

Fibers of point cloud persistence

**ICERM Workshop: Rigidity Theory meets Geometric Data Science
for Applications in Chemistry and Biology (July 2026)**

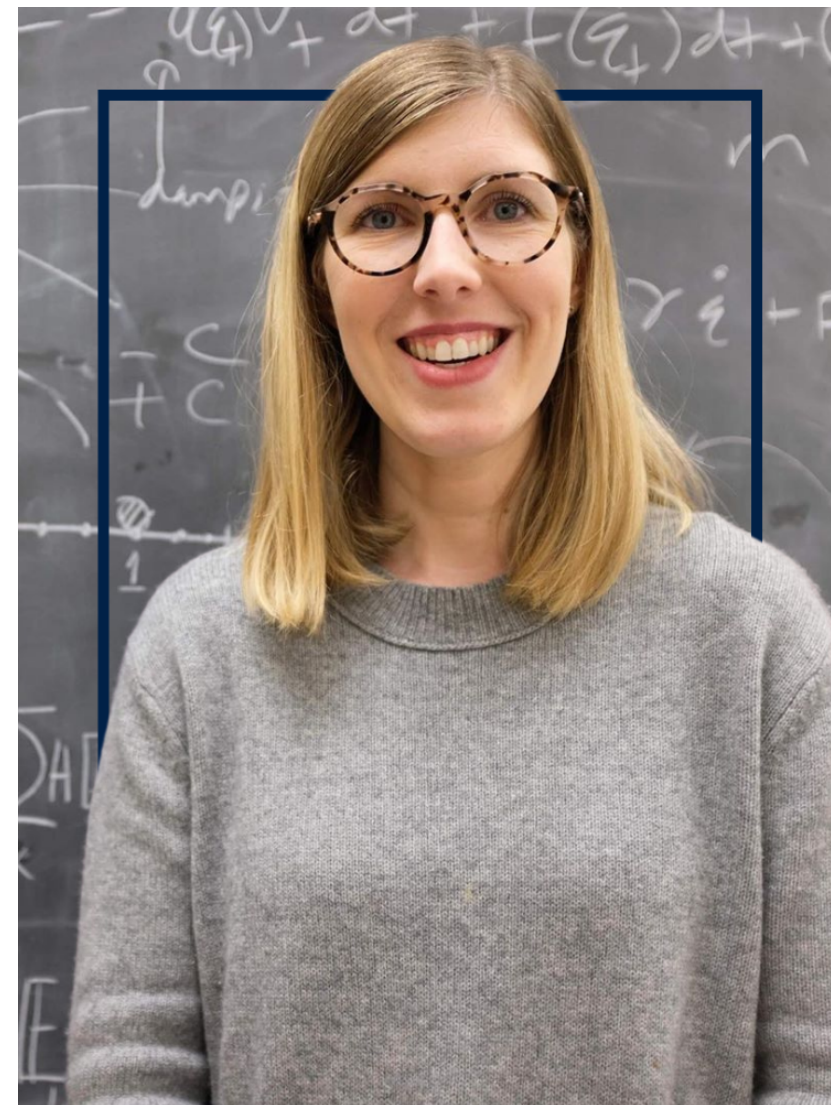
Louis Theran (St Andrews)

Credits

TDA results joint with



David Beers (UCLA)



Heather Harrington (MPI)



Jacob Leygoine (Oxford)



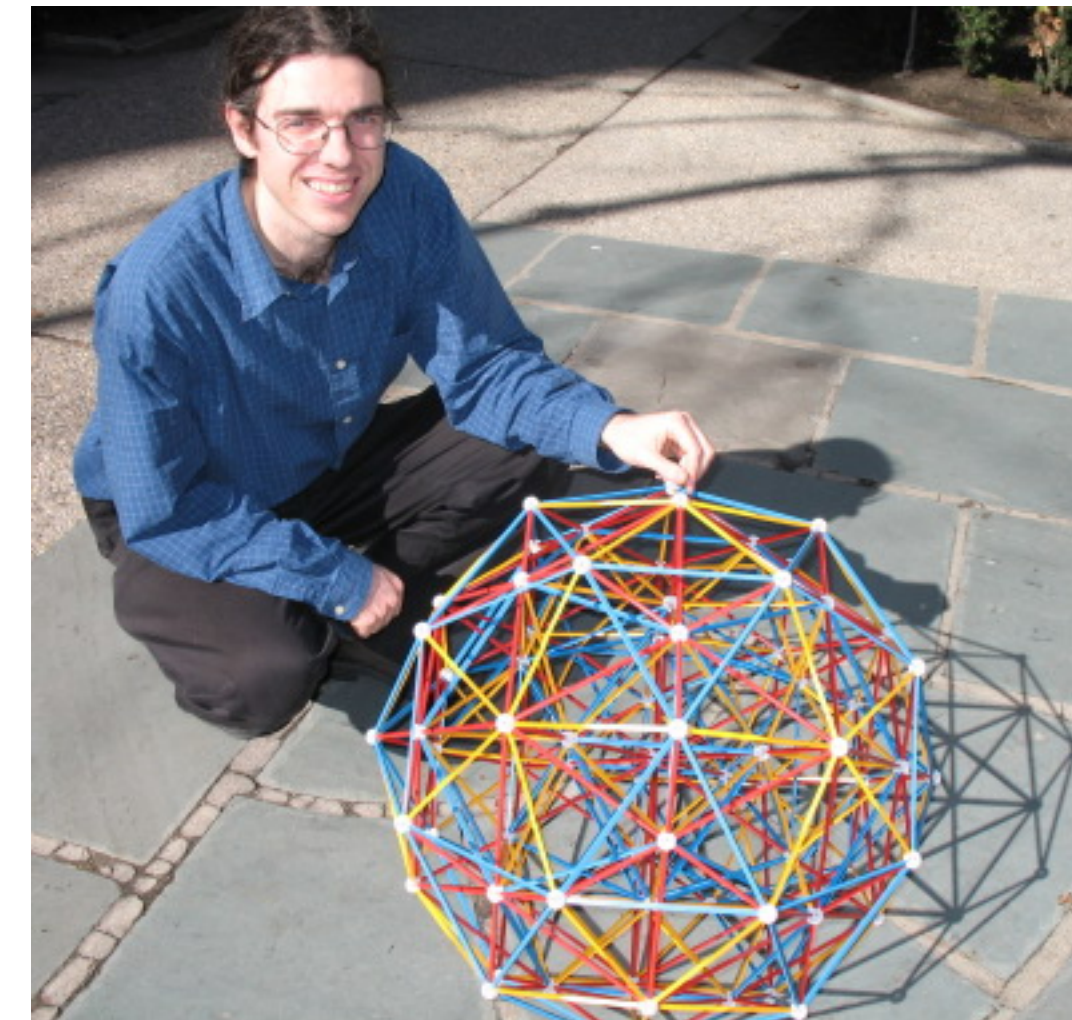
Uzu Lim (QMUL)

