BiLipschitz Invariant Theory

Motivation:

- ML algorithms (such as nearest neighbor search) work on data in Euclidean space $V=\mathbb{R}^n$
- data with symmetry lies in $V/G=\{G\cdot v \mid v\in V\}$ where group G acts on $V=\mathbb{R}^n$

  $G\cdot v$: G-orbit of v
- to apply ML algs, find embedding $\phi: V/G \hookrightarrow \mathbb{R}^m$

Goal: find ϕ with small distortion of metric (and small m)

§1. Metric Geometry.

(X, d_X) (Y, d_Y) metric spaces

def. $\phi : X \longrightarrow Y$ is bi-Lipschitz if there exist $\alpha, \beta > 0$ such that $\forall\, x_1, x_2 \in X$

$$\alpha\, d_X(x_1, x_2) \le d_Y(\phi(x_1), \phi(x_2)) \le \beta\, d_X(x_1, x_2)$$

choose α maximal, β minimal

$\mathrm{dist}(\phi) = \beta/\alpha$ called the distortion of ϕ.

$\mathrm{dist}(\phi) \geq 1$, $\quad \mathrm{dist}(\phi) = 1 \Leftrightarrow \phi$ is isometry

$c_2(X) :=$

$\inf\{\mathrm{dist}(\phi) \mid \phi : X \hookrightarrow \mathcal{H}$ where $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ real Hilbert space$\}$

$c_2(X) = \sup\{\, c_2(F) \mid F \subseteq X$ finite$\}$

suppose $X = \{x_1, x_2, \dots, x_n\}$

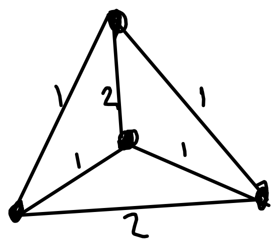
$D = (\, d_X^2(x_i, x_j))_{1 \leq i,j \leq n} \in \mathbb{R}^{n \times n}$

Example: $c_2\left(\{0,1\}^n, d_{\mathrm{Hamming}}\right) = \sqrt{n}$

Prop (Linial, London, Rabinovich): Suppose

- $Q \in \mathbb{R}^{n \times n}$ pos. semidef., $Q \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $Q = Q_+ - Q_-$, $Q_+, Q_- \in \mathbb{R}_{\geq 0}^{n \times n}$

Then: $\mathrm{Trace}(DQ_+) \leq c_2(X)^2 \, \mathrm{Trace}(DQ_-)$

Example: $X =$ 

$$D = \begin{bmatrix} 0 & 1 & 4 & 1 \\ 1 & 0 & 1 & 4 \\ 4 & 1 & 0 & 1 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

$$16 \leq c_2(X)^2 \, 8$$

$$c_2(X) \geq \sqrt{2}.$$

§2 Euclidean embeddings

$V = \mathbb{R}^n$, $E(V) = O(V) \ltimes V = O(n) \ltimes \mathbb{R}^n$

Euclidean group.

$G \subseteq E(V)$ closed subgroup.

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$V/G = \{ G\cdot v \mid v \in V \}$

define metric: $d_{V/G}(G\cdot v, G\cdot v) =$

$$\min_{g,h\in G} d(g\cdot v, h\cdot w) = \min_{g\in G} d(g\cdot v, w)$$

Example:

point clouds: $\{a_1, a_2, \ldots, a_n\} \subseteq \mathbb{R}^d$

...

$$\updownarrow$$

S_n orbits of $A = [a_1 a_2 \ldots a_n] \in \mathbb{R}^{d \times n}$.

$$d^2_{V/S_n}(A,B) = \min_{\sigma \in S_n} \left\{ \sum_{i=1}^{n} \| a_i - b_{\sigma(i)} \|^2 \right\} \quad \text{2-Wasserstein}$$

Theorem: For a representation V of a compact Lie group G there exists a bi-Lipschitz embedding $\phi : V/G \longrightarrow \mathbb{R}^m$ for some m

Theorem is not explicit!

Theorem (Mixon-Parshall): if G finite, then

$$c_2(V/G) \leq 4e^{\frac{3}{2}}|G|^{\frac{5}{2}}\sqrt{\log(e|G|)}$$

randomized construction.

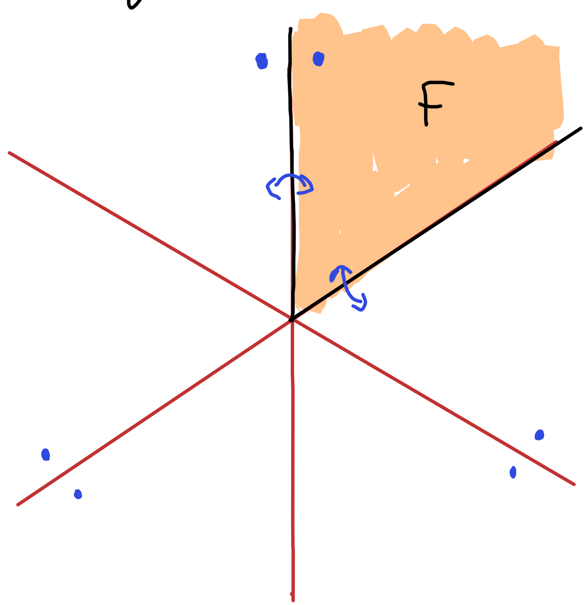
§3 some examples (Mixon-Packer)

$G \subseteq O(n)$ reflection group

$F \subseteq \mathbb{R}^n$ fundamental domain.

