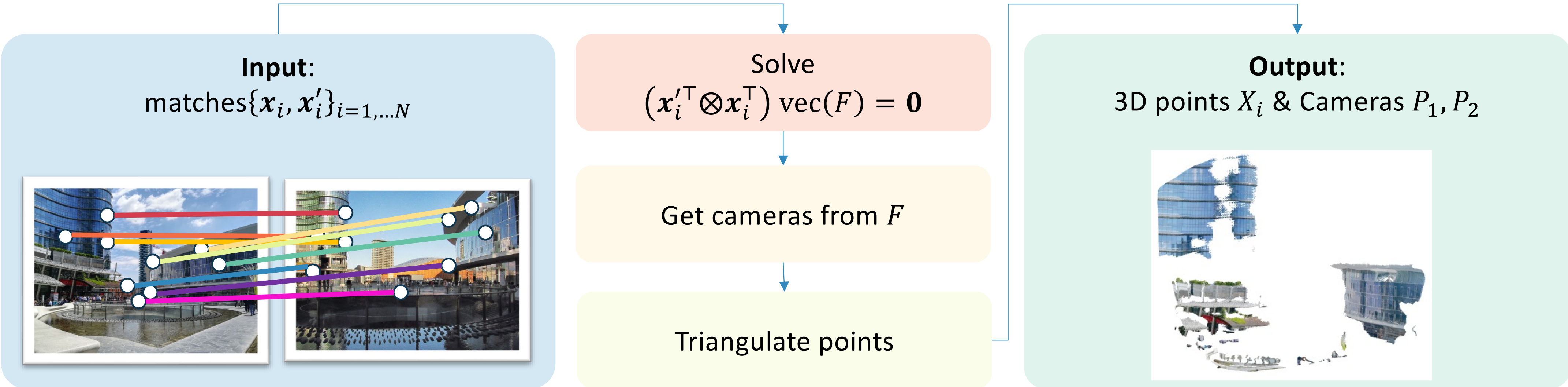
Correspondences yielding $\dim(N) > 1$ do not determine a unique F.

Near these configurations, estimates may be numerically unstable!

Erin Connelly, Sameer Agarwal, Alperen Ergur, Rekha R. Thomas.
The Geometry of Rank Drop in a Class of Face-Splitting Matrix Products. Advances in Geometry 2024

Robust fitting and degeneracy test

In practice, the estimation is framed as a robust fitting problem to handle both outliers and (some) degenerate configurations.



Robust fitting and degeneracy test (idea of Plunder, Degensac)

Input:
matches$\{\boldsymbol{x}_i, \boldsymbol{x}'_i\}_{i=1,\ldots N}$

Sampling $S \subseteq \{\boldsymbol{x}_i, \boldsymbol{x}'_i\}$

Is S degenerate?

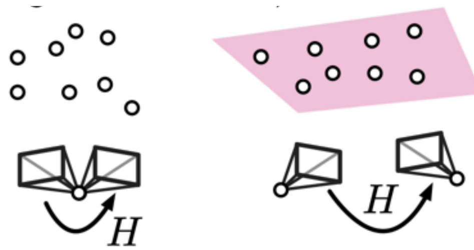
Solve
$(\boldsymbol{x}'^{\top}_i \otimes \boldsymbol{x}^{\top}_i)\ \mathrm{vec}(F) = \mathbf{0}$

Keep F maximizing the inliers

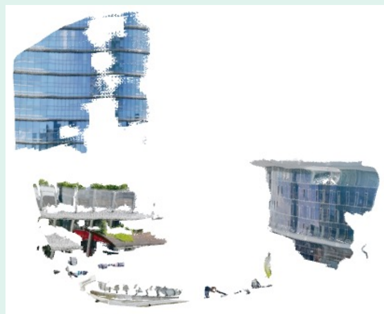
Get cameras from F

Triangulate points

Degeneracies are ***typically*** handled ***only*** for planar scene / pure rotation



Output:
3D points X_i & Cameras P_1, P_2



For the general critical surface there is no analogous test.

Chum, et al: Two-view Geometry Estimation Unaffected by a Dominant Plane; CVPR 2005

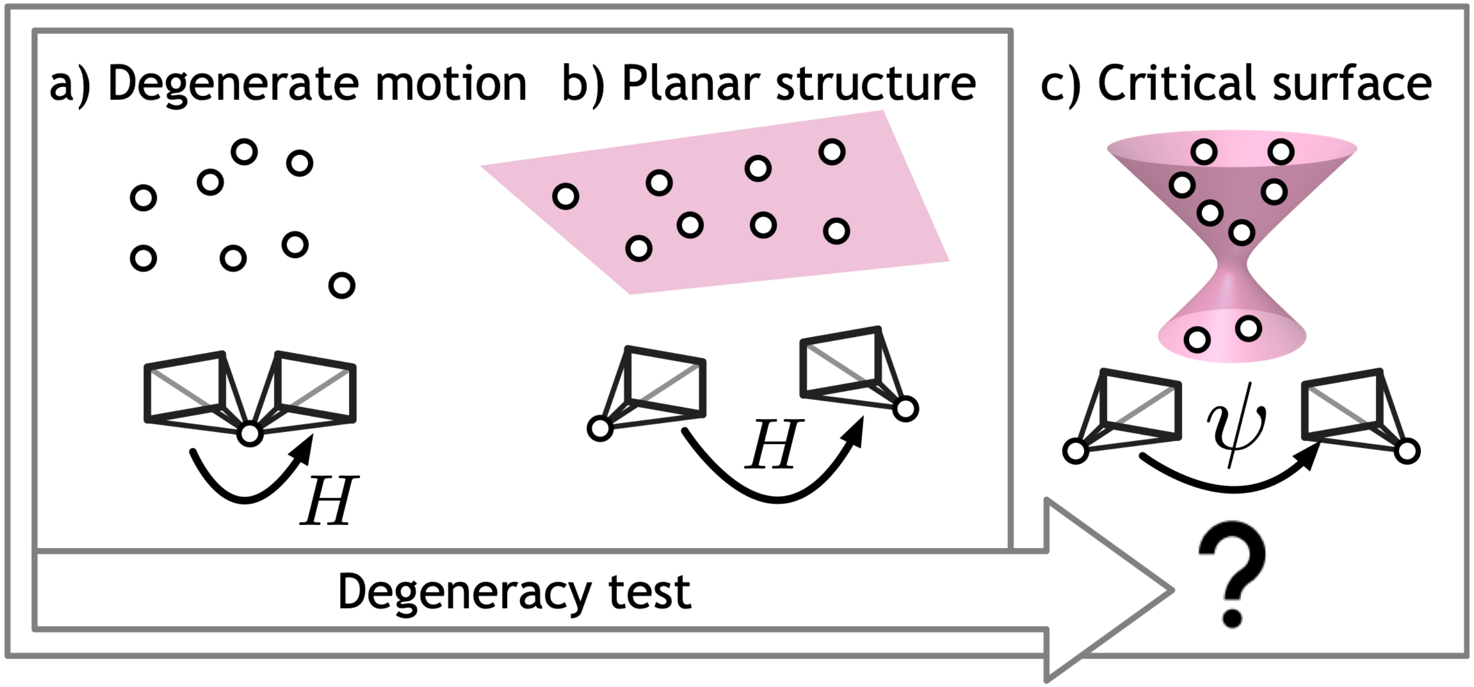
Torr, et al: Robust detection of degenerate configurations while estimating the fundamental matrix. *CVIU 1998*

Degeneracy tests

✓ $\dim(N) = 1$, nothing to do

• $\dim(N) = 2$, this talk

✓ $\dim(N) = 3$, homography test (DEGENSAC, PLUNDER…)



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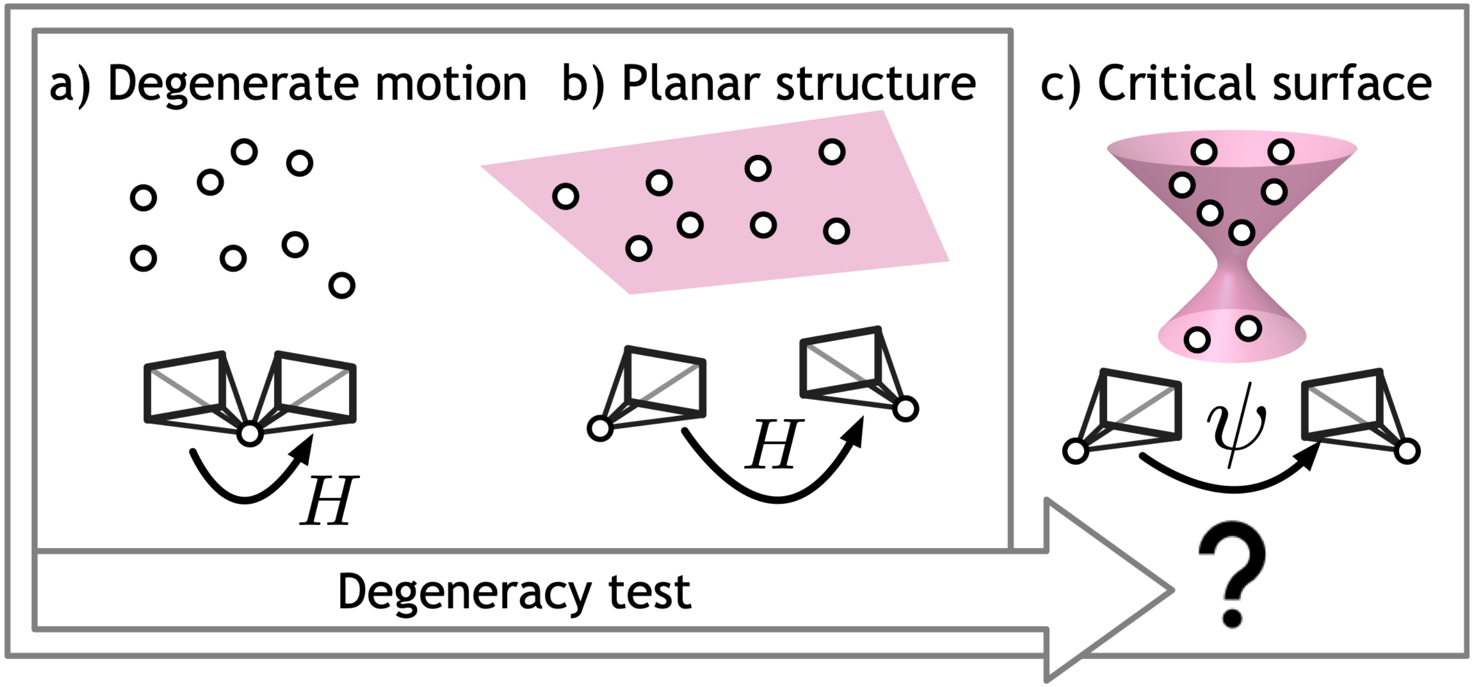
Prior art:

- **Luong–Faugeras** (1996) solve a related test but require F and a non-linear fit. It's unstable near criticality;
- **Fan–Kileel–Kimia** (2022) study the **minimal** (7-pt) case.