Credits

Rigidity results joint with



Steven J. Gortler (Harvard)



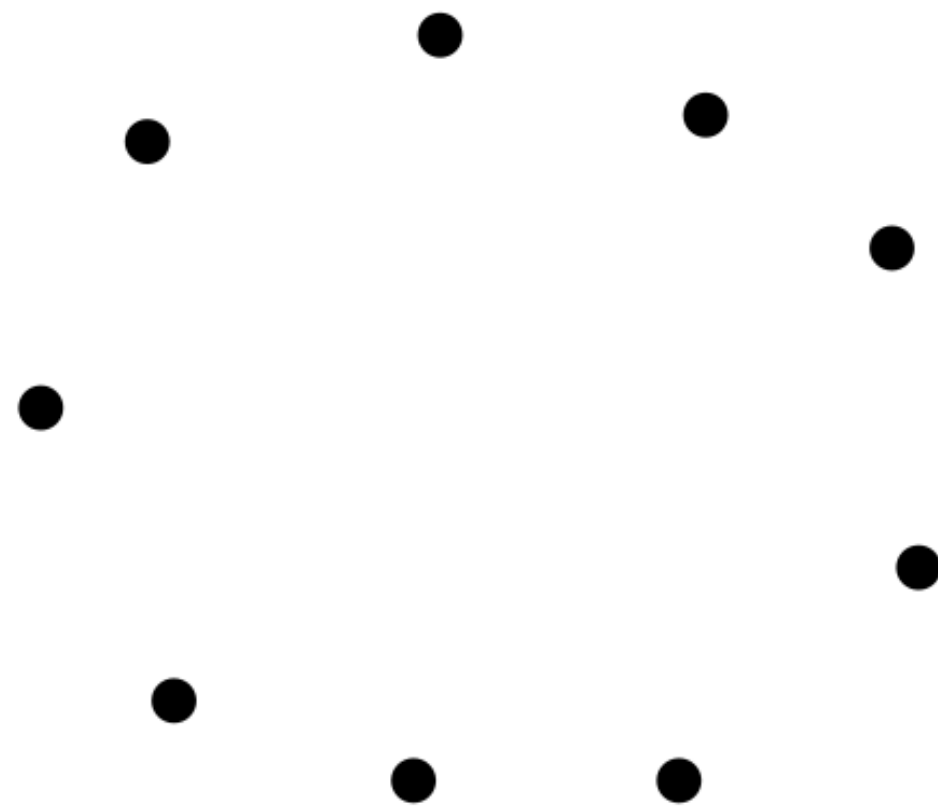
Dylan Thurston (Indiana)

1. Motivation

Topological data analysis

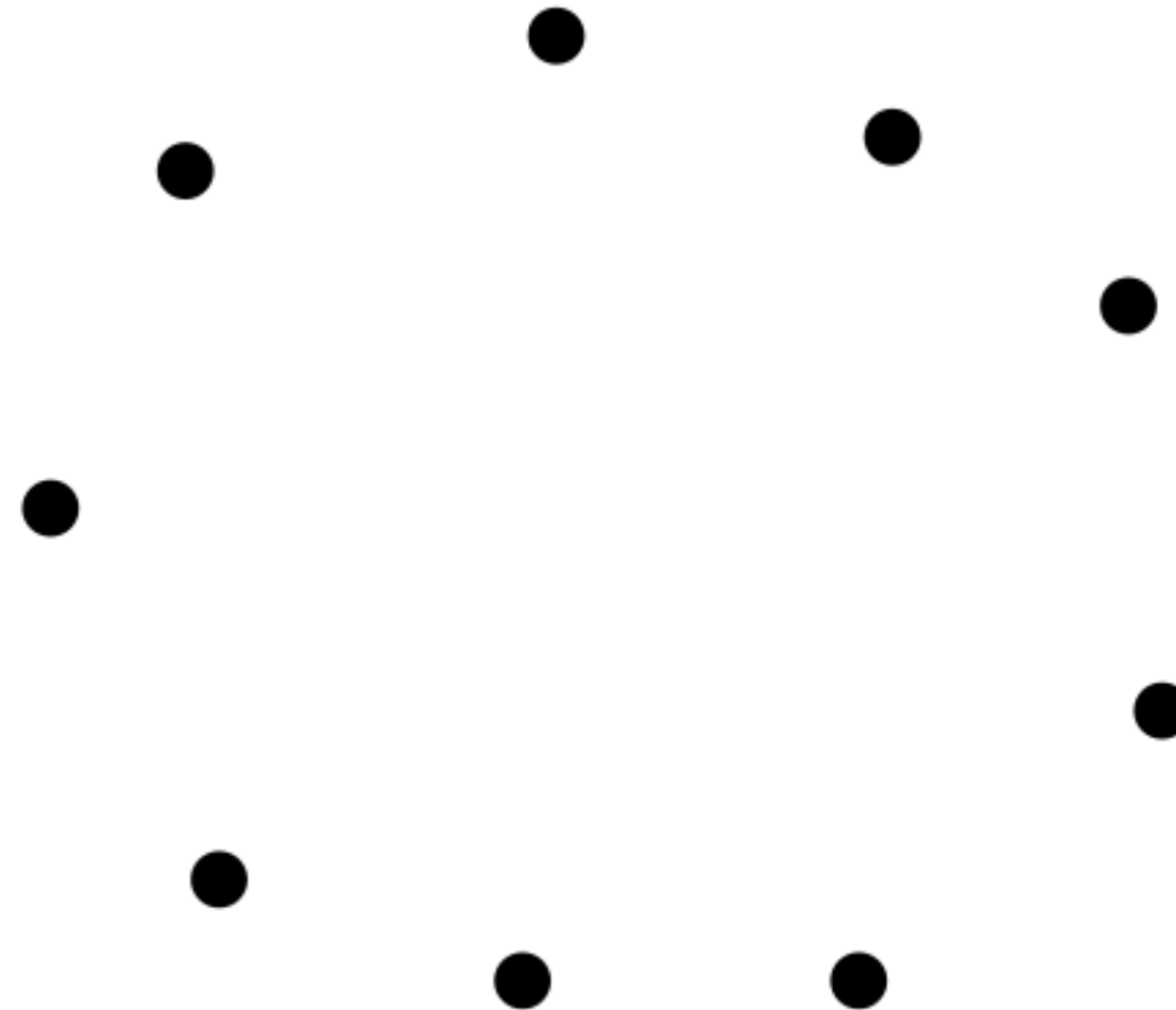
Motivating question: what is the topology of data?

