A Survey on Distance Measures and Matching for Discrete Geometric Shapes

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Overview

1. **Shape Matching Introduction**
   Geometric shapes: points, line segments, curves, geometric graphs, …

2. **Distance Measures for Geometric Shapes**
   Hausdorff, Fréchet, graph distances, …

3. **Matching Shapes under Transformations: Exact**
   Configuration space approaches, lower bounds, …

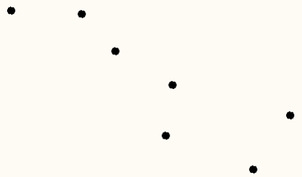
4. **Matching Shapes under Transformations: Approximate**
   Reference points, …

5. **Inverse Problem: Fréchet Realizability**
   Distance geometry, …

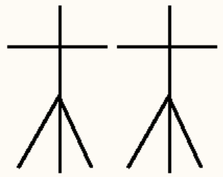
1. **Shape Matching Introduction**
   Geometric shapes: points, line segments, curves, geometric graphs, …

Discrete Geometric Shapes & Shape Matching

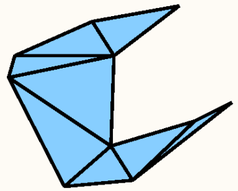
Discrete Geometric Shapes



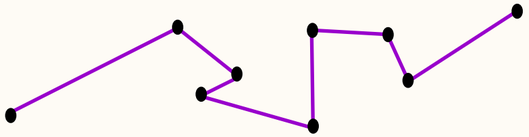
points

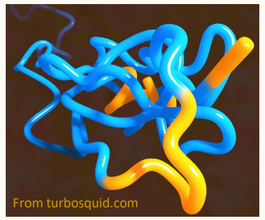


line segments

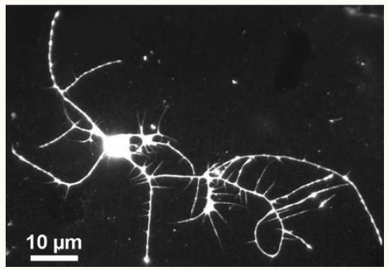


triangles

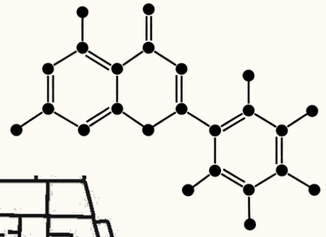


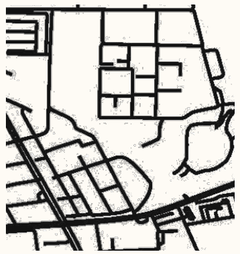


curves (GPS trajectories, protein chains)



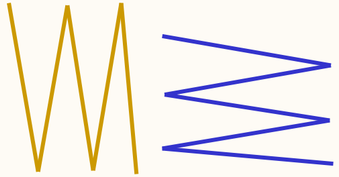
geometric trees (plant morphology, neuron)



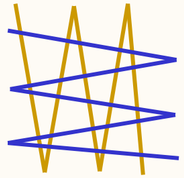


geometric graphs

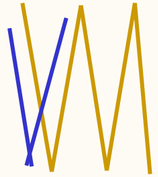
How Similar are Two Shapes?



Transformations (translations, rotations)



Choice of distance measure



Full or partial matching

Shape Matching

Let $\mathcal{T}$ be a set (or group) of transformations. E.g.:
- translations
- rigid motions (= translations & rotations)
- isometries (= rigid motions and reflections)

Let $\mathcal{S}$ be a set (or group) of shapes.
Let d be a metric or distance on $\mathcal{S}$.

Then given two shapes $A, B \in \mathcal{S}$, the **shape matching problem** is to find an optimal transformation $T^* \in \mathcal{T}$ such that:

$$d(T^*(A), B) = \min_{T \in \mathcal{T}} d(T(A), B)$$

2. **Distance Measures for Geometric Shapes**
   Hausdorff, Fréchet, graph distances, …

Hausdorff Distance for Compact Sets

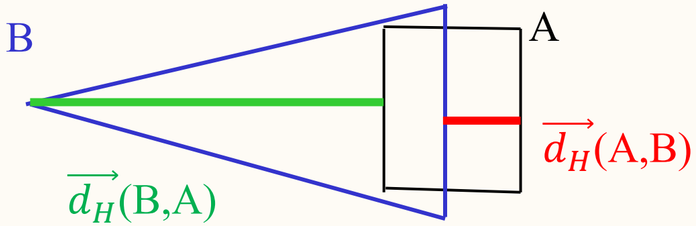
Let $A, B \in \mathbb{R}^D$ be two **compact sets**, and let d be a point metric on $\mathbb{R}^D$.

- The directed Hausdorff distance is:

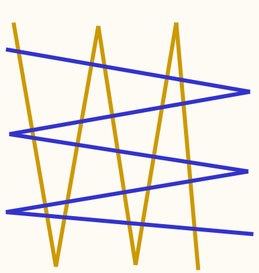
$$\overrightarrow{d_H}(A,B) = \max_{a \in A} \min_{b \in B} d(a,b)$$

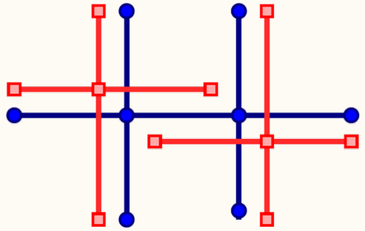
- The undirected Hausdorff distance (metric) is:

$$\mathrm{d}_H(A,B) = \max(\overrightarrow{d_H}(A,B), \overleftarrow{d_H}(A,B))$$



Con: Does not take continuity of curves or topology of graphs into account





Runtimes: (for shapes A, B of complexity m, n; $\mathrm{N} = \max(\mathrm{m}, \mathrm{n})$):

- For point sets in $\mathbb{R}^2$: $O(N \log N)$ via nearest neighbor search in Voronoi diagrams
- For point sets in $\mathbb{R}^D$: $O(N^2)$ trivial
- For sets of line segments in $\mathbb{R}^2$: $O(N \log N)$ [ABB95]
- For sets of k-simplices in $\mathbb{R}^D$: $O\left(N^{k+1+\varepsilon}\right)$ expected, for any $\varepsilon > 0$ [ABGKW03]

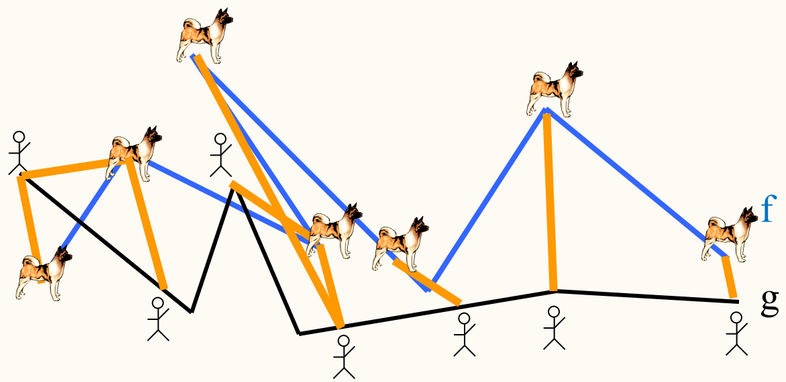
[ABB95] H. Alt, B. Behrends and J. Blömer. Approximate matching of polygonal shapes. Ann. Math. Artif. Intell. 13:251–266, 1995.
[ABGKW03] H. Alt, P. Braß, M. Godau, C. Knauer, C. Wenk), Computing the Hausdorff Distance of Geometric Patterns and Shapes, Disc. & Comp. Geom., 65-76, 2003..