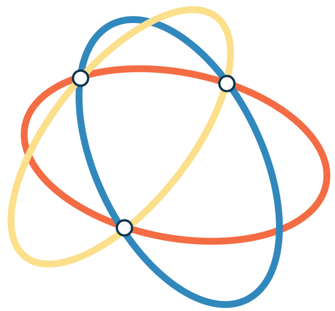


Contributions

When F is not unique, three questions:

1. **Detect:** recognize it from image correspondences alone, without a pre-computed F.
2. **Classify:** characterize which critical surface we are on.
3. **Do it practically:** stable, linear, from the non-minimal set $(N \geq 8)$.

One object answers all three:
the quadratic cremona transformation ψ between images, and its **homaloidal** structure.



Geometry of critical loci in 3D

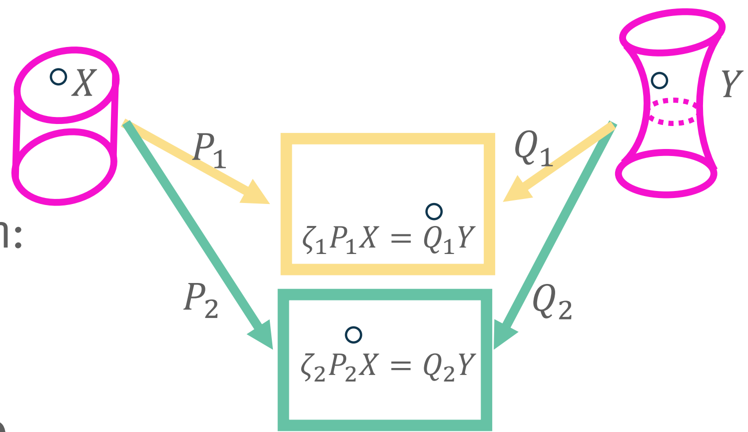
Setup

- We consider the uncalibrated, two views setup
- Cameras $P_{\mathrm{i}} \in \mathbb{P}(\mathbb{R}^{3x4})$, rank $P_{\mathrm{i}} = 3$; up to $\mathbb{P}GL(4)$.
  The only thing about the cameras that matters: the center $C_{\mathrm{i}} = \ker(P_{\mathrm{i}})$.
  Assume $C_1 \neq C_2$.
- $N \geq 8$ points $X_j \in \mathbb{P}^3$, up to $\mathbb{P}GL(4)$.

The critical quadric

A conjugate point Y of X must land on the **same two image points** *(equal up to scale, one scale ζ_i per view)*:

$$\zeta_1 P_1 X = Q_1 Y$$
$$\zeta_2 P_2 X = Q_2 Y$$



Collect the 6 unknowns (Y, ζ_1, ζ_2) into one homogeneous system:

$$\underbrace{\begin{bmatrix} Q_1 & P_1 X & \mathbf{0} \\ Q_2 & \mathbf{0} & P_2 X \end{bmatrix}}_{M(X)} \begin{bmatrix} Y \\ -\zeta_1 \\ -\zeta_2 \end{bmatrix} = \mathbf{0}.$$

A conjugate exists $\Leftrightarrow$ the system has a **nonzero** solution $\Leftrightarrow$ $M(X)$ is **singular**: $\det M(X) = 0$

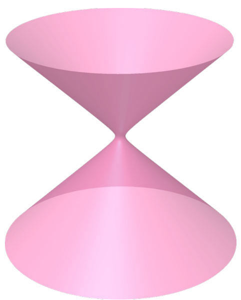
Only two columns of M depend on X, each linearly $\Longrightarrow$ $\det M(X)$ has **degree 2 in** X
$\Longrightarrow$ the critical points X form a **quadric** $\mathcal{S}$.

Critical ruled quadric for two uncalibrated view

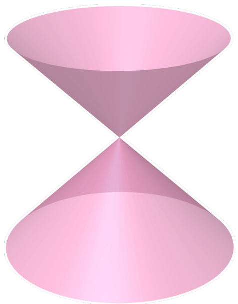
Lemma (Hartely & Kahl): Consider the quadrics $\mathcal{S}_P = P_2^\top F_Q P_1$ and $\mathcal{S}_Q = Q_2^\top F_P Q_1$.

$\mathcal{S}_P, \mathcal{S}_Q$ contains the camera centers of P_j and Q_j respectively and are ruled quadric.

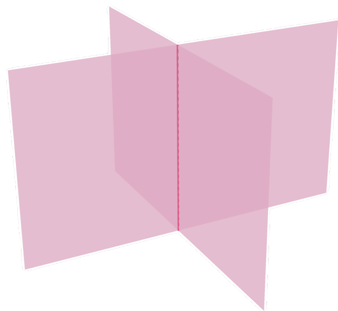
Theorem: All (non-trivial) critical configurations have their points and camera centers on a ruled quadric.



Hyperboloid
$\text{rank}(\mathcal{S}) = 4$



Cone
$\text{rank}(\mathcal{S}) = 3$



Two planes
$\text{rank}(\mathcal{S}) = 2$

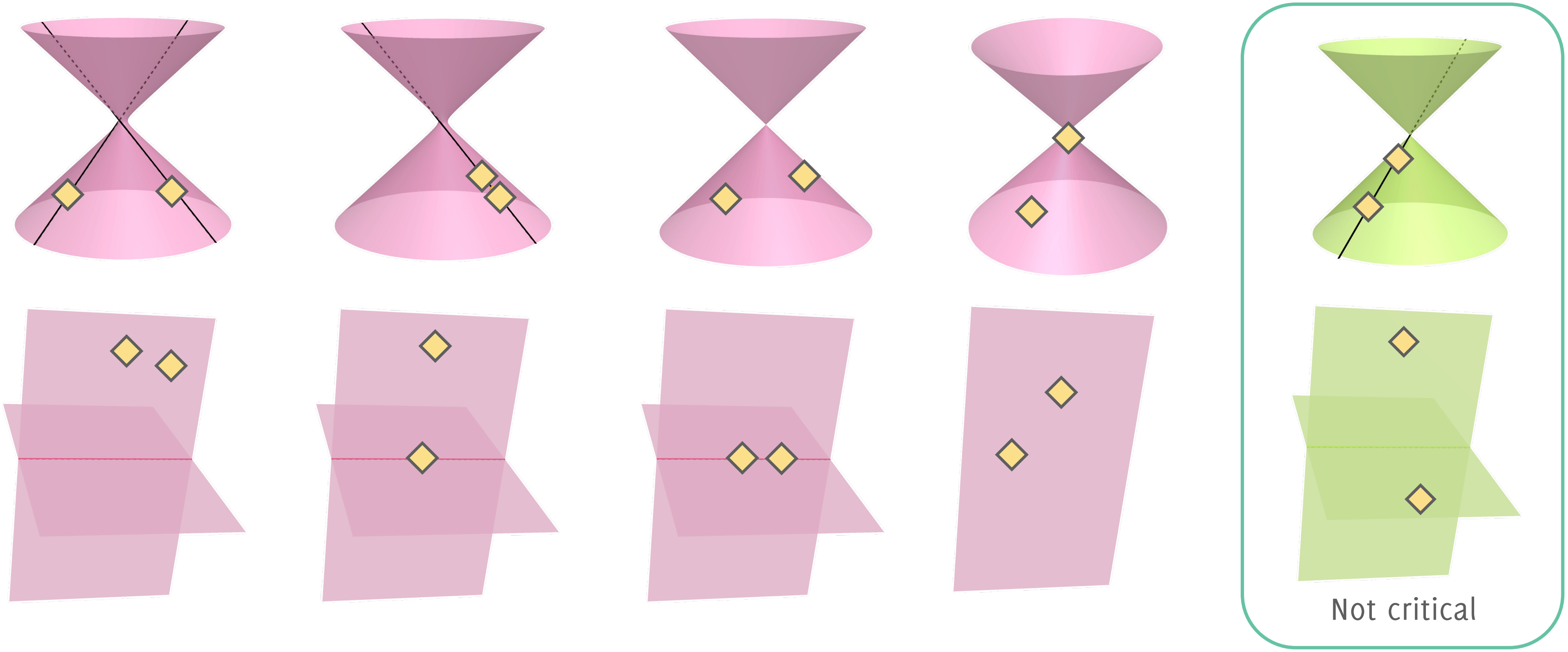
One plane
$\text{rank}(\mathcal{S}) = 1$

Remark: the converse is not true (counter-example in green next slide)

Hartley, R., Kahl, F. Critical Configurations for Projective Reconstruction from Multiple Views. IJCV 2007

Bråtelund's complete classification

Bråtelund: complete taxonomy for two views. For two views there are 8 possible configurations.



Martin Bråtelund, Critical configurations for two projective views, a new approach, Journal of Symbolic Computation,2024,

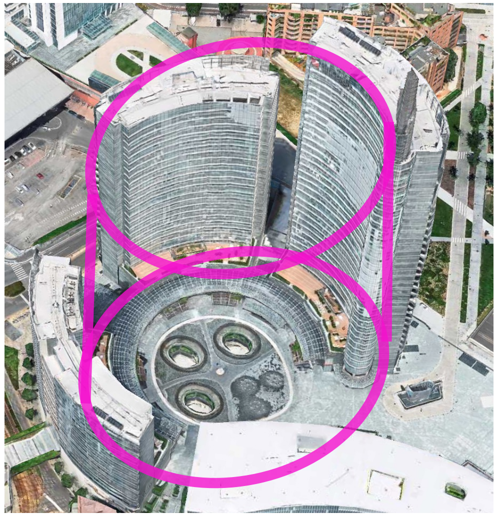
Båtelund's complete classification

Quadric $\mathcal{S}_P$	Conjugate quadric $\mathcal{S}_Q$
Smooth, cameras not on a line	Same
Smooth, cameras on a line	Cone, cameras not on a line
Cone, cameras not on a line	Smooth, cameras on a line
Cone, one camera at vertex	Same
Two planes, cameras in same plane	Same
Two planes, one camera on singular line	Same
Two planes, cameras on singular line	Same
Double plane, cameras in the plane	Same

Martin Båtelund, Critical configurations for two projective views, a new approach,
Journal of Symbolic Computation,2024,