- “Data” in this talk is a configuration $\mathbf{p}$ of n *distinct* points in $\mathbb{R}^d$
 - Called “point clouds” in TDA. Example in $\mathbb{R}^2$



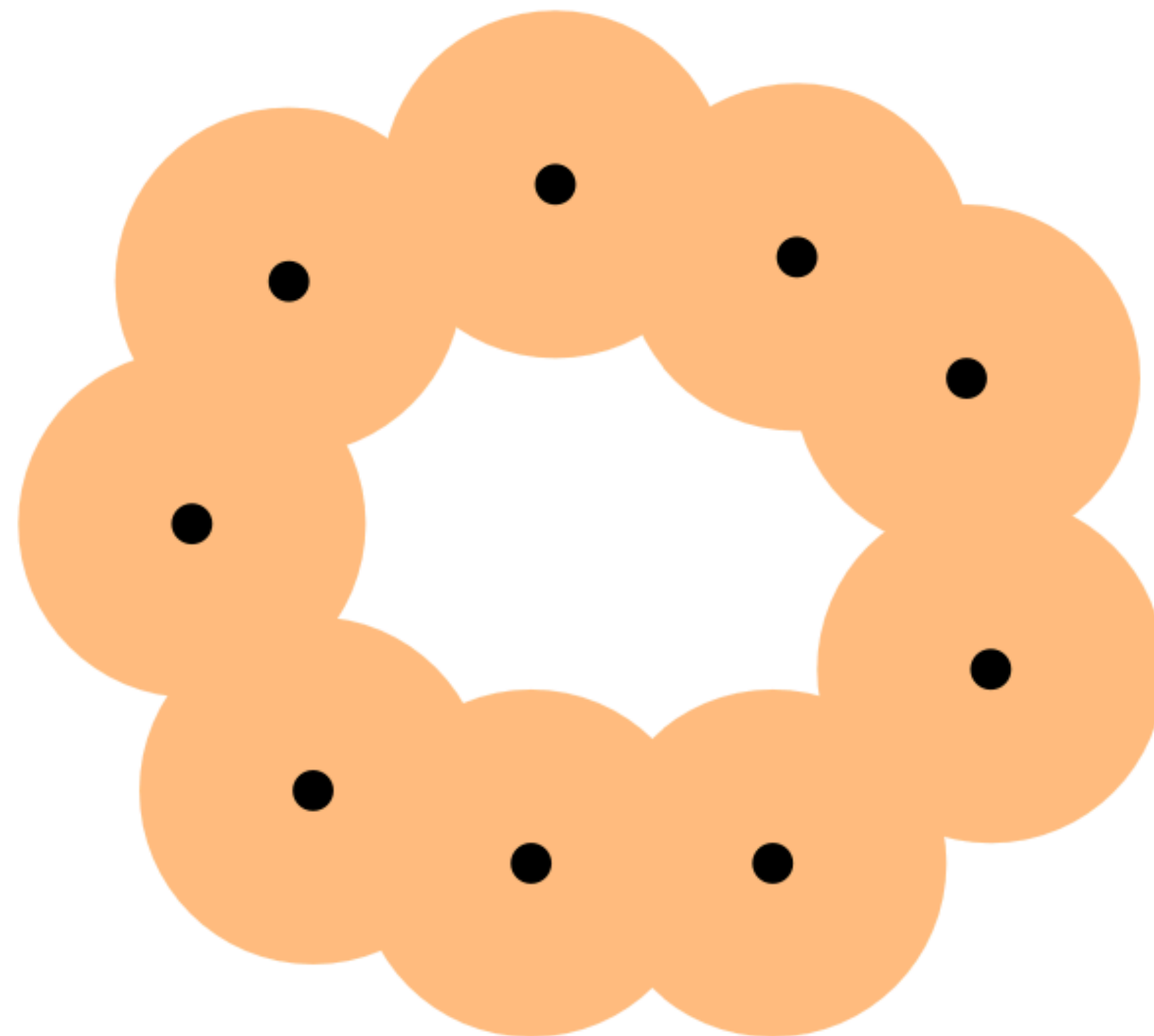
Topology of this point set?

Not very interesting!