Fréchet Distance for Curves

Let $f, g\colon [0,1] \to \mathbb{R}^D$ be two **curves**. The Fréchet distance is:

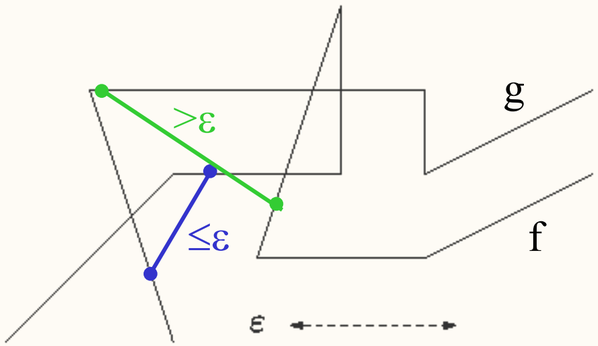
$$\mathrm{d}_F(f,g) = \inf_{\alpha,\beta:[0,1]\mapsto[0,1]} \max_{t\in[0,1]} d\big(f(\alpha(t)), g(\beta(t))\big),$$

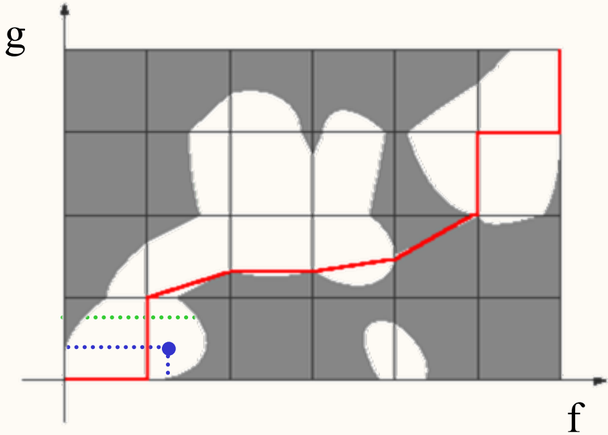
where α, β range over continuous monotone increasing reparameterizations only. (Generalizes to oriented manifolds using orientation-preserving homeomorphisms).



- Man and dog walk on one curve each
- They hold each other at a leash
- They are only allowed to go forward
- d_F is the minimal possible leash length

Free space





- Let $\varepsilon > 0$ (eventually solve decision problem)
- $F_\varepsilon(f,g) = \{(s,t) \in [0,1]^2 : d\big(f(s), g(t)\big) \le \varepsilon\}$ *is the free space of f and g (white points)*
- *The free space in one cell is an ellipse*

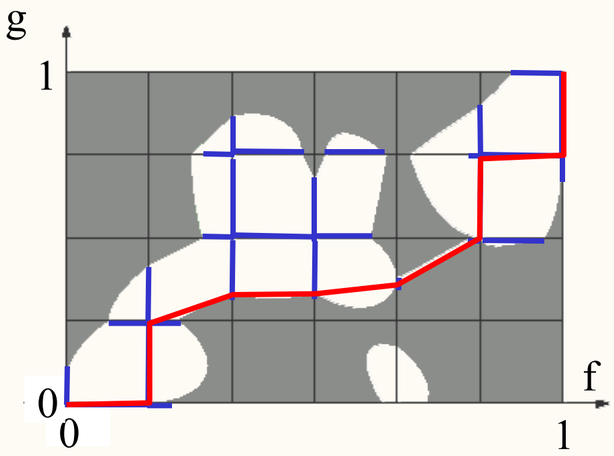
- Monotone path encodes reparameterizations of f, g
- $d_F(f,g) \le \varepsilon$ *iff there is a monotone path in the free space from* $(0,0)$ *to* $(1,1)$.

[FO6] M. Fréchet, Sur quelques points de calcul fonctionel, Rendiconti del Circolo Mathematico di Palermo 22: 1–74, 1906.

Computing the Fréchet Distance

Solve the decision problem
$d_F(f,g) \leq \varepsilon$ in $O(mn)$ time:

- Find monotone path using DP:
- On each cell boundary compute the interval of all points that are reachable by a monotone path from $(0,0)$
- Compute a monotone path by backtracking



Solve the optimization problem

- In practice in O(mn log b) time with binary search and b-bit precision
- In O(mn log mn) time [AG95] using parametric search (using Cole's sorting trick)

[AG95] H. Alt, M. Godau, Computing the Fréchet distance between two polygonal curves, IJCGA 5: 75-91, 1995

$d_H(f,g) \le d_{wF}(f,g) \le d_F(f,g) \le d_{dF}(f,g)$

Fréchet Variants

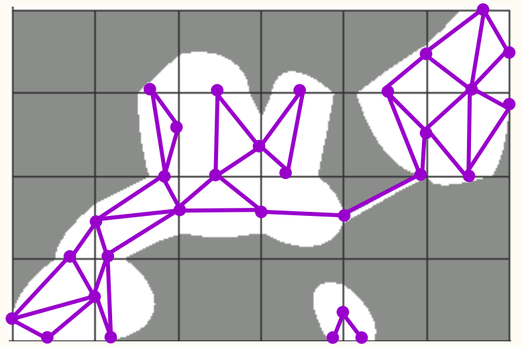
Weak Fréchet Distance $d_{wF}(f,g)$:

Allow any continuous reparameterizations α and β
$\Rightarrow$ Any continuous path in free space (not necessarily monotone)

Use free space graph to encode free space connectivity.