From 3D quadric to image map

From quadric to image map



3D quadric in $\mathbb{P}^3$

Theory

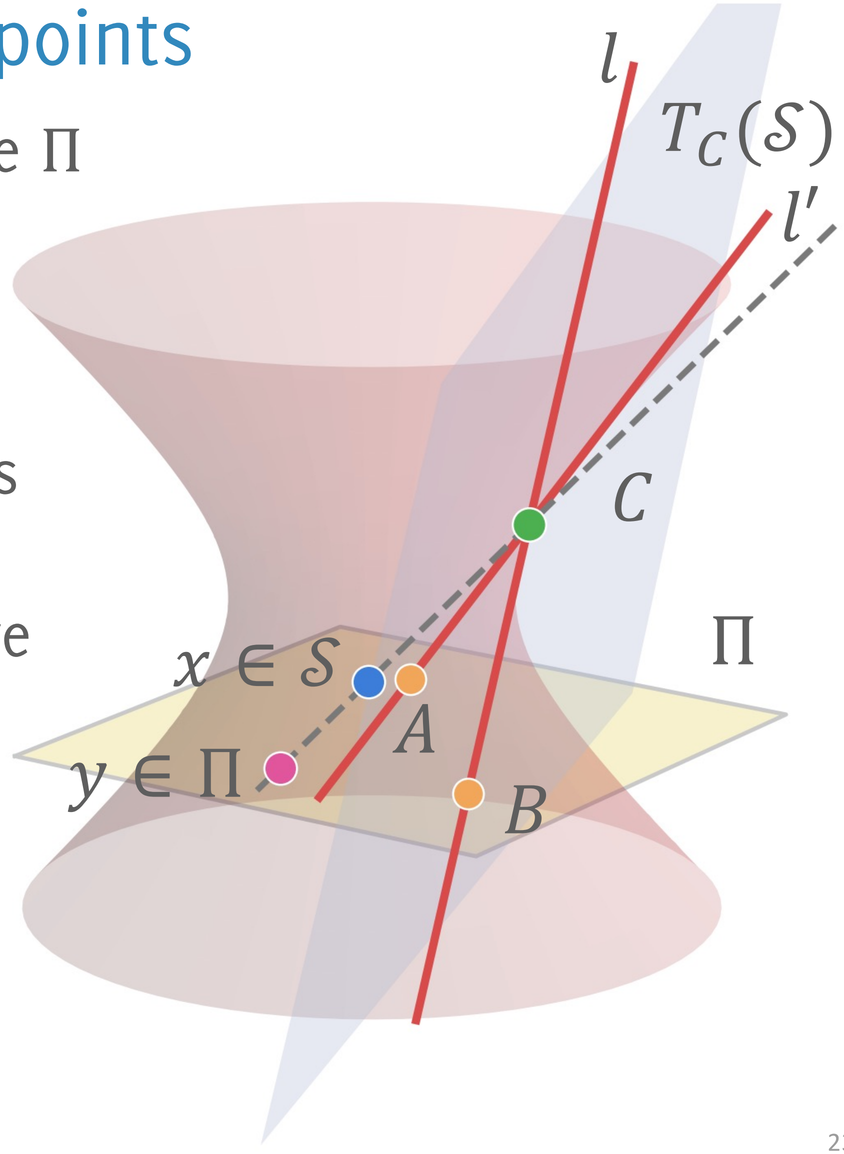
Image correspondences in $\mathbb{P}^2 \times \mathbb{P}^2$

Practical test

Projecting a quadric from one of its points

Let $\mathcal{S}$ be a smooth ruled quadric, $C \in \mathcal{S}$, plane Π not through C.

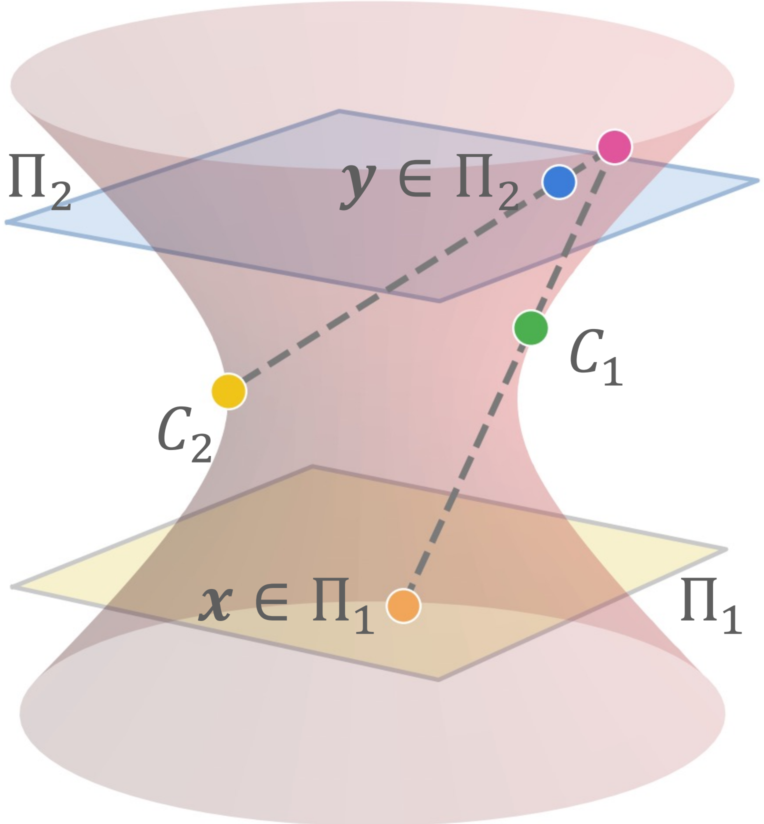
- The projection $P_C\colon \mathcal{S} \dashrightarrow \Pi$ is birational, undefined at C.
- The tangent plane $T_C(\mathcal{S})$ meets $\mathcal{S}$ in two lines l, l'.
- P_C contracts l, l' to two points A, B; elsewhere biregular.
- $P_C{}^{-1}\colon \Pi \dashrightarrow \mathcal{S}$ is undefined at A, B, and contracts line AB to C.



Projecting from two points

Proposition: Given two projections P_{C_1} and P_{C_2} of the same smooth quadric $\mathcal{S}$ from two distinct points $C_1, C_2 \in \mathcal{S}$, on two target planes Π_1 and Π_2, the composite map $\psi = P_{C_2} \circ (P_{C_1})^{-1} \colon \Pi_1 \dashrightarrow \Pi_2$ is a birational **quadratic transformation.**

We can apply this to the critical quadric $\mathcal{S}$ to get a parametric transformation between the two image planes.



Bertolini, M., Magri, L. & Turrini, C. Critical Loci for Two Views Reconstruction as Quadratic Transformations Between Images. J Math Imaging Vis, 2019

Transformation induced by the projections of a quadric

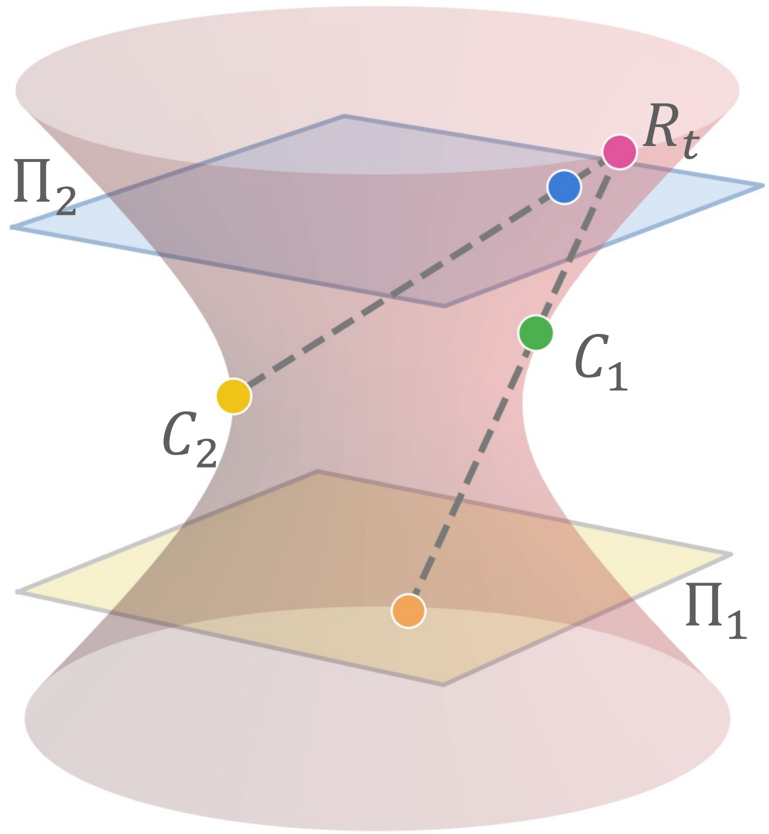
Sketch of the proof:

- Take a general point $R_t = C_{Q_1} + tQ_1^+ x$ along the optical ray from the first camera
- Solve for t by imposing $R_t \in \mathcal{S}$:

$$\det\begin{bmatrix} P_1 & Q_1 R_t & \mathbf{0} \\ P_2 & \mathbf{0} & Q_2 R_t \end{bmatrix} = 0$$

- Project R_t into the second image

You get $\boldsymbol{x}' = \det(N_2)\, Q_2 C_{Q_1} - \det(N_1)\, Q_2 Q_1^+\, \boldsymbol{x}$, which defines a quadratic transformation Ψ.

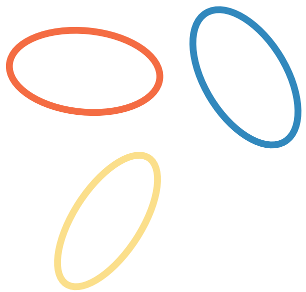


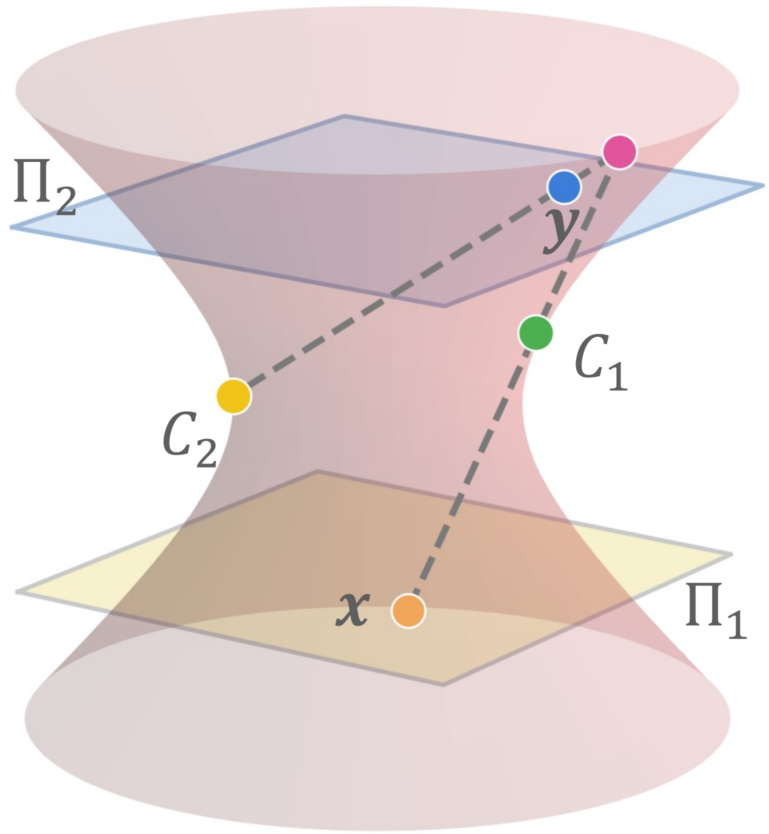
A quadratic map

A general quadratic transformation $\psi: \Pi_1 \rightarrow \Pi_2$ sends $\boldsymbol{x} = (x_0 : x_1 : x_2)$ to $\boldsymbol{y} = (y_0 : y_1 : y_2)$ with each component y_i a **quadric in $\boldsymbol{x}$**:

$$\begin{cases} \rho y_0 = \boldsymbol{x}^\top C_0 \boldsymbol{x} \\ \rho y_1 = \boldsymbol{x}^\top C_1 \boldsymbol{x} \\ \rho y_2 = \boldsymbol{x}^\top C_2 \boldsymbol{x} \end{cases}$$

Each C_k is a symmetric 3x3 matrix → 6 coefficients → 3 x 6 = 18 parameters.