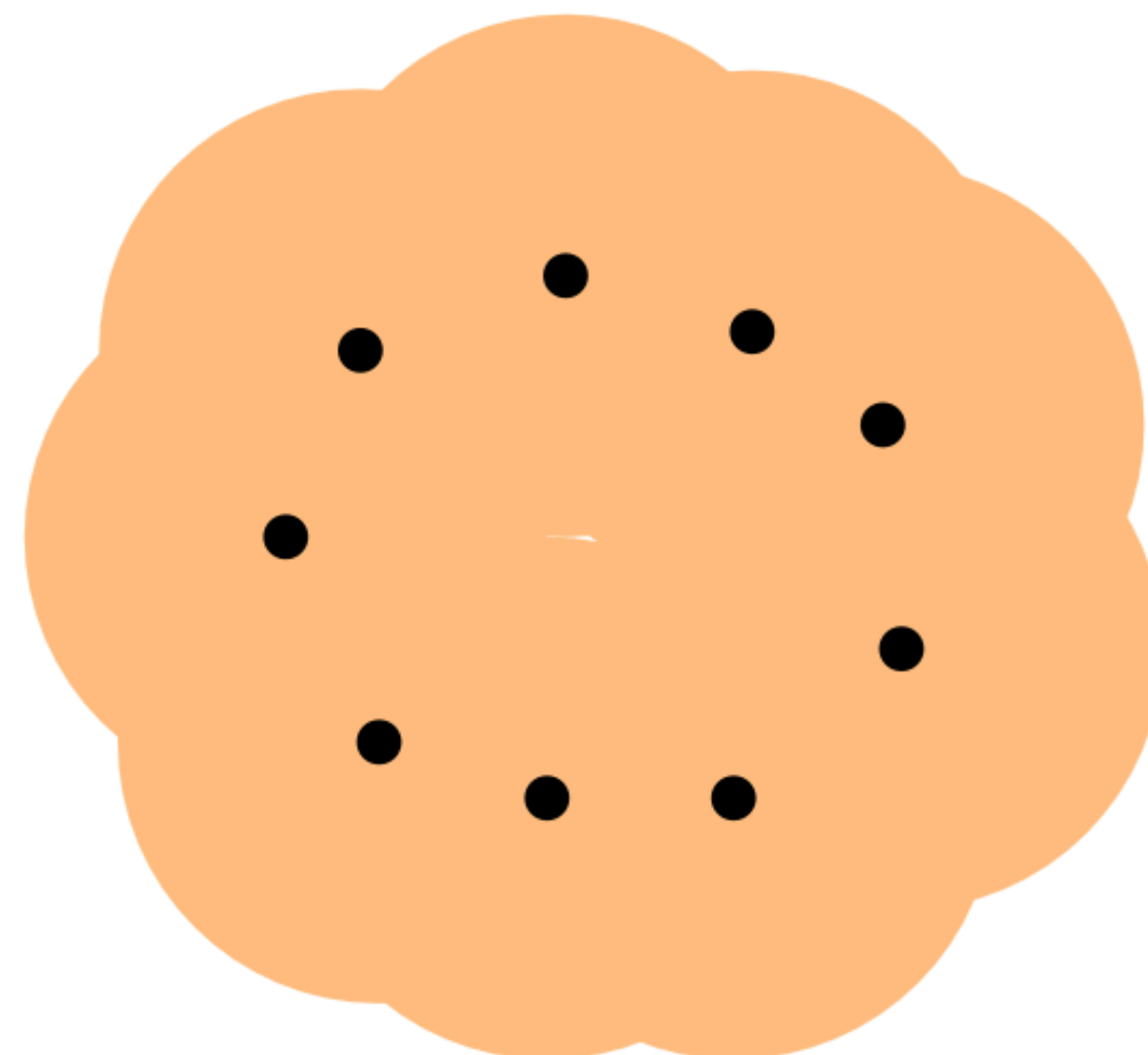
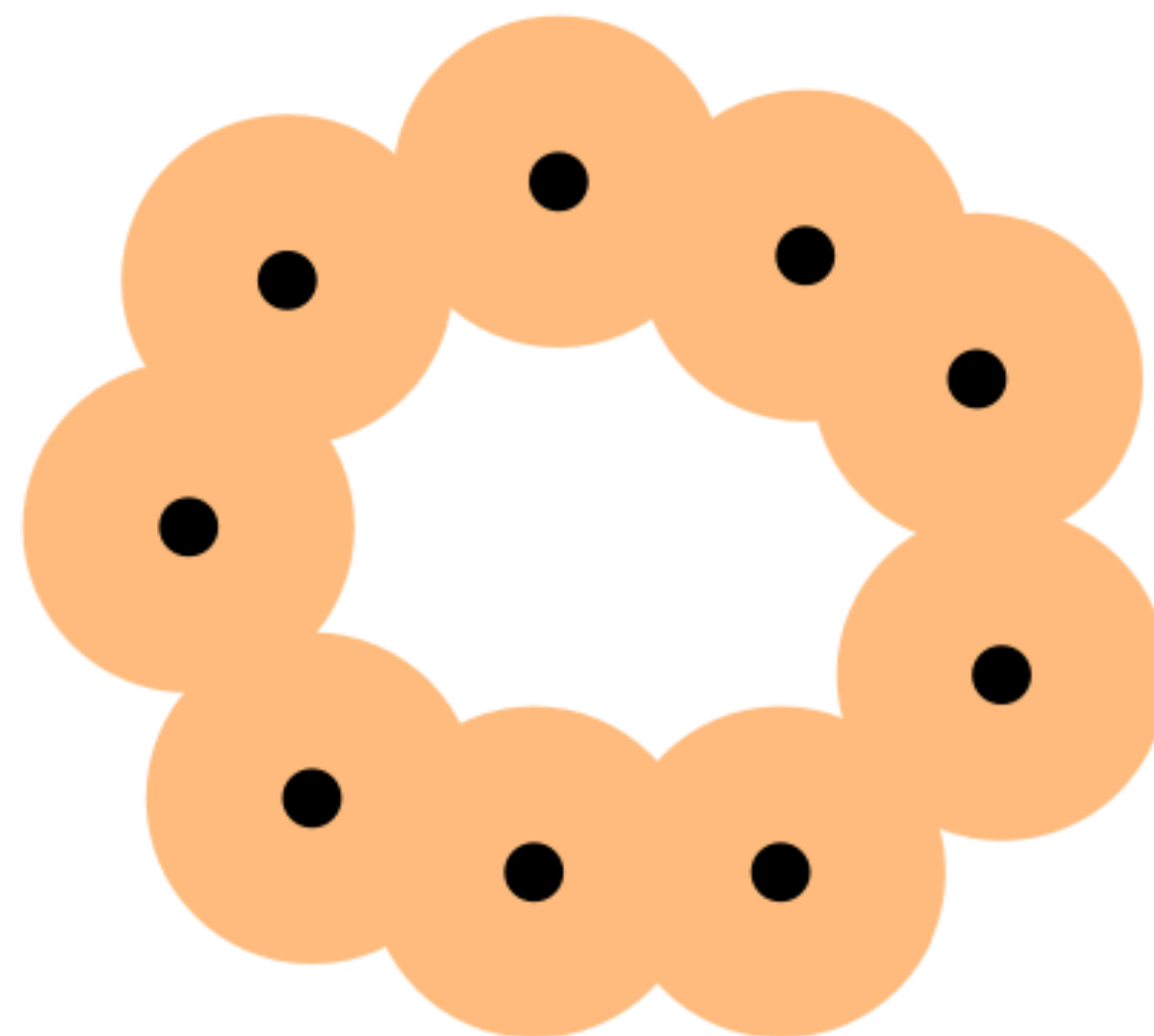