For each cell boundary store locally optimal ε as its weight

- Weak Fréchet distance = maximum of all ε's along an optimal path in the free space graph
- MST/Dijkstra's runs in $O(N^2)$ (for planar graphs)

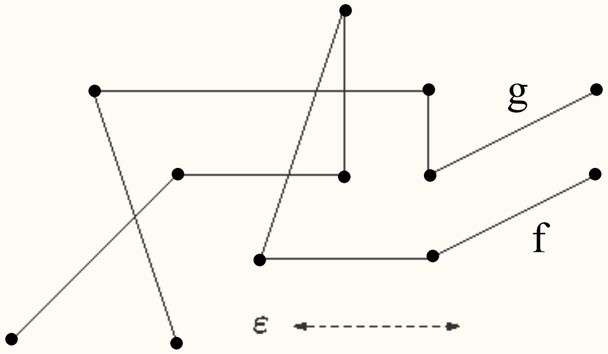


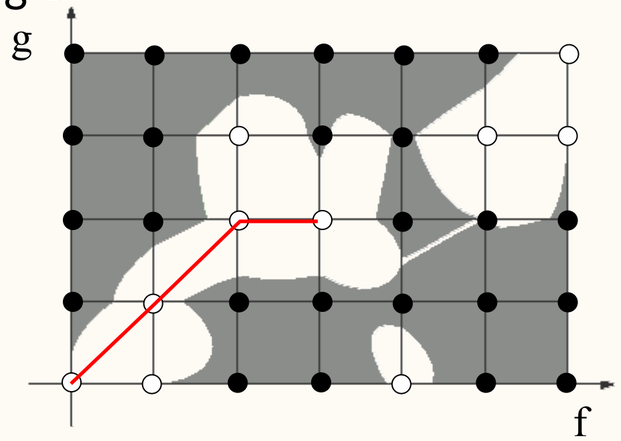
Discrete Fréchet Distance $d_{dF}(f,g)$:

Curves = sequences of discrete points

Allow only monotone assignments between points

Compute in $O(N^2)$ time using DP.





Integral Fréchet Distance = Continuous Dynamic Time Warping Distance $d_{CDTW}(f,g)$:

Replaces the max with a path integral over $(\alpha(t),\ \beta(t))$

- Less affected by outliers; but no triangle inequality.
- Hard to compute exactly for L_2 (transcendentals)
- Poly-time $(5+\varepsilon)$-approx. for $L_2, \mathbb{R}^2$ [BBSW26]
- Poly-time 5 -approx. for $L_1, \mathbb{R}^2$ [BBSW26]

[EM94] Eiter, Mannila, Computing discrete Fréchet distance, technical report, Technical University Vienna, 1994.
[B07] M. Buchin, On the Computability of the Frechet Distance Between Triangulated Surfaces, PhD thesis, Free University Berlin, 2007.
[BBSW26] K. Buchin, M. Buchin, J.E. Swiadek, S. Wong, A Constant-Factor Approximation for Continuous Dynamic Time Warping in 2D. ICALP: 47:1-47:22, 2026.

Fréchet-Like Distances for Graphs I

Geometric graph $G = (V_G, E_G, \Phi_G)$ with continuous $\Phi_G: \mathrm{G} \to \mathbb{R}^{\mathrm{D}}$ that is homeomorphic (embedded graph) or locally homeomorphic (immersed graph) onto its image.

Fréchet distance $d_F(G,H)$

If G, H are homeomorphic, apply Fréchet distance definition on Φ_G , Φ_H, but drop „orientation-preserving" requirement. Equivalent definition:

$$\mathrm{d}_F(G,H) = \inf_{\alpha} \max_{e \in E_G} d_F(e, \alpha(e)),$$

where α ranges over all edge mappings corresponding to homeomorphisms between G, H.

- Is graph-isomorphism hard. Can be computed in poly time for trees and for graphs of bounded tree-width. [BKN20]
- For planar graphs, can enumerate orientation-preserving homeomorphisms in polynomial time. [FW21]

Path-based distance [AFHW14]

Directed Hausdorff distance on path sets

$$d_{PD}(G,H) = \overrightarrow{d_H}(\pi_G, \pi_H) = \max_{p_G \in \pi_G} \min_{p_H \in \pi_H} d_F(p_G, p_H),$$

where π_G, π_H are path sets (potentially all paths)

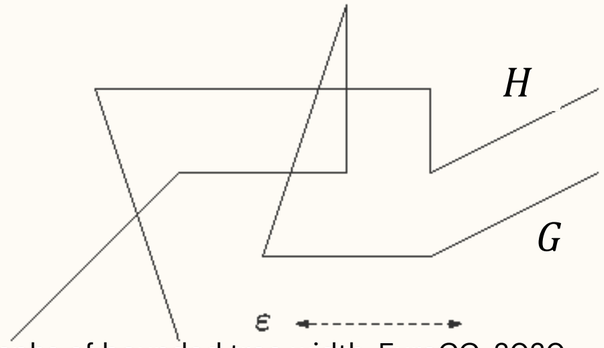
Contour-tree distance [BOS17]

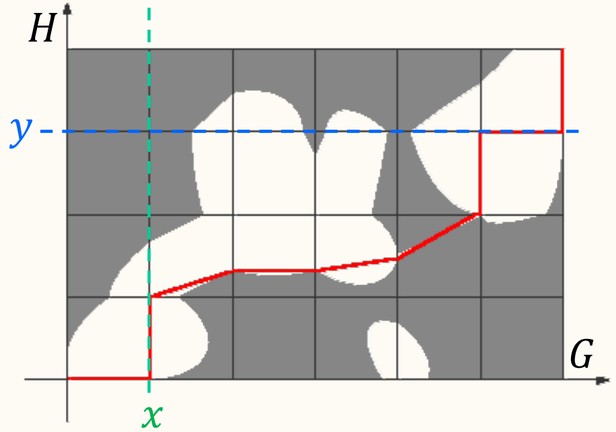
$$d_C(G,H) = \inf_{\tau \subseteq G \times H} \sup_{(x,y) \in \tau} d(x,y),$$

where τ are correspondences between G, H s.t.:

- For all $x \in G$: $\tau \cap (\{x\} \times H) \neq \emptyset$ and connected
- For all $y \in H$: $\tau \cap (G \times \{y\}) \neq \emptyset$ and connected

Is a metric. But NP-complete already for trees.





[FW21] P. Fang, C. Wenk. The Fréchet distance for plane graphs. EuroCG 21.
[BKN20] M. Buchin, A. Krivosija, A. Neuhaus. Computing the Fréchet distance of trees and graphs of bounded tree width. EuroCG. 2020
[AFHW14] M. Ahmed, B. Fasy, K. Hickmann, C. Wenk, Path-based distance for street map comparison, TSAS 1(1): article 3, 28 pages, 2015.
[BOS17] K. Buchin, T. Ophelders, B. Speckmann. Computing the Fréchet distance between real-valued surfaces. SODA: 2443–2455, 2017.
[BCFFGMW23] M. Buchin, E. Chambers, P. Fang, B.T. Fasy, E. Gasparovic, E. Munch, C. Wenk, Distances Between Immersed Graphs: Metric Properties, LaMatematica 2: 197-222, 2023.