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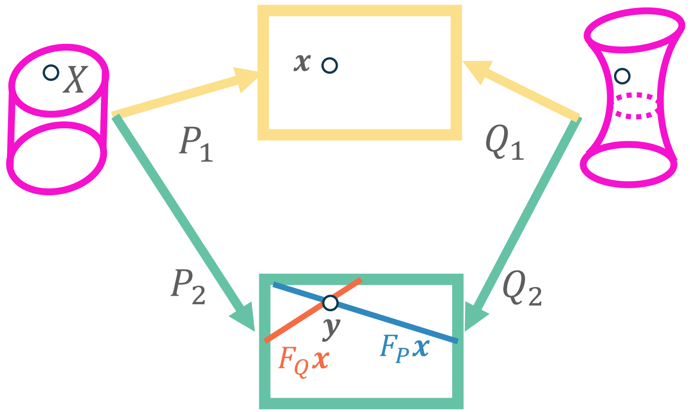
But, in our setup ψ is birational, **we have fewer degrees of freedom** =14



Cremona quadratic transformation

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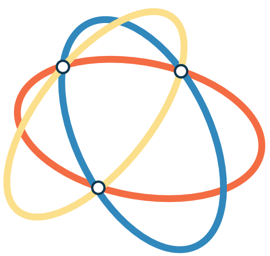
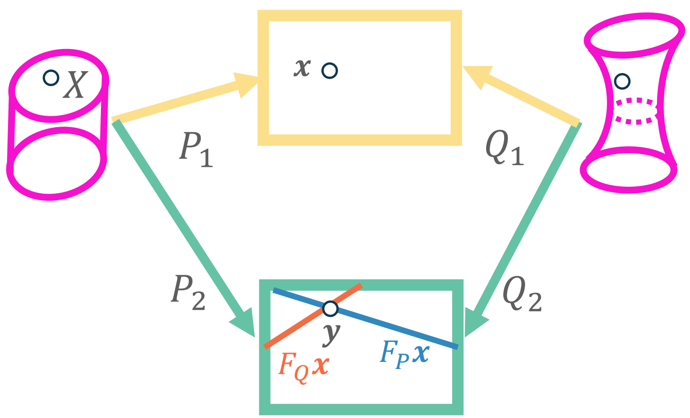
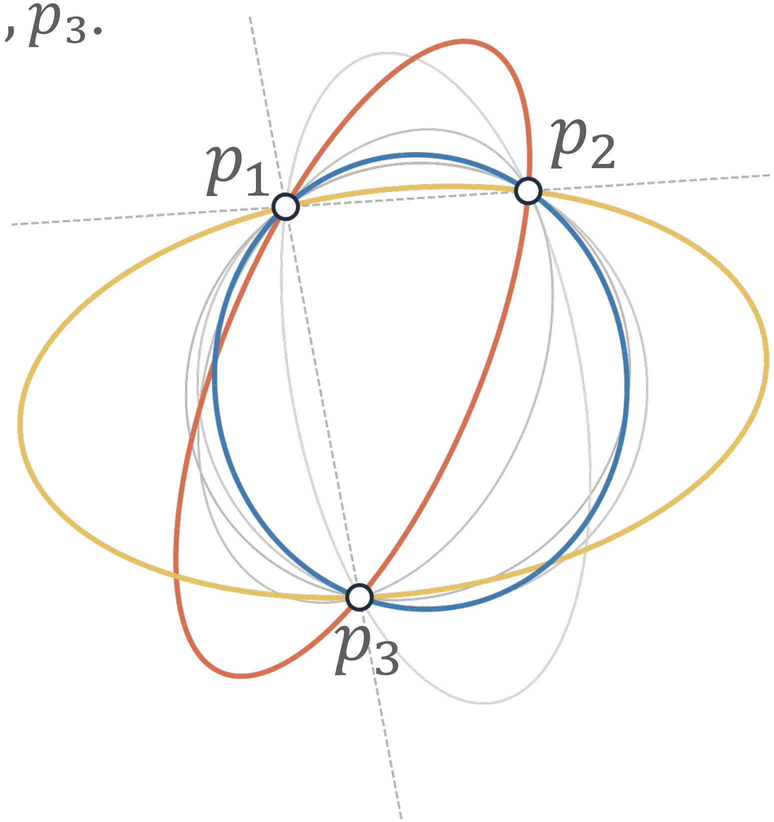
The conic C_k are not independent!





Cremona quadratic map and homaloidal net

- **The three conics form a net.**
  The conics $\psi_0 = 0, \psi_1 = 0, \psi_2 = 0$
- For ψ to be **birational** (a cremona map), the net must be **homaloidal:** the conics share **3 common base points** p_1, p_2, p_3.



Where ψ is undefined $\Longleftrightarrow$ the base points of the net

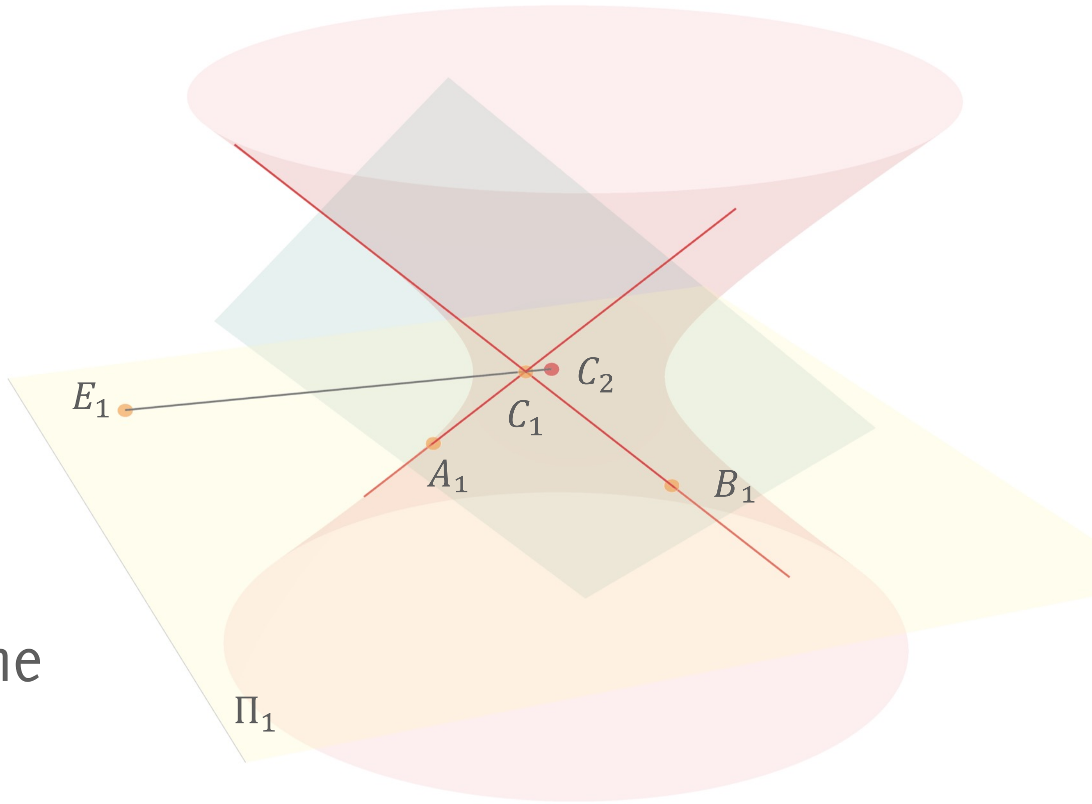
$$\psi = P_{C_2} \circ \left(P_{C_1}\right)^{-1}: \Pi^1 \dashrightarrow \Pi^2$$

ψ is undefined at its 3 **base points**:

- A_1, B_1 the images of the tangent lines contracted by the projection
- $E_1 = P_1(C_2)$ the epipole;

(similarly, $A_2, B_2, E_2\ in\ \Pi_2$)

The **configuration of base points** (3 / 2 / degenerate) provide a classification of the possible configurations.