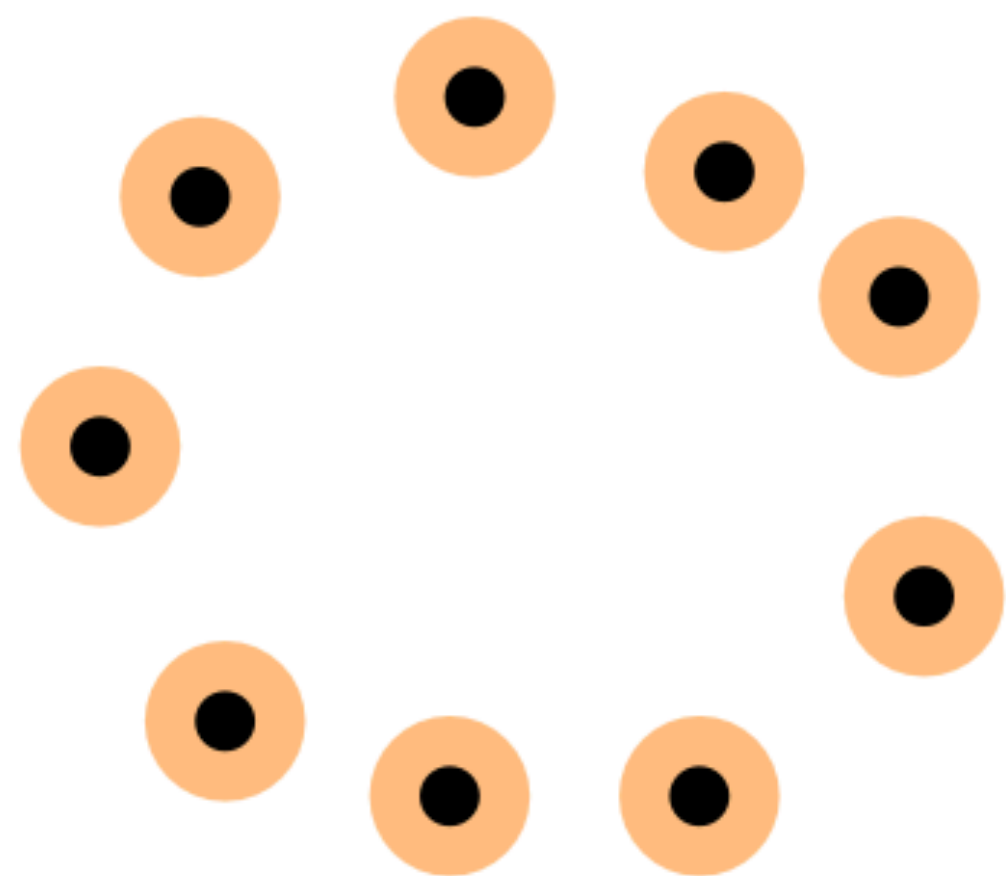
Topology of this point set?