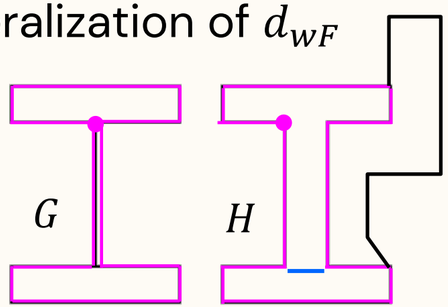
Fréchet-Like Distances for Graphs II

Traversal distance [AERWO3]

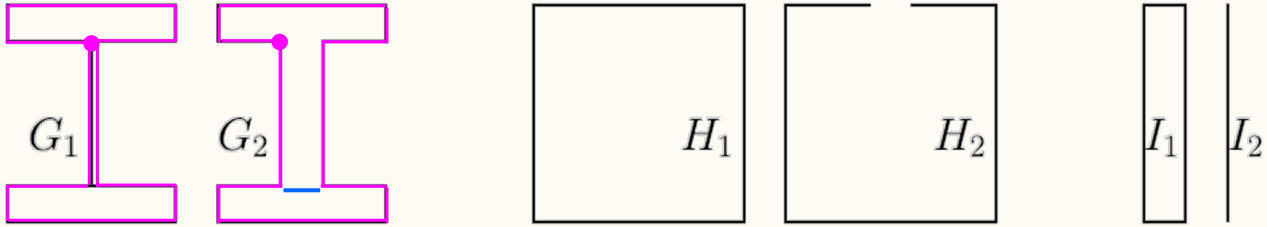
$d_T(G,H) = \inf_{g,h} \max_{t\in[0,1]} d(g(t),h(t))$, where

g: $[0,1] \to G$ are traversals (continuous, surjective), and h: $[0,1] \to H$ partial traversals (continuous).

- Compute using free space diagram ($G \times H$) in $O(N^2 \log N)$
- Generalization of d_{wF}



Small traversal distance



- Small traversal distance

Strong (and weak) graph distances [ABKSW21]

$d_G(G,H) = \inf_{s:G\to H} \max_{e\in E_G} d_F(e,s(e))$, where

graph mappings s: $G \to H$ are continuous maps, i.e.:

- send each $v \in V_G$ to a point $s(v) \in H$, and
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Weak graph distance uses d_{wF} instead.

[AERWO3] H. Alt, A. Efrat, G. Rote, C. Wenk, Matching Planar Maps, Journal of Algorithms 49: 262-283, 2003.
[ABKSW21] H.A. Akitaya, M. Buchin, B. Kilgus, S. Sijben, C. Wenk, Distance Measures for Embedded Graphs, CGTA 95, 101743, 2021.
[BKK26] M. Buchin, Wolf Kißler, F. Kubon, Hardness and Parameterized Tractability of the Weak Graph Distance. WALCOM 2026: 64-78.
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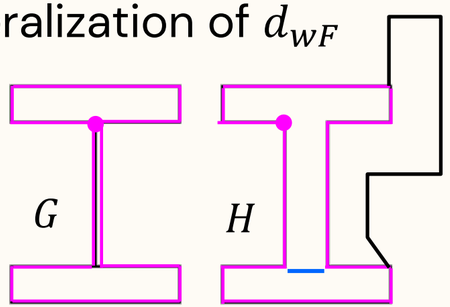

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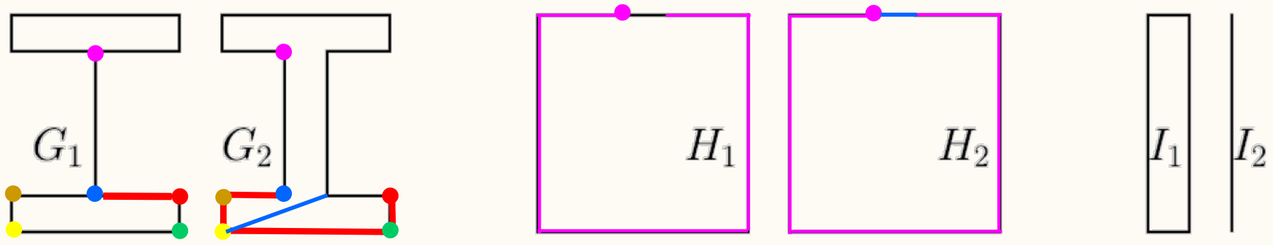

- Small traversal distance
- Large graph distance

- Small traversal distance

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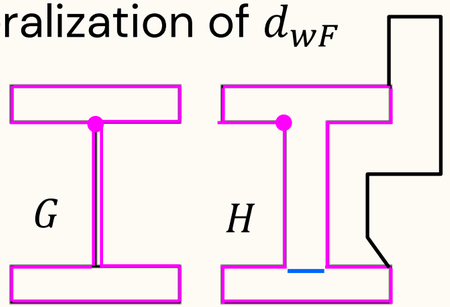

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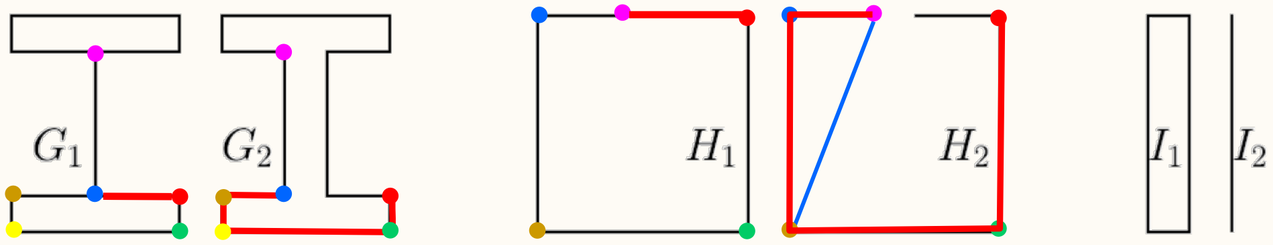

- Small traversal distance
- Large graph distance

- Small traversal distance
- Large graph distance

- Both small

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- send each $e \in E_G$ to a simple path from $s(u)$ to $s(v)$ in H

Weak graph distance uses d_{wF} instead.

- We have: $d_T(G,H) \le d_{wG}(G,H) \le d_G(G,H)$
- Strong distance is NP-hard, but poly time for trees.
- Weak distance is poly time for (spike-free) plane graphs

[AERWO3] H. Alt, A. Efrat, G. Rote, C. Wenk, Matching Planar Maps, Journal of Algorithms 49: 262-283, 2003.
[ABKSW21] H.A. Akitaya, M. Buchin, B. Kilgus, S. Sijben, C. Wenk, Distance Measures for Embedded Graphs, CGTA 95, 101743, 2021.
[BKK26] M. Buchin, Wolf Kißler, F. Kubon, Hardness and Parameterized Tractability of the Weak Graph Distance. WALCOM 2026: 64-78.
[BCFFGMW23] M. Buchin, E. Chambers, P. Fang, B.T. Fasy, E. Gasparovic, E. Munch, C. Wenk, Distances Between Immersed Graphs: Metric Properties, LaMatematica 2: 197-222, 2023.