Classification by base points

Quadric $\mathcal{S}$	Line C_1C_2	# Fundamental points pts of ψ
smooth	$\not\subset \mathcal{S}$	3 distinct
smooth	$\subset \mathcal{S}$	2 distinct
cone	$\not\subset \mathcal{S}$	2 distinct
two planes	C_1, C_2 same plane	ψ is a homography

In the image plane, Bråtelund's 3D taxonomy collapses into a smaller number of base-point configurations; the last case appears as further degeneration (homography)

Madhavan, Bertolini, Turrini, Arrigoni, Magri. Classification and detection of two-view uncalibrated critical configuration from images. Manuscript in preparation

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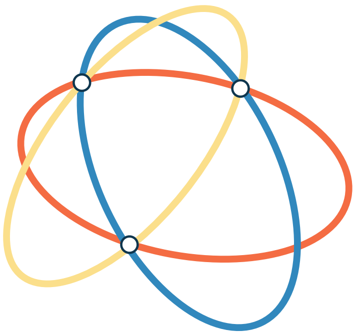

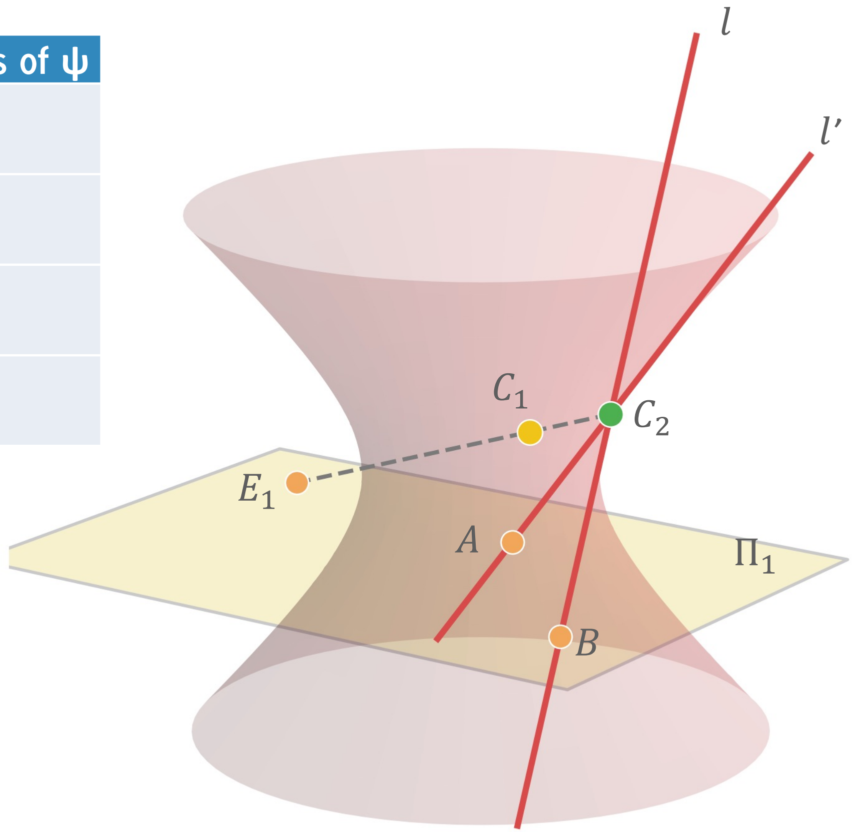

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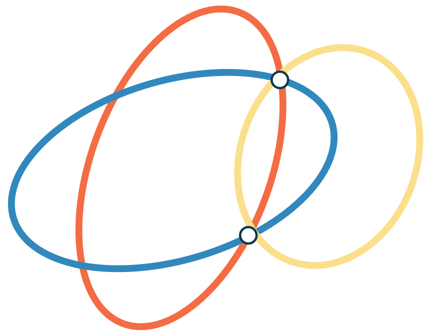

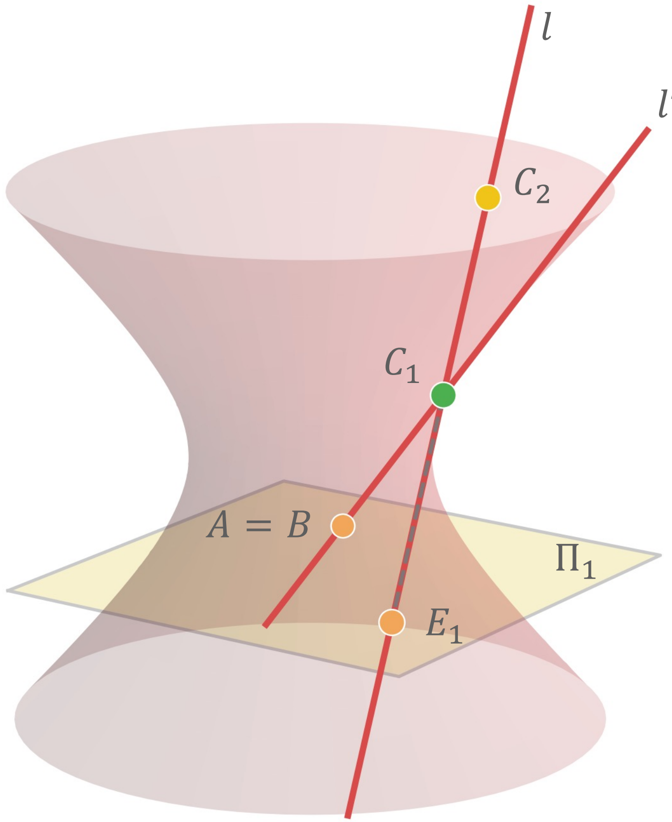

Madhavan, Bertolini, Turrini, Arrigoni, Magri. Classification and detection of two-view uncalibrated critical configuration from images. Manuscript in preparation

Classification by base points

Quadric $\mathcal{S}$	Line C_1C_2	# Fundamental points pts of ψ
smooth	$\not\subset \mathcal{S}$	3 distinct
smooth	$\subset \mathcal{S}$	2 distinct
cone	$\not\subset \mathcal{S}$	2 distinct
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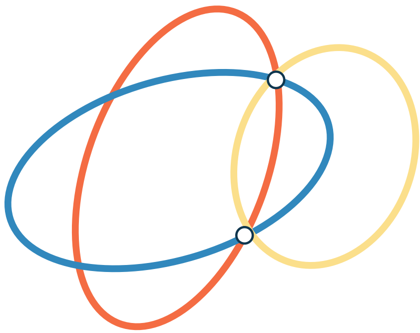

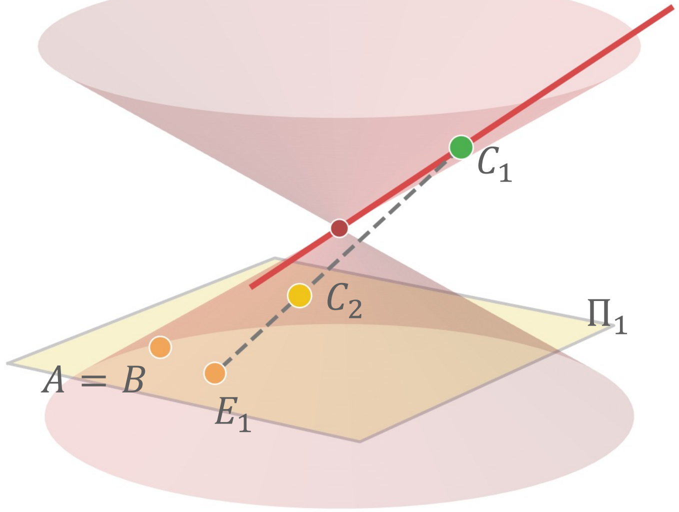

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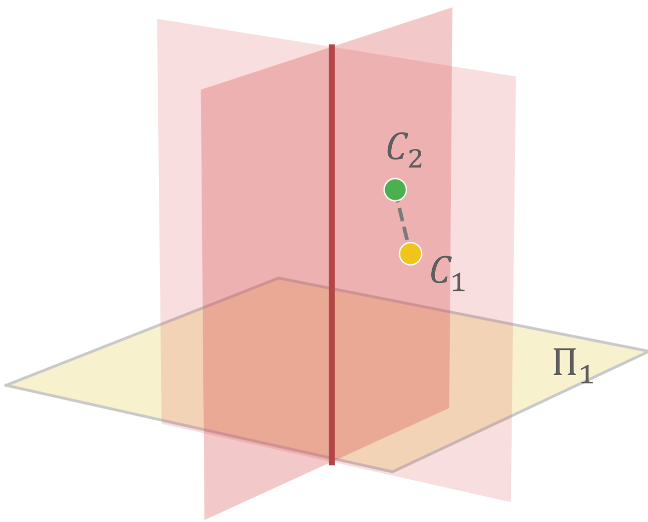

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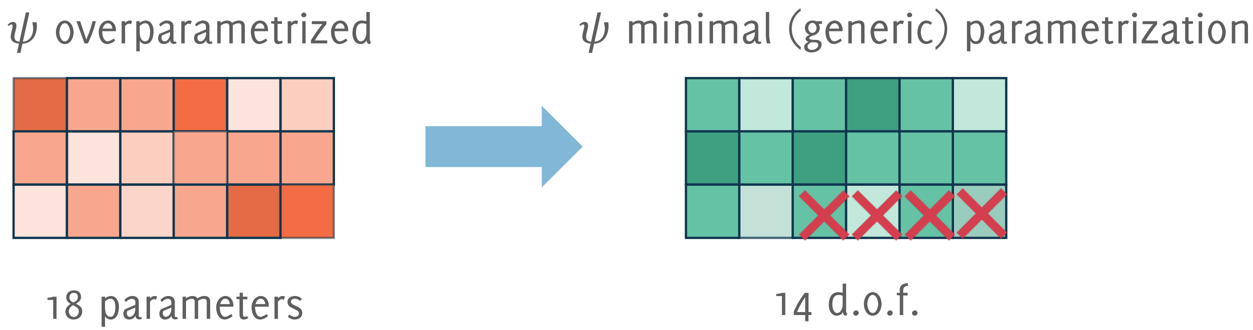
Detecting critical configurations

Looking for a minimal parametrization of ψ

To derive a practical critical test *from image only,* we look for a **minimal parametrization** of ψ.

ψ has 18 parameters, but the d.o.f. are just 14.

The idea is to exploit the geometry of the homaloidal net of conics to remove the redundant parameters.



Homaloidal net generators

Proposition: a quadratic transformation $\psi\colon \Pi_1 \dashrightarrow \Pi_2$ between projective planes can be written as

$$\begin{cases} \rho y_0 = L_1 M_2 - L_2 M_1 \\ \rho y_1 = L_2 M_0 - L_0 M_2 \\ \rho y_2 = L_0 M_1 - L_1 M_0 \end{cases}$$

with L_i, M_i linear form in $\boldsymbol{x}$ $(3\times 6 = 18$ parameters),
i.e., $L_i(x_0\colon x_1\colon x_2) = l_{i0}x_0 + l_{i1}x_1 + l_{i2}x_2$,
$M_i(x_0\colon x_1\colon x_2) = m_{i0}x_0 + m_{i1}x_1 + m_{i2}x_2$

Equivalently, ψ is defined by the 2×2 minors of the matrix

$$G = \begin{bmatrix} L_0 & L_1 & L_2 \\ M_0 & M_1 & M_2 \end{bmatrix}$$

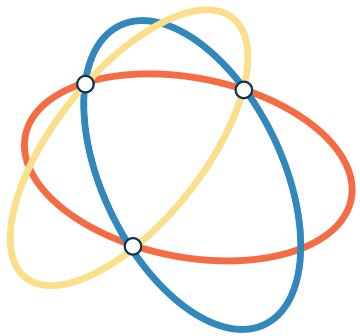
Beltrametti et al.. Lectures on Curves, Surfaces and Projective Varieties: A Classical View of Algebraic Geometry. 2009.

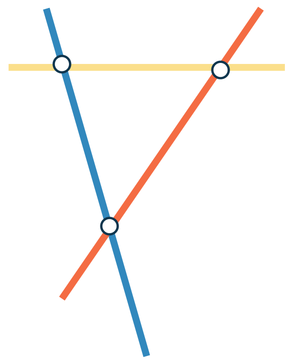
Geometric interpretation

The linear forms L_i, M_i define auxiliary lines entering a determinantal representation of the homaloidal net.