Make it more interesting: fatten by a radius r



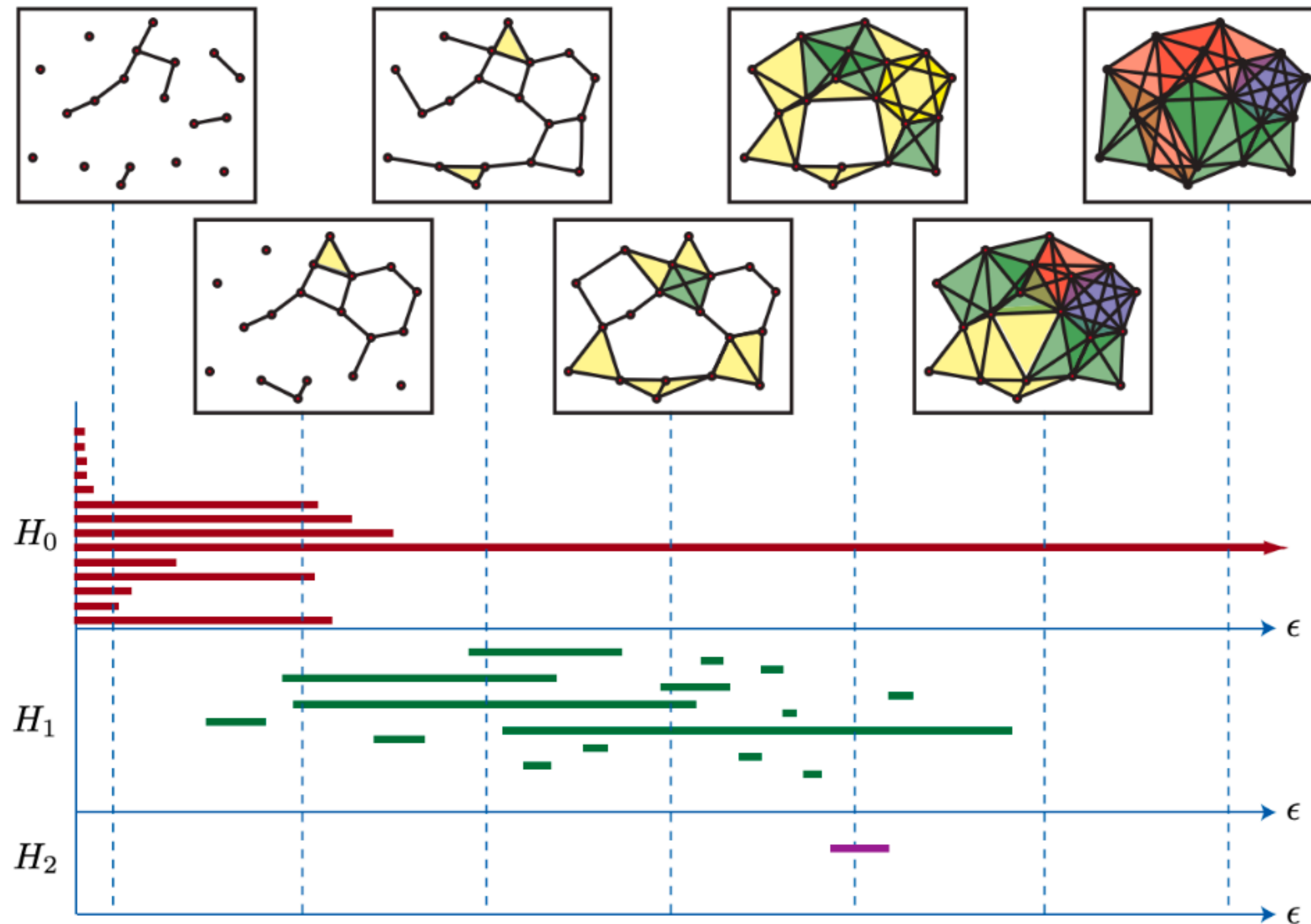
Topology of this point set?

Which choice of r is the right one?



Topological persistence

Track *homology* as r varies instead



[Image: Ghrist 2007]

2. Barcode

Filtrations

First examples

- $T \subseteq \mathbb{R}$ any subset
- **Definition:** A (single parameter) *filtration* is a collection of topological spaces $\{X_t\}_{t \in T}$ satisfying $X_s \subseteq X_t$ whenever $s \leq t$.
- Examples from $f : X \rightarrow \mathbb{R}$:
 - Sublevel sets: $X_t = f^{-1}((-\infty, t])$
 - Superlevel sets: $X_t = f^{-1}([t, \infty))$

Filtrations

Union of balls

- $T \subseteq \mathbb{R}$ any subset
- **Definition:** A (single parameter) *filtration* is a collection of topological spaces $\{X_t\}_{t \in T}$ satisfying $X_s \subseteq X_t$ whenever $s \leq t$.
- **Union of balls filtration:** $T = [0, \infty)$, $\mathbf{p}$ a point cloud in $\mathbb{R}^d$

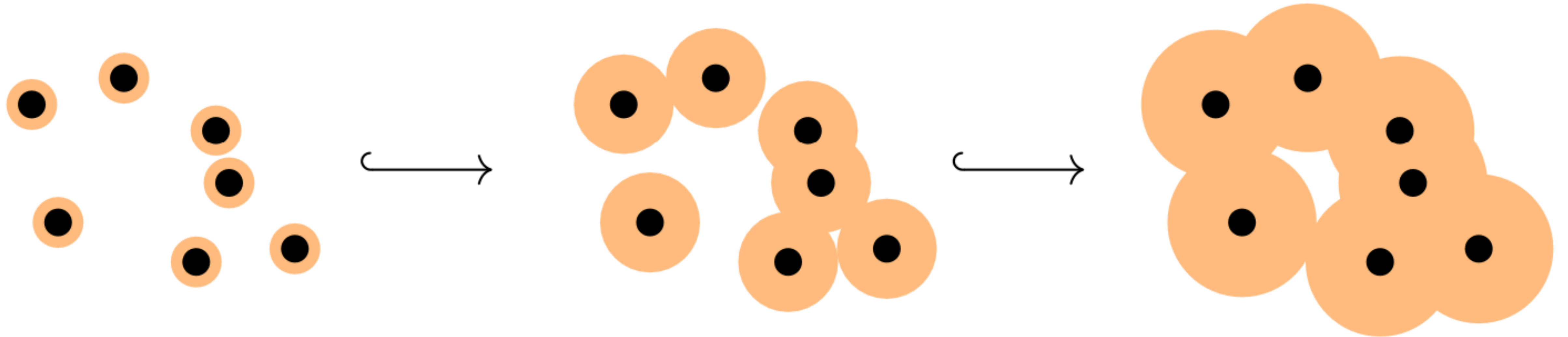
$$X_t = \mathbf{p}(t) := \bigcup_{i=1}^n D(\mathbf{p}_i, t)$$