3. **Matching Shapes under Transformations: Exact**
   Configuration space approaches, lower bounds, …

Hausdorff Distance under Translations

Find a translation t^* such that

$$d_H(A + t^*, B) = \min_{t \in \mathbb{R}^D} d_H(A + t, B)$$

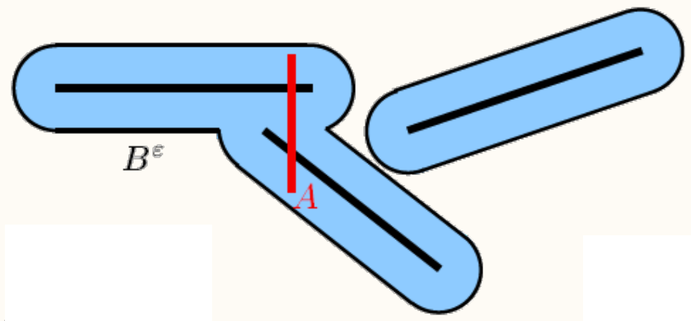
For the directed Hausdorff distance:

$$\overrightarrow{d_H}(A + t^*, B) = \min_{t \in \mathbb{R}^D} \overrightarrow{d_H}(A + t, B)$$

Configuration space approach: Answer the **decision problem for given ε** by constructing the set of valid translations.

Ex.: A, B sets of segments in $\mathbb{R}^2$

$$\vec{\delta}_{\mathrm{H}}(A, B) \leq \varepsilon \iff A \subseteq B^\varepsilon$$



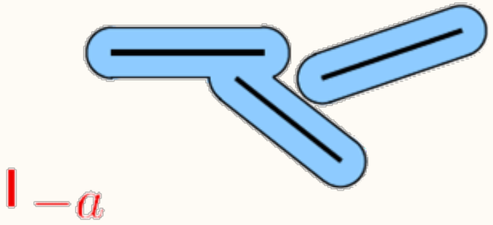
Configuration space lemma: $\vec{\delta}_{\mathrm{H}}(A + t, B) \leq \varepsilon \iff t \in \bigcap_{a \in A} \overline{\overline{B^\varepsilon} \oplus (-a)}$

Proof: $\vec{\delta}_{\mathrm{H}}(A + t, B) \leq \varepsilon \Leftrightarrow A + t \subseteq B^\varepsilon$

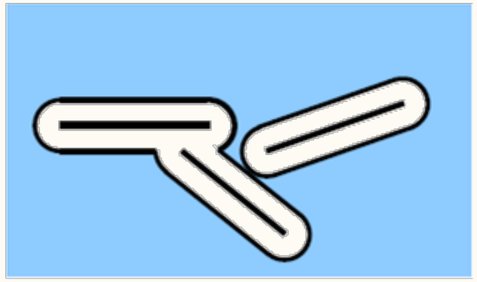
$$\Leftrightarrow \bigwedge_{a \in A} \bigwedge_{x \in a} x + t \in B^\varepsilon \Leftrightarrow t \in \bigcap_{a \in A} \bigcap_{x \in a} B^\varepsilon - x$$

$$\Leftrightarrow t \in \bigcup_{a \in A} \bigcup_{x \in a} \overline{B^\varepsilon} - x \Leftrightarrow t \in \bigcap_{a \in A} \overline{\overline{B^\varepsilon} \oplus (-a)}$$

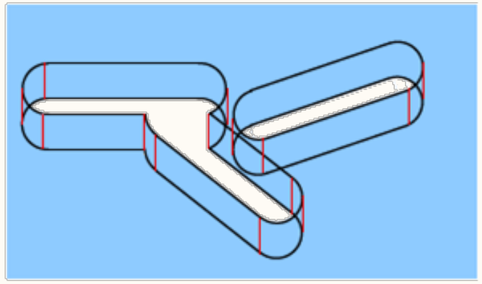
B^ε



$\overline{B^\varepsilon}$



$\overline{B^\varepsilon} \oplus (-a)$



$\overline{\overline{B^\varepsilon} \oplus (-a)}$

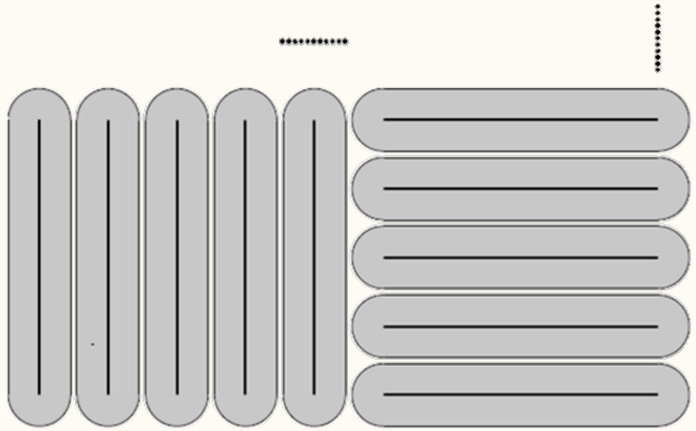
Hausdorff Distance under Transformations

Transformations	Metric	Points	Segments	
Rigid motions	L_2	$O(m^2n^2N\log(mn))$	$O(m^3n^3\log(mn))$	[CGHKKK97]
	L_2	$\Omega(N^5)$	$\Omega(N^6)^\dagger$	[R96]
Similarities	L_2, L_∞	random. expected for convex polygons: $O(n+m)$		[A94]
2 scaling factors	L_2	$\Omega(N^7)$	$\Omega(N^8)^\dagger$	[R96]
2 scaling factors	L_∞	$\Omega(N^7)^\dagger$	$\Omega(N^8)^\dagger$	[R96]
Affine maps	L_2, L_∞	$\Omega(N^9)$	$\Omega(N^{12})^\dagger$	[R96]

$\dagger$ Only for $\overrightarrow{d_H}$

Configuration space

$$\bigcap_{a\in A} \overline{\overline{B^\varepsilon} \oplus (-a)}$$



Translations

Dim.	Metric	Points	Segments
$\mathbb{R}^1$	–	$O((m+n)\log(m+n))$ [R96]	–
$\mathbb{R}^2$	L_∞	$O(nm\log^2(nm))$ [CK92]	$O(m^2n^2\alpha(mn))$ [HKS93]
	L_2	$O(nm(n+m)\log(nm))$ [HKS93]	$O(m^2n^2\log^3(mn)$ [AST94]
	L_p	$\Omega(N^3)$ [R96] $\Omega(N^{2-\delta})$ cond. [BN21]	$\Omega(N^4)$ [R96] 3-SUM hard, conj. $\Omega(N^2)$ [BHO1]
$\mathbb{R}^3$	L_∞	$O(n^3\log^2 n)$ [CDEK99]	–
	L_2	$O(n^2m^2(n+m)^{1+\delta})$ [HKS93]	–
$\mathbb{R}^d$	L_∞	$O(N^{\frac{4d-2}{3}}\log^2 N)$ [CDEK99]	–
		$\Omega(N^{\lfloor\frac{3d}{2}\rfloor})$ [R96]	–
	L_2	$O(N^{\lceil\frac{3d}{2}\rceil+1}\log^3 N)$ [CDEK99]	–