Remark: While a homaloidal net is uniquely identified by its 3 base points, the lines L_i and M_i are not uniquely defined.

A different choice of the lines results in the **very same** homaloidal net.

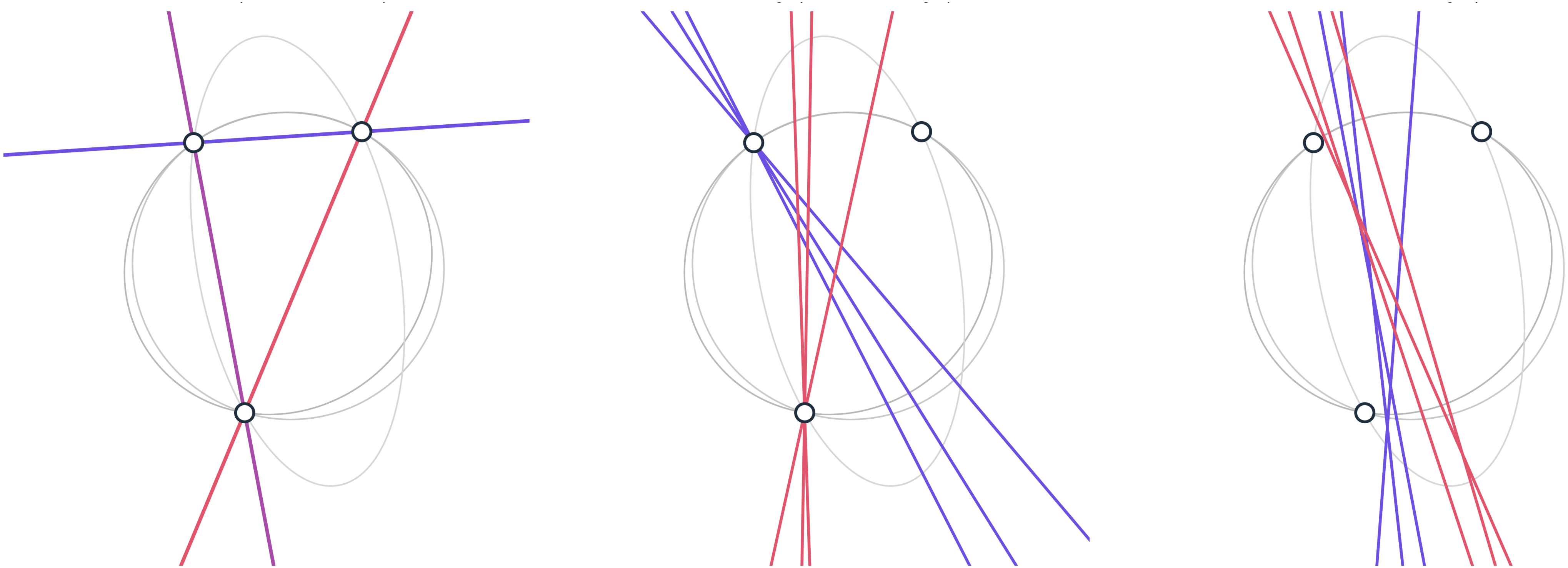




Beltrametti et al.. Lectures on Curves, Surfaces and Projective Varieties: A Classical View of Algebraic Geometry. 2009.

Concrete example

Different choice of generators L_i, M_i,
same base points p_1, p_2, p_3, same homaloidal net



Homaloidal parametrization

Proposition: The three generating conics are invariant under any invertible 2x2 change $H = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ applied to G

Proof. The 2x2 minor becomes $(\alpha\delta - \beta\gamma)(L_0M_1 - L_1M_0)$; it vanishes iff the original did (same for all minors).

By applying H we can fix 4 parameters out of 18. **We have 14 free parameters.**

The coefficient of ψ are linear in the monomial

$$[x_0y_2, x_1y_0, x_1y_1, x_1y_2, x_2y_0, x_2y_1, x_2y_2].$$

Every match $(\boldsymbol{x}, \boldsymbol{y})$ provides two independent equations,

7 points are enough to fit $\boldsymbol{\psi}$.

Madhavan et al. Homaloidal parametrization for detecting critical two-view configurations. CVPR 2026

Fitting the quadratic transformation

Algorithm 1 Fitting the quadratic transformation ψ

Require: Seven correspondences $\{(\mathbf{x}_i, \mathbf{y}_i)\}_{i=1}^{7}$

7 points are enough to fit ψ

Ensure: A quadratic transformation ψ

Initialize $l_{00}, l_{10}, m_{00}, m_{10}$ to some random values.

Fix 4 d.o.f.

Build a 14×7 design matrix A :

for $i = 1, \ldots, 7$ **do**

  Set $\mathbf{x}_i = (x_0, x_1, x_2)$ and $\mathbf{y}_i = (y_0, y_1, y_2)$

  Compute the monomials

  $\mathbf{u} = [x_0y_2, x_1y_0, x_1y_1, x_1y_2, x_2y_0, x_2y_1, x_2y_2]$,

  Compute $v = -(l_{00}x_0y_0 + l_{10}x_0y_1)$,

  Compute $w = -(m_{00}x_0y_0 + m_{10}x_0y_1)$,

  Compute $A_i = \begin{pmatrix} \mathbf{u} & \mathbf{0}_7 \\ \mathbf{0}_7 & \mathbf{u} \end{pmatrix}$,

  Set $\mathbf{b}_i = [v, w]^\top$,

  Append A_i to matrix A e $\mathbf{b}_i$ in a vector $\mathbf{b}$.

end for

Solve for $\mathbf{z}$ such that $A\mathbf{z} = \mathbf{b}$

The estimation of ψ boils down to solve a linear system

Construct the quadratic transformation as in Eq. (8), using the initial $l_{00}, l_{10}, m_{00}, m_{10}$ and $\mathbf{z} = [l_{20}\ l_{01}\ l_{11}\ l_{21}\ l_{02}\ l_{12}\ l_{22}\ m_{20}\ m_{01}\ m_{11}\ m_{21}\ m_{02}\ m_{12}\ m_{22}]^\top$

Free parameters of ψ

Criticality test

Input:
matches$\{\boldsymbol{x}_i, \boldsymbol{y}_i\}_{i=1,\dots 8}$

$\tau > 0$ threshold

Get 7 matches

Fit ψ (solve linear system)

Compute $r = ||\psi(x_8) - y_8||$
in the second image

$r < \tau$

no

yes

Non-critical

Critical

Criticality test

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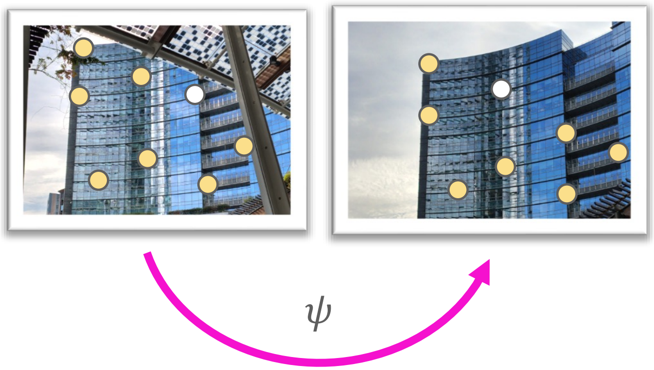

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x_8

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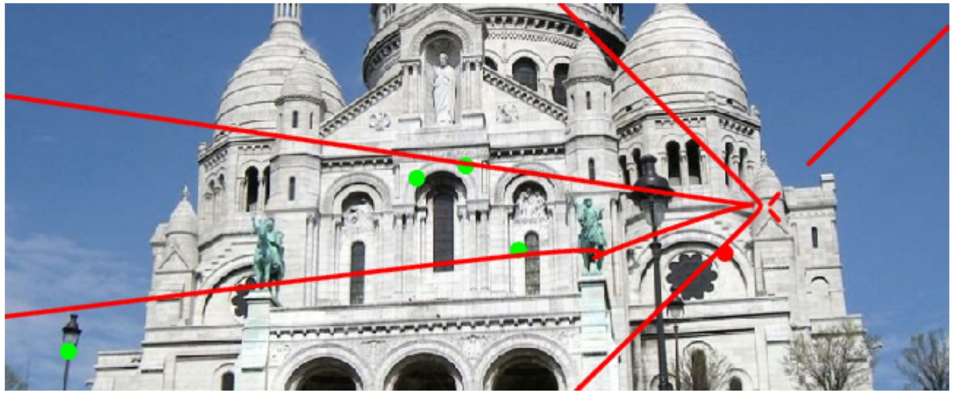
x_8 $\quad$ $||\psi(x_8) - y_8||$

Experiments

Alternative methods

- Luong Faugeras method
- Critical test in the minimal regime Fan, Kileel & Kimia (CVPR 2022): they study *ill-conditioning* of the 7-point problem. Instability is measured as a distance to a degenerate *6.5-point curve* in the image plane.



The critical curve for 7 point algorithm, image form Fan et al. CVPR 2022

Luong and Faugeras. The fundamentalmatrix: Theory, algorithms, and stability analysis. IJCV, 1996

Fan, Kileel, Kimia. On the Instability of Relative Pose Estimation and RANSAC's Role CVPR 2022

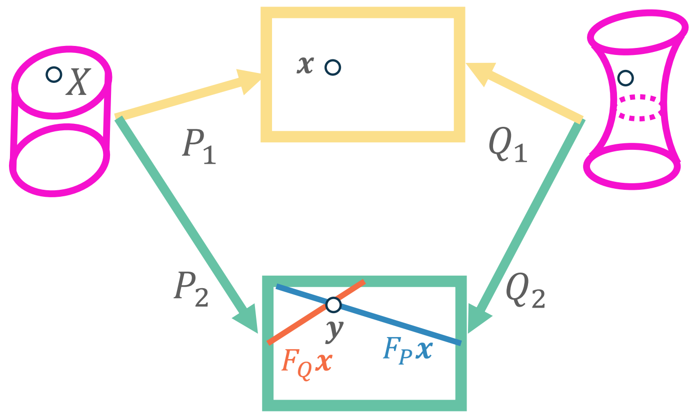
Luong-Faugeras Algorithm

Algorithm 2 Luong-Faugeras fitting of ψ

Require: $n \geq 7$ corresponding points $\{(\mathbf{x}_i, \mathbf{y}_i)\}_{i=1}^n$
Ensure: A quadratic transformation ψ

1: Compute fundamental matrices F_P^j from the corresponding points $(\mathbf{x}_i, \mathbf{y}_i)$ using the 7-point algorithm (up to 3 possible solutions F_P^1, F_P^2, F_P^3)
2: **for** each F_P^j **do**
3: Construct the 7×9 matrix A such that the i-th row A_i is given by $A_i = \mathbf{x}_i^\top \otimes [\mathbf{y}_i]_\times [F_P^j \mathbf{x}_i]_\times$
4: Solve for F_Q^j such that $A\text{vec}(F_Q^j) = 0$
5: Refine F_Q^j by minimizing the error in Eq. (4)
6: **end for**
7: Choose F_P, F_Q corresponding to $\min e_j$.
8: $\psi(\mathbf{x}) = F_P\mathbf{x} \times F_Q\mathbf{x}$.