- $D(\mathbf{x}, r)$ closed Euclidean ball

Filtrations

Union of balls

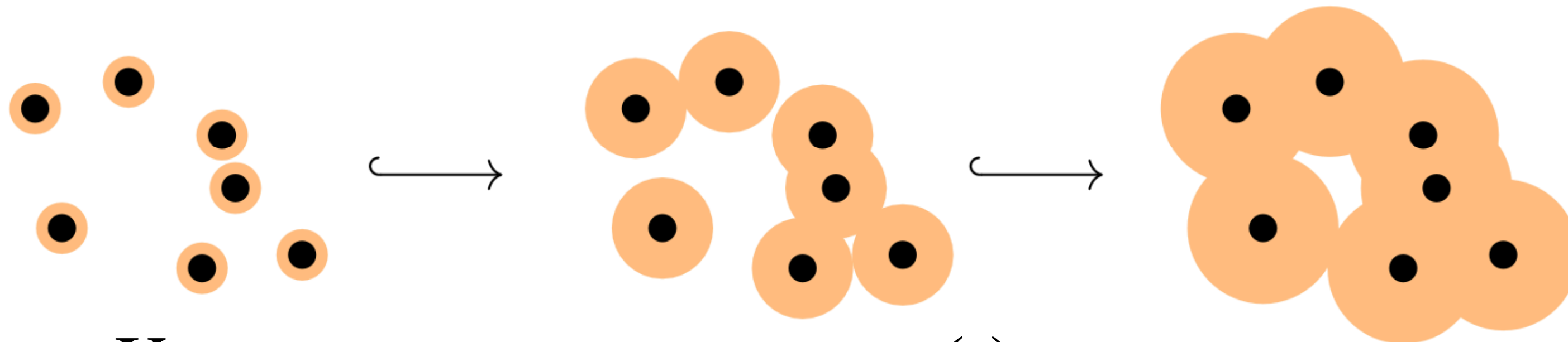
- Union of balls filtration: $T = [0, \infty)$, $\mathbf{p}$ a point cloud in $\mathbb{R}^d$, $X_t = \mathbf{p}(t)$



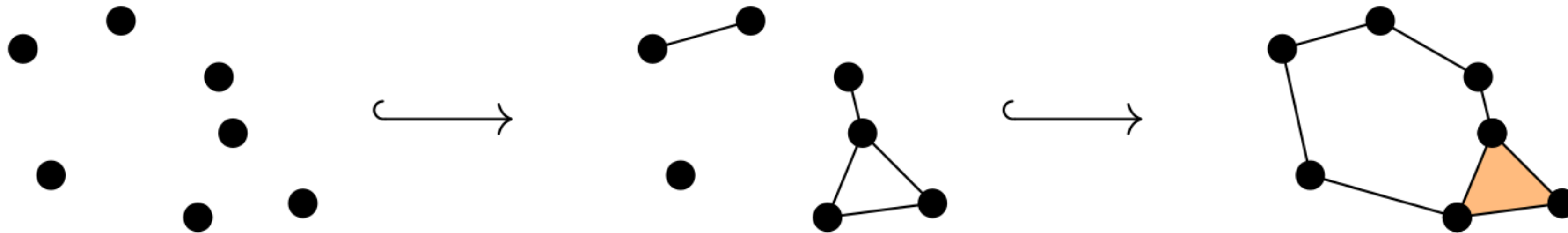
Filtrations

Simplicial complexes: Čech filtration

- Union of balls filtration: $T = [0, \infty)$, $\mathbf{p}$ a point cloud in $\mathbb{R}^d$, $X_t = \mathbf{p}(t)$



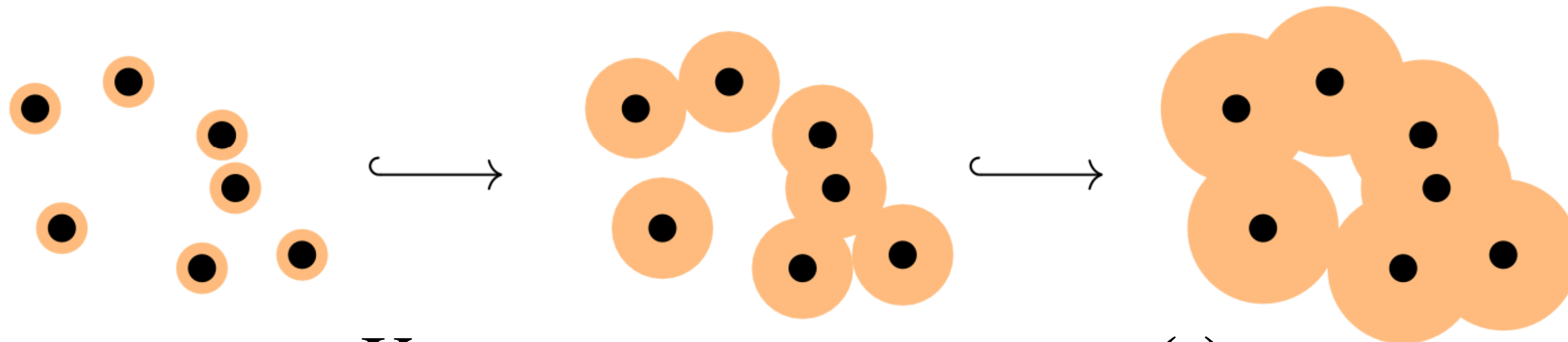
- Čech filtration: X_t the nerve complex of $\mathbf{p}(t)$