[BHO1] G. Barequet, S. Har-Peled, Polygon Containment and Translational Min-Hausdorff-Distance Between Segment Sets are 3SUM-Hard, IJCGA 11(4): 465-474, 2001
[BN21] K. Bringmann, A. Nusser, Translating Hausdorff is hard: fine-grained lower bounds for Hausdorff distance under translation. J. Comput. Geom. 13(2): 30-50 (2021)
[AST94] P. K. Agarwal, M. Sharir, S. Toledo. Applications of parametric searching in geometric optimization. Journal of Algorithms 17:292–318, 1994
[CDEK99] L. P. Chew, D. Dor, A. Efrat and K. Kedem. Geometric pattern matching in d-dimensional space. DCG 21:257–274, 1999.
[A94] N. Amenta. Bounded boxes, …. SoCG 1994, 340–347.
[CGHKKK97] Chew, Goodrich, Huttenlocher, Kedem, Kleinberg & Kravets. Geometric pattern matching under Euclidean motion. CGTA 7:113-124, 1997.
[R96] W. J. Rucklidge. Lower bounds…, DCG 16:135–153, 1996.
[CK92] L. P. Chew, K. Kedem. Improvements on geometric…, SWAT 1992.
[HKS93] Huttenlocher, Kedem & Sharir. The upper envelope of Voronoi surfaces and its applications. DCG 9:267–291, 1993.

Hausdorff Distance under Translations

A: m points, B: n polyhedral sites

Metric	Dim.	Sites	Complexity	Runtime
polyh.	$d = 3$	line segments	$O(m^3 n^2 \log n)$, $\Omega(m^3 n^2)^\dagger$	$O(m^4 n^2 \log^2 n)$
polyh.	$d = 3$	triangles	$\theta(m^3 n^3)$	$O(m^4 n^3 \log n)$
L_2	$d = 3$	line segments	$O_\delta(m^3 n^2)$, $\Omega(m^3 n^2)$	$O_\delta(m^4 n^2)$
polyh.	$d \geq 4$	simplices	$\theta(m^d n^d)$	$O(m^{d+1} n^d \log n)$
convex	$d \geq 2$	$(d-1)$-simplices	$\Omega(m^d n^d)$	–
L_∞, L_2	$d \geq 2$	k-simplices	$\Omega(m^d n^{k+1})$	–

† Only for L_∞

A, B: both polyhedral sites, and polyhedral metric

Problem	Dim.	Complexity and Runtime
decision	$d = 3$	$O(N^9 \log^3 N)$
decision	$d \geq 4$	$O(N^{d^2+d})$
opt.	$d \geq 3$	$O(N^{d^2+d} \log^2 N)$

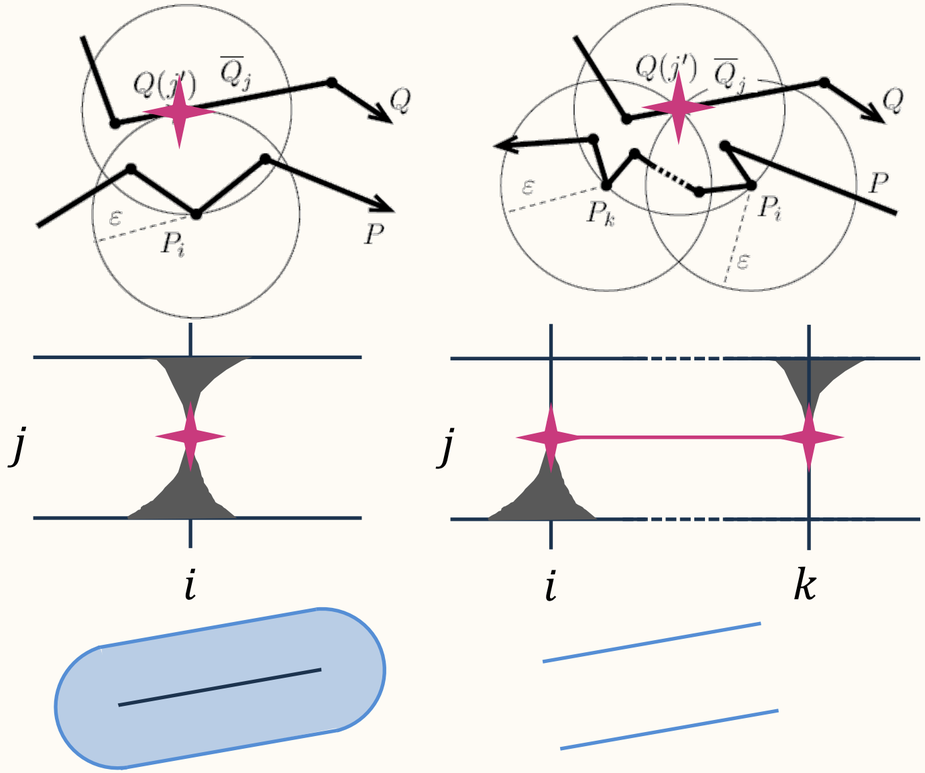
[W02] C. Wenk, Shape Matching in Higher Dimensions, PhD thesis, Freie Universität Berlin, 2002.

Fréchet Distance under Transformations

(Strong) Fréchet

Transformations	Dim.	DOF	Time	Space
Translations	d	d	$O(N^{3d+2}\log N)$	$O(N^{3d})$
Transformations	d	k	$O(N^{3k+2}\log N)$	$O(N^{3k})$
Affine Transf.	d	d^2+d	$O(N^{3(d^2+d)+2}\log N)$	$O(N^{3(d^2+d)})$
Scalings	d	1	$O(N^5\log N)$	$O(N^3)$
Rotations	2	1	$O(N^5\log N)$	$O(N^3)$
Rotations	3	3	$O(N^{11}\log N)$	$O(N^9)$
Rigid Motions	2	3	$O(N^{11}\log N)$	$O(N^9)$
Rigid Motions	3	6	$O(N^{20}\log N)$	$O(N^{18})$
Similarities	2	4	$O(N^{14}\log N)$	$O(N^{12})$
Similarities	3	7	$O(N^{23}\log N)$	$O(N^{21})$
Translations	2	2	$O(N^8\log N)$	$O(N^8)$ [AKW03]

Weak Fréchet or discrete Fréchet:
$O(N^{2k+2}\log N)$



[W02] C. Wenk, Shape Matching in Higher Dimensions, PhD thesis, Freie Universität Berlin, 2002.
[AKW03] H. Alt, C. Knauer and C. Wenk. Comparison of distance measures for planar curves. Algorithmica, 2003.
[BKN21] K. Bringmann, M. Künnemann, A. Nusser, Discrete Fréchet Distance under Translation: Conditional Hardness and an Improved Algorithm, ACM Trans. Algorithms 17(3): 25:1-25:42, 2021
[BBHNW25] K. Buchin, M. Buchin, Z. Huang, A. Nusser, S. Wong: Faster Fréchet Distance Under Transformations. ICALP: 36:1-36:20, 2025.