Idea: given F_P, compute F_Q



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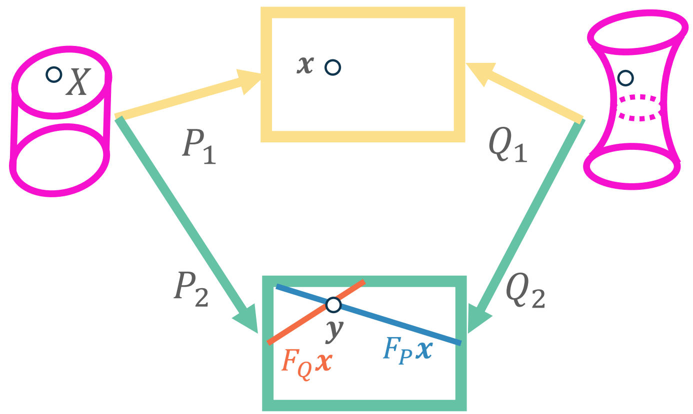


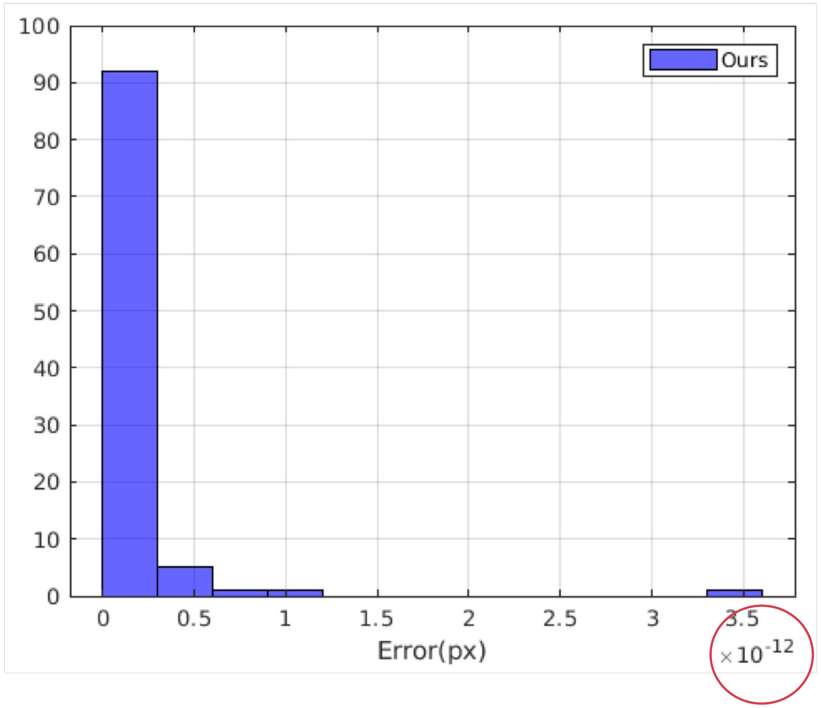
Needs F_P **in advance**: the very object that is ill-conditioned near
Plus a non-linear, iterative optimization.
Our method needs no F and is linear.

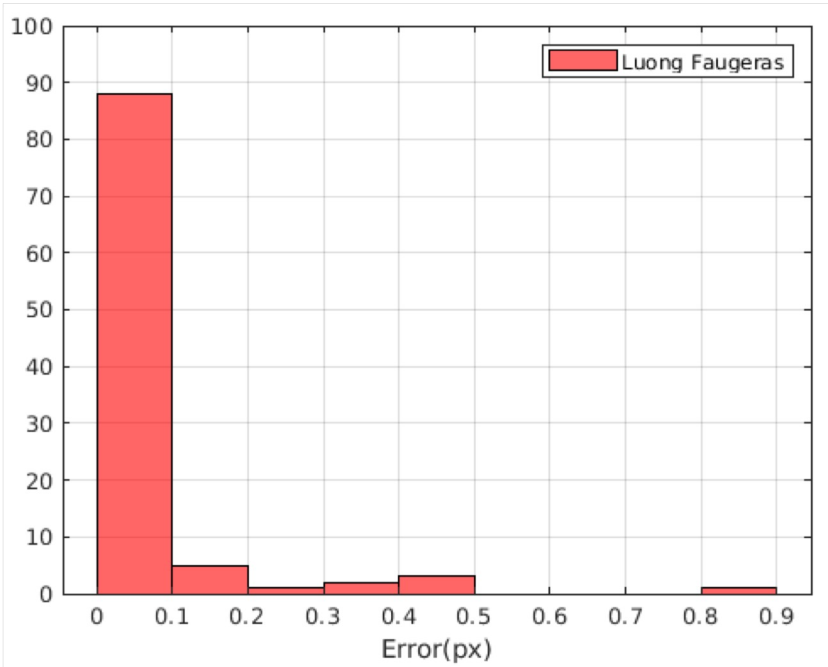
Stability & sensitivity

Random critical hyperboloids, 100 trials. No noise.

ours $\in [10^{-16}, 7\times10^{-12}]$ (median 9×10^{-15});

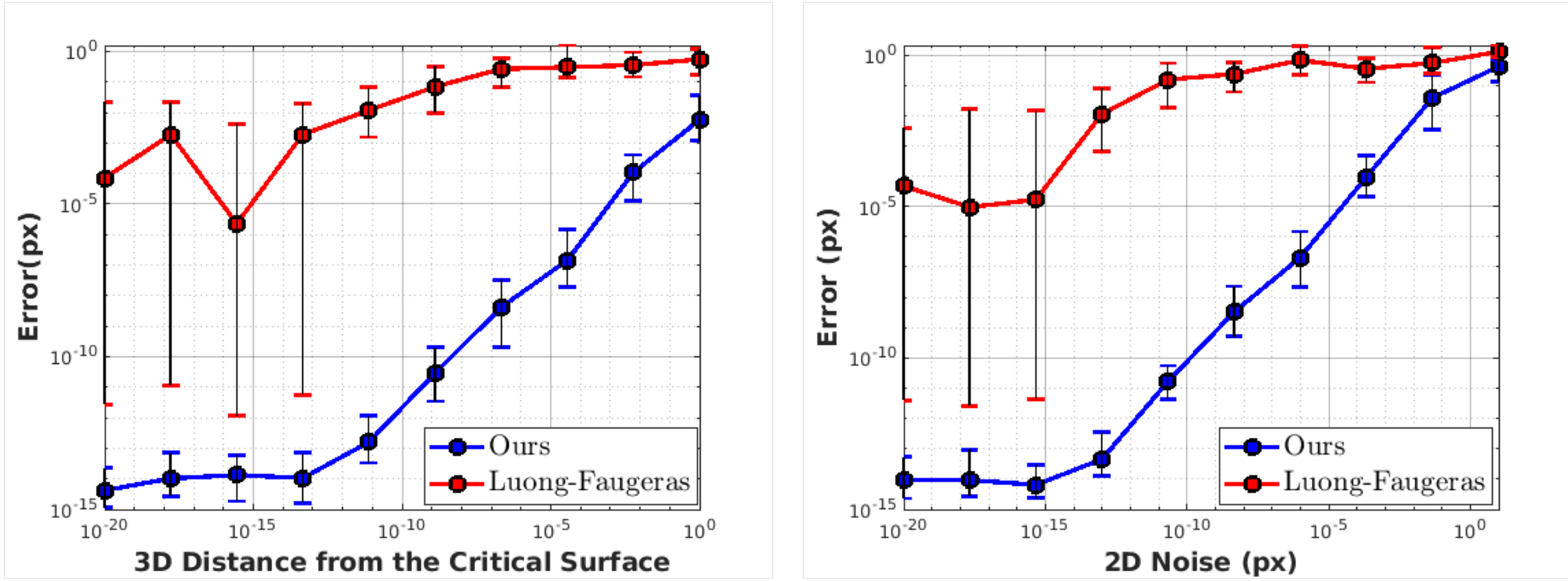
$LF \in [6\times10^{-15}, 0.96]$ (median 3×10^{-5}).





Stability & sensitivity

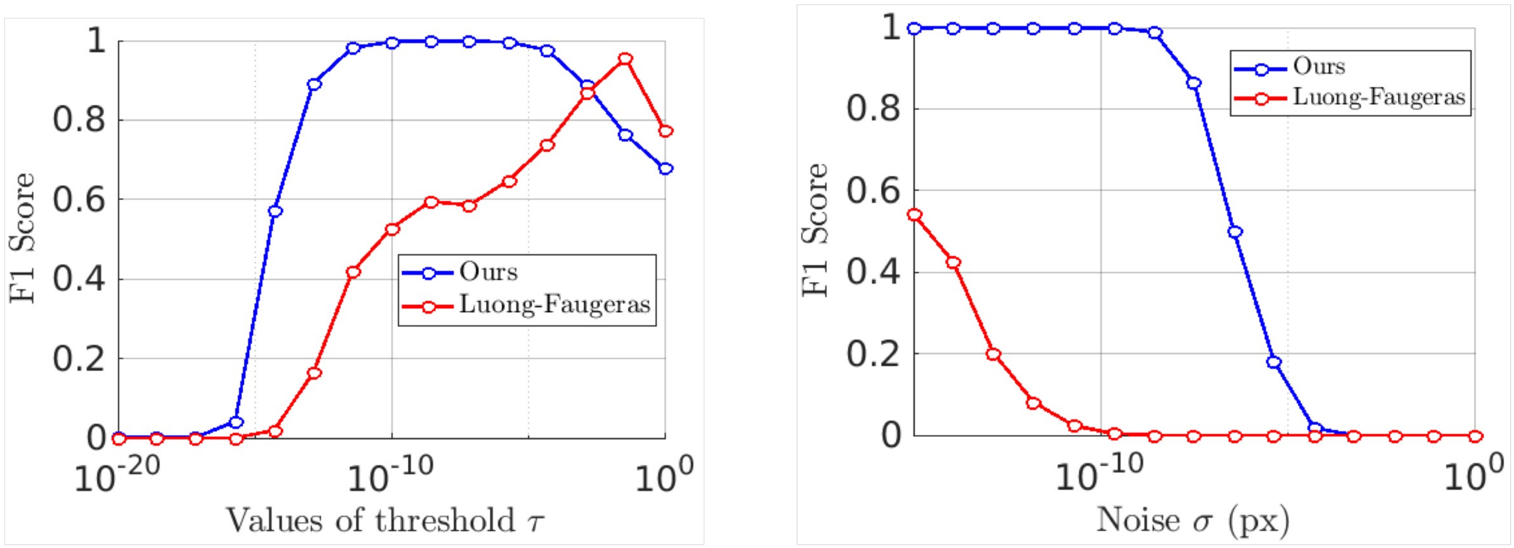
Sensitivity to 3D distance θ and to 2D noise σ (log-spaced):
ours degrades gracefully, LF is unstable even for tiny deviations.