E.g. $G = D_n \subseteq O(2)$ Dihedral gp.

$n=3$



$\phi(G\cdot v) = (G \cdot v) \cap F$.

ϕ is isometric!

Ex. $G = S_n \circlearrowright \mathbb{R}^n$ permutes coordinates

if $x = (x_1, x_2, \ldots, x_n)$ then

$$\phi(S_n \cdot x) = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$$ where

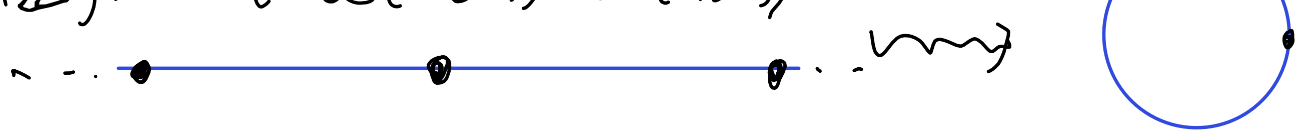
$$x_{\sigma(1)} \geq x_{\sigma(2)} \geq \cdots \geq x_{\sigma(n)}$$

Example: $G = \mathbb{Z}$ acts on $\mathbb{R}$ by translation.

$X = \mathbb{R}/\mathbb{Z}$

$$\phi : X \hookrightarrow \mathbb{R}^2$$

$$\phi(x+\mathbb{Z}) = (\cos(2\pi x), \sin(2\pi x))$$



$$c_2(\mathbb{R}/\mathbb{Z}) = \frac{\pi}{2}$$

more generally: $\Lambda \subseteq \mathbb{R}^n$ lattice

Haviv - Regev: $c_2(\Lambda) = O(\sqrt{n \log(n)})$

there exists a family $\{\Lambda_n \subseteq \mathbb{R}^n\}$ s.t.

$$c_2(\Lambda_n) = \Omega(\sqrt{n})$$

in particular $c_2(\Lambda_n) \longrightarrow \infty$ as $n \longrightarrow \infty$

Cahill - Iverson - Mixon:

$G = \{\pm 1\}$ acts on $V = \mathbb{R}^n$ by scalar mult.

$\phi : V/G \longrightarrow \mathbb{R}^{n \times n} \cong \mathbb{R}^{n^2}$

$\phi(\{\pm v\}) = \frac{1}{\|v\|} v v^T$ least distortion, $c_2(V/G) = \sqrt{2}$

§4 new results (Blum-Smith, D., Mixon, Qaddura, Vose):

Example: $\zeta_d = e^{2\pi i/d} = \cos\left(\frac{2\pi}{d}\right) + i \sin\left(\frac{2\pi}{d}\right)$

$G = \langle \zeta_d \rangle = \{1, \zeta_d, \ldots, \zeta_d^{d-1}\} \cong \mathbb{Z}/d\mathbb{Z}$.

acts on $V = \mathbb{C}^n \cong \mathbb{R}^{2n}$ by scalar mult.

optimal embedding

$\phi : \mathbb{C}^n/G \longrightarrow (\mathbb{C}^n)^{\otimes d} \oplus (\mathbb{C}^n \otimes \mathbb{C}^n)$

$\phi([v]) = \left(\sin\left(\frac{\pi}{2d}\right) \frac{v^{\otimes d}}{\|v\|^{d-1}}, \cos\left(\frac{2\pi}{d}\right) \frac{v \otimes \bar{v}}{\|v\|} \right)$

$c_2(\mathbb{C}^n/G) = d \sin\left(\frac{\pi}{2d}\right)$

ϕ is optimal

Example: n labeled points in $\mathbb{R}^d$ with $O(d)$-action. $G = O(d) \curvearrowright V = \mathbb{R}^{d \times n} \ni A, B$

$d^2_{V/G}(A, B) = \|A\|^2 + \|B\|^2 - 2\|AB^T\|_*$

$$\|A\|^2 = \mathrm{Tr}(AA^T)$$

$\|A\|_*$ sum of singular values

$$\phi : V/O(d) \longrightarrow \mathbb{R}^{n\times n}$$

$$\phi(O(d)\cdot A) = \sqrt{A^T A}$$

$$\mathrm{dist}(\phi) = c_2(V/G) = \sqrt{2}$$

$$c_2(V/E(d)) = \sqrt{2}$$

$$\tilde{\phi} : V/E(d) \longrightarrow \mathbb{R}^{n\times n}, \quad \tilde{\phi}(E(d)\cdot[v_1\ v_2 \cdots v_n]) = \phi([v_1-\bar{v}, \dots, v_n-\bar{v}])$$

$$\bar{v} = \frac{1}{n}\sum_{i=1}^{n} v_i$$

$$c_2(V/SO(d)) \in [\sqrt{2}, 2\sqrt{2}]$$