Fréchet Distance under Transformations

(Strong) Fréchet

Transformations	Dim.	DOF	Time	Space
Translations	d	d	$O(N^{3d+2} \log N)$	$O(N^{3d})$
Transformations	d	k	$O(N^{3k+2} \log N)$	$O(N^{3k})$
Affine Transf.	d	d^2+d	$O(N^{3(d^2+d)+2} \log N)$	$O(N^{3(d^2+d)})$
Scalings	d	1	$O(N^5 \log N)$	$O(N^3)$
Rotations	2	1	$O(N^5 \log N)$	$O(N^3)$
Rotations	3	3	$O(N^{11} \log N)$	$O(N^9)$
Rigid Motions	2	3	$O(N^{11} \log N)$	$O(N^9)$
Rigid Motions	3	6	$O(N^{20} \log N)$	$O(N^{18})$
Similarities	2	4	$O(N^{14} \log N)$	$O(N^{12})$
Similarities	3	7	$O(N^{23} \log N)$	$O(N^{21})$
Translations	2	2	$O(N^8 \log N)$	$O(N^8)$ [AKW03]
Translations	d	k	$O(N^{3k+4/3} \log N)$	$O(N^{3k})$ [BBHNW25]

Weak Fréchet or discrete Fréchet:
$O(N^{2k+2} \log N)$

Offline dynamic grid reachability

Discrete Fréchet:
$\tilde{O}(N^{4+2/3})$, $\Omega(n^{4-o(1)})$ cond. [BKN21]

[W02] C. Wenk, Shape Matching in Higher Dimensions, PhD thesis, Freie Universität Berlin, 2002.
[AKW03] H. Alt, C. Knauer and C. Wenk. Comparison of distance measures for planar curves. Algorithmica, 2003.
[BKN21] K. Bringmann, M. Künnemann, A. Nusser, Discrete Fréchet Distance under Translation: Conditional Hardness and an Improved Algorithm, ACM Trans. Algorithms 17(3): 25:1-25:42, 2021
[BBHNW25] K. Buchin, M. Buchin, Z. Huang, A. Nusser, S. Wong: Faster Fréchet Distance Under Transformations. ICALP: 36:1-36:20, 2025.

4. **Matching Shapes under Transformations: Approximate**
   Reference points, …

Reference Points

Definition (Approximate matching / Loss factor) *Let A, B be two shapes, δ a distance measure, and $\mathcal{T}$ a class of transformations. Then the* approximate matching problem *with* loss factor *a, for some $a > 1$, is to find a $T' \in \mathcal{T}$ such that*

$$\delta(T'(A), B) \leq a \cdot \min_{T \in \mathcal{T}} \delta(T(A), B) \ .$$

Definition (Reference point) *Let $\mathcal{C}^d$ be the set of all convex subsets of $\mathbb{R}^d$, δ a metric on $\mathcal{C}^d$, and $\mathcal{T}$ be a set of transformations on $\mathbb{R}^d$. Then a mapping $r : \mathcal{C}^d \to \mathbb{R}^d$ is called a* reference point with respect to $\mathcal{T}$ *if it has the following two properties:*

- **Equivariance:** $r(T(A)) = T(r(A))$ *for all $A \in \mathcal{C}^d$ and $T \in \mathcal{T}$*
- **Lipschitz continuity:** *There exists a $c \geq 0$ such that for all $A, B \in \mathcal{C}^d$*

$$||r(A) - r(b)|| \leq c \cdot \delta(A, B) \ .$$

The Lipschitz constant c is called the quality *of the reference point r.*

Approximate Hausdorff Matching

Segment or triangle sets; using Steiner point as reference point [AAR97]

Transformations	Dimension	Runtime	Loss Factor
Translations	2	$O((m+n)\log(m+n))$	$4/\pi+1$
	3	$O(H(m,n))$	2.5
Rigid motions	2	$O(mn\log(mn)\log^*(mn))$	$4/\pi+1$
	3	$O((mn)^3 H(m,n))$	2.5
Similarities	2	$O(mn\log(mn)\log^*(mn))$	$4/\pi+3$
	3	$O((mn)^3 H(m,n))$	4.5

Exact:

$\tilde{O}(n^4)$

$\tilde{O}(n^6)$
$\tilde{O}(n^9)$

Translations	2	$\tilde{O}(N \log N)$ with indecision interval	[CKS94]
Rigid motion	2	$\tilde{O}(N^2 \log N)$ with indecision interval	[CKS94]

Point sets [GMO99]

Transformations	Dimension	Runtime	Loss Factor
Translations	2	$O(nm\log n)$	2
	d	$O(nm\log n)$	$2+\varepsilon$
Rigid motions	2	$O(n^2 m\log n)$	4
	3	$O(n^3 m\log n))$	$8+\varepsilon$
Rotations	3	$O(n^2 m\log n)$	$4+\varepsilon$

Exact:

$\tilde{O}(n^3)$; or $\tilde{O}(n^2)$ for L_∞

$\tilde{O}(n^5)$

Translations	d	$\tilde{O}(N \log N)$ Monte Carlo projections, w.h.p	[CS98]
Rigid motion	d	$\tilde{O}(N^d \log N)$ Monte Carlo projections, w.h.p	[CS98]

[AAR97] Alt, Aichholzer & Rote. Matching shapes with a reference point. IJCGA 7:349–363, 1997.
[GMO99] Goodrich, Mitchell & Orletsky. Approximate geometric pattern matching under rigid motions. PAMI 1999, 371–379.
[CKS94] L. P. Chew, K. Kedem, S. Schirra , On characteristic points and approximate decision algorithms for the minimum Hausdorff distance. Tech report MPI-I-94-150,, 1994
[CS98] D.E. Cardoze, L.J. Schulman, Pattern Matching for Spatial Point Sets. FOCS: 156-165, 1998.

(Approximate) Matching for Curves and Graphs

Approximate Matching for Curves / Fréchet

- Trivial approximation for translations:
  Mapping start points of curves onto each other gives a 2-approximation

- Reference points for d_H are automatically reference points for d_F, since $d_H \leq d_F$