(a) 3D noise θ (b) 2D noise

Degeneracy detection

- Binary classification, 500 critical vs. 500 non-critical point sets.
- Ours reaches $\boldsymbol{F1} = \mathbf{1}$ across a wide range of thresholds τ, both noiseless and under increasing noise.
- LF never reaches F1 = 1 (its reliance on F propagates instability).

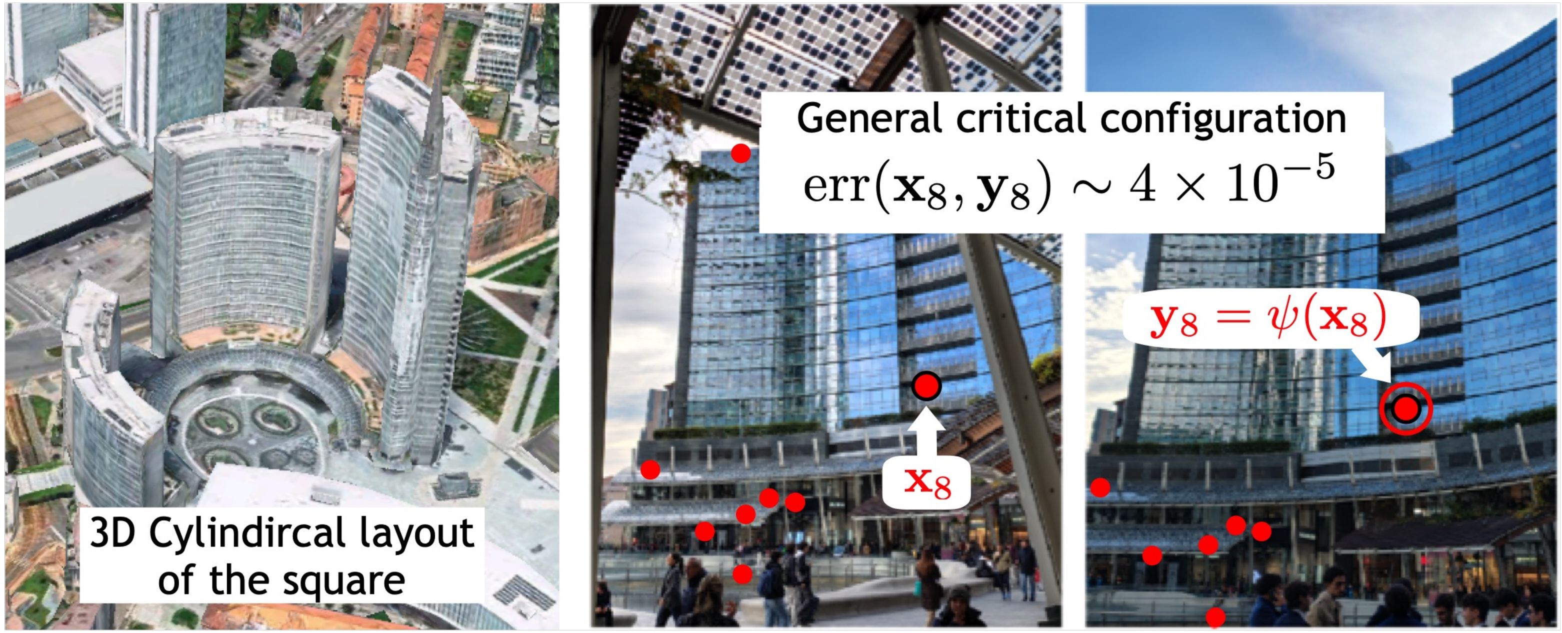


(a) F1 score vs varying thresholds (b) F1 score vs increasing noise

Real world degeneracy

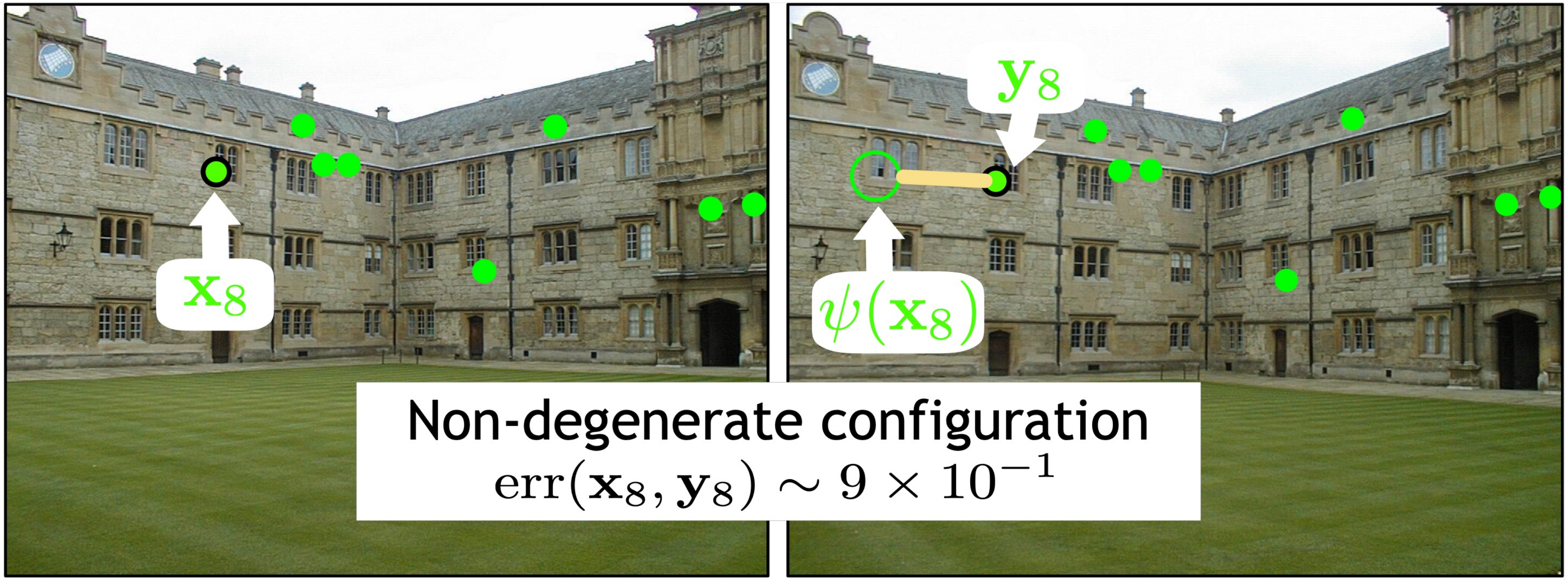
A square with cylindrical façade layout, viewpoints near the surrounding walls
$\Longrightarrow$ points and centers near the critical surface.
Test on a random 8-tuple: $\psi(x_8)$ matches $y^8, err \approx 4\times10^{-5}px$

The configuration is correctly flagged critical.



The planar case

Also validated on a planar scene, correctly detected as degenerate when all the matches come from the very same plane.



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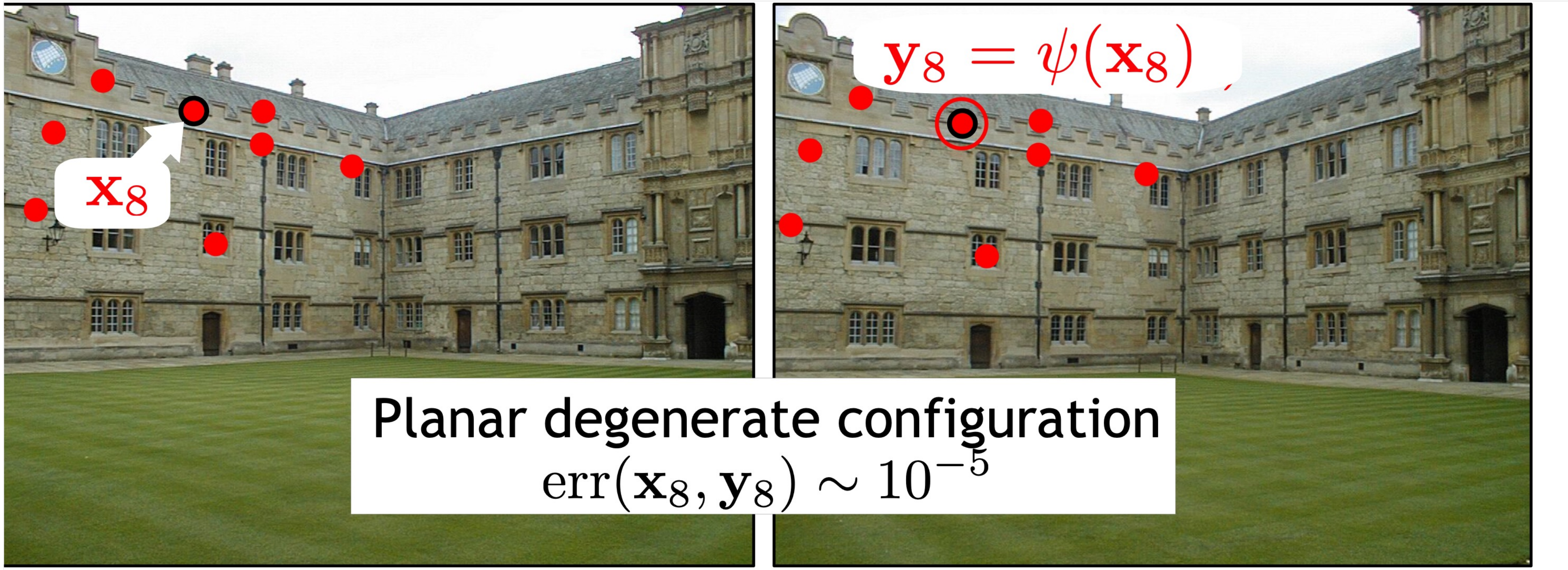

Conclusions

Geometry rules!

All in all, moving from **quadric in 3D** to a **homaloidal map in 2D.**
That buys the test:

- Linear / cheap
- Stable (no fundamental matrix needed)
- General (handle planar degeneracy as well)

Future work:

- Embed it in robust estimation framework
- After flagging criticality / recover usable geometry (epipoles, scene structure)
- Bridge the non minimal case

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