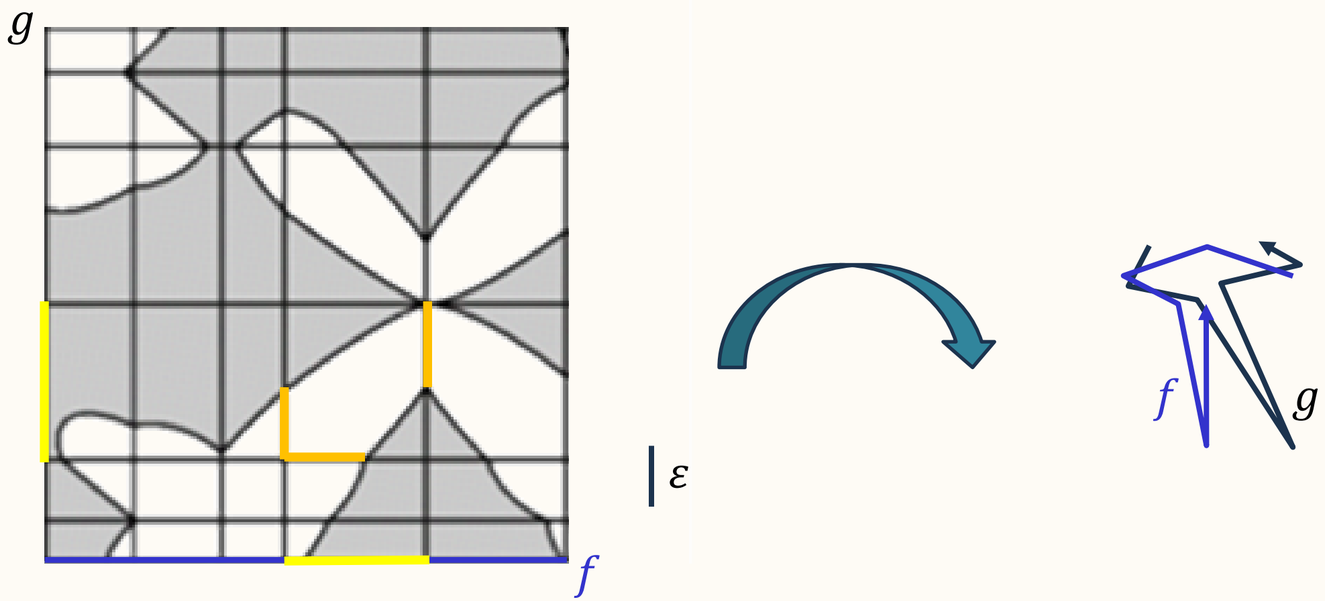
Graphs: Open

- Graph distances are fairly new.
- There are no exact or approximate algorithms for matching under transformations yet.
- Some of the configuration-space approaches from Fréchet matching probably carry over to Fréchet-like graph distances, such as the traversal distance

5. **Inverse Problem: Fréchet Realizability**
   Distance geometry, …

Inverse Problem: Fréchet Realizability

Given a free space diagram D_ε, are there curves f, g in $\mathbb{R}^d$ and $\varepsilon > 0$ such that $F_\varepsilon(f, g) = D_\varepsilon$?

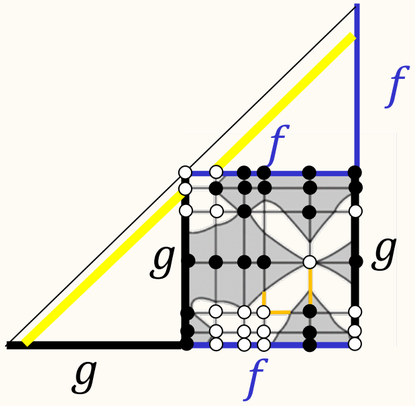


We show: [ABMRW25]

1) $\exists\mathbb{R}$-completeness via linkage realization for $d \geq 2$
2) Weak NP-completeness for $d = 1$, and FPT and pseudo-polynomial algorithms
3) Discrete curves: poly-time algo for $d = 1$, $\exists\mathbb{R}$-completeness for $d \geq 2$ (using binary free space matrix only)

(1) and (2) assumes that cell/edge lenths and precise free intervals on cell boundaries are given.

Distance matrix:



What if less information is given?

- No cell/edge lengths
- Only relative order of free intervals (no specified values)
- Or in the discrete case: Only the order of distances for each point

[ABMRW25] H. Akitaya, M. Buchin, M. Mirzanezhad, L. Ryvkin, C. Wenk, Realizability of Free Spaces of Curves, Journal of Computational Geometry: Theory and Applications 127: 102151, 2025.

Open Problem

Problem 1: Point set realization of distance orders (bipartite graph)

Given: Fixed dimension D, distance function d (can be $\|.\|_2^2$), and

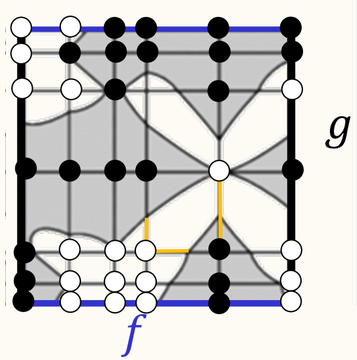
- permutations $\pi_i: [m] \to [m]$ for all $i \in [n]$
- permutations $\sigma_j: [n] \to [n]$ for all j$\in [m]$

Are there point sets $P, Q \subseteq \mathbb{R}^D$, $|P| = m, |Q| = n$, such that

$$d(p_i, \pi_i(1)) < d(p_i, \pi_i(2)) < \cdots < d(p_i, \pi_i(n)) \text{ for all } i \in [m] \text{ and}$$

$$d\left(q_j, \sigma_j(1)\right) < d\left(q_j, \sigma_j\,(2)\right) < \cdots < d(q_j, \sigma_j(m)) \text{ for all j} \in [n]$$

Relaxation of discrete Fréchet realization problem

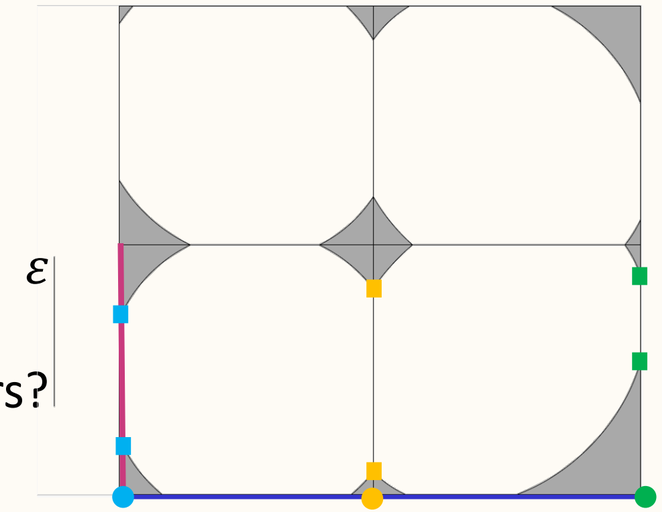
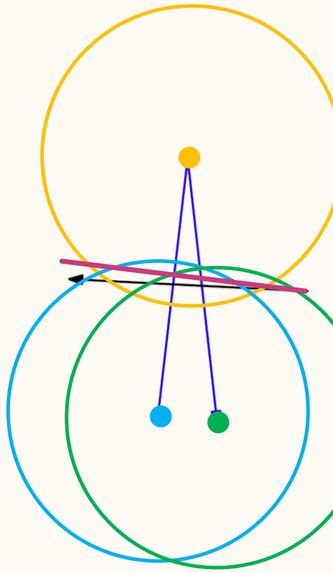


Variant 1: May prescribe some of the "<" to be "="

Variant 2 (complete graph): $P = Q$ and $m = n$

Problem 2: Curve realization from free interval orders

Given: m, n, dimension D, optionally ε, and

the order of free space intervals along every row and every column.

Are there polygonal curves f, g that realize these free interval orders?