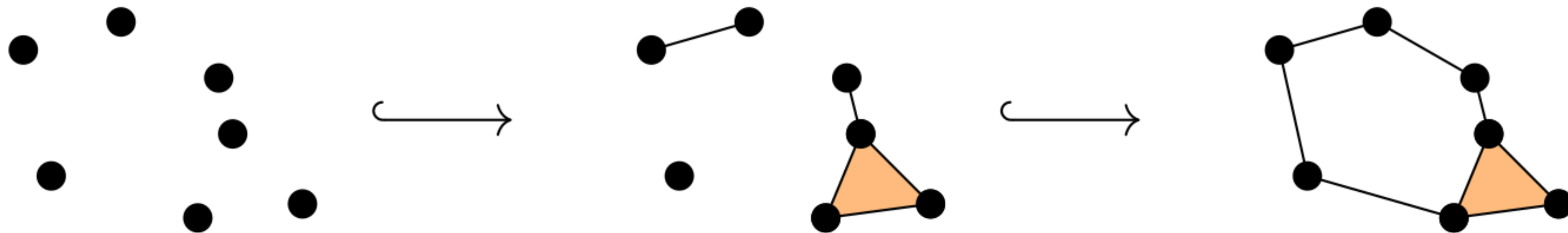
Filtrations

Simplicial complexes: Rips filtration

- Union of balls filtration: $T = [0, \infty)$, $\mathbf{p}$ a point cloud in $\mathbb{R}^d$, $X_t = \mathbf{p}(t)$



- Vietoris–Rips filtration: X_t the Rips complex of $\mathbf{p}(t)$



Filtrations

Simplicial complexes: formulas

- Nerve complex $\mathbf{p}^{\check{C}}(r)$

$$\sigma \in \mathbf{p}^{\check{C}}(r) \quad \Longleftrightarrow \quad \bigcap_{j \in \sigma} D(\mathbf{p}_j, r) \neq \emptyset$$

- Rips complex $\mathbf{p}^{\text{VR}}(r)$

$$\sigma \in \mathbf{p}^{\text{VR}}(r) \quad \Longleftrightarrow \quad \text{diam}(\mathbf{p}_j)_{j \in \sigma} \leq 2r$$

Persistence

Persistence modules

- **Definition:** A (single parameter) *filtration* is a collection of topological spaces $\{X_t\}_{t \in T}$ satisfying $X_s \subseteq X_t$ whenever $s \leq t$.
- Take (singular) deg i homology with F coefficients:

$$H_i(X_s; F) \xrightarrow{l_*} H_i(X_t; F) \text{ for all } s \leq t$$

- **Definition:** A (single parameter) *persistence module* is any functor

$$V : (T, \leq) \rightarrow \mathbf{Vect}_F$$

pointwise finite dimensional (pfd) if $\dim V_t < \infty$ for all $t \in T$

Persistence

Persistence modules

- **Definition:** A (single parameter) *persistence module* is any functor

$$V : (T, \leq) \rightarrow \mathbf{Vect}_F$$

pointwise finite dimensional (pfd) if $\dim V_t < \infty$ for all $t \in T$

- More concretely need:
 - F -vector spaces V_t
 - Linear maps $V(s \leq t) : V_s \rightarrow V_t$ with $V(s \leq t) \circ V(r \leq s) = V(r \leq t)$

Persistence

Persistence modules

- **Definition:** A (single parameter) *persistence module* is any functor

$$V : (T, \leq) \rightarrow \mathbf{Vect}_F$$

- Example: $I \subseteq \mathbb{R}$ an interval determines an *interval module* F_I

$$(F_I)_t = \begin{cases} F & t \in I \\ 0 & t \notin I \end{cases} \quad F_I(s \leq t) = \begin{cases} \text{id} & s, t \in I \\ 0 & \text{o.w.} \end{cases}$$

Persistence

Indecomposable persistence modules

- *Direct sum and isomorphism* inherited from $\mathbf{Vect}_F$

$$(V \oplus W)_t = V_t \oplus W_t$$

- **Definition:** V is *indecomposable* if $V \cong W \oplus X$ implies $W = 0$ or $X = 0$
- Interval modules are indecomposable

Persistence

Structure Theorem

- **Theorem (Zomorodian–Carlsson 2004):** Every pfd persistence module V has an essentially unique direct sum decomposition into interval modules

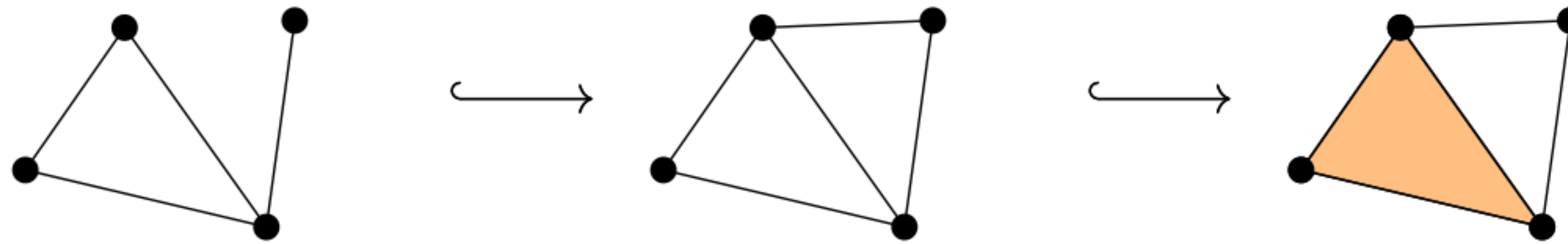
$$V = \bigoplus_{I \in \mathcal{D}(V)} F_I$$

- **Definition:** The multi-set of intervals $\mathcal{D}(V)$ is called the *barcode* of V .
- The barcode may be infinite, but if V is “tame”, it is finite.
 - For Čech and Vietoris–Rips filtrations, always tame

Persistence

Structure Theorem: Example

- **Definition:** The multi-set of intervals $\mathcal{D}(V)$ is called the *barcode* of V .



- Persistence module from $H_1(\cdot; F)$:

$$F \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} F^2 \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} F = (F \xrightarrow{\text{id}} F \rightarrow 0) \oplus (0 \rightarrow F \xrightarrow{\text{id}} F)$$

Persistence

Structure Theorem: Example

- **Definition:** The multi-set of intervals $\mathcal{D}(V)$ is called the *barcode* of V .
- Persistence module from $H_1(\cdot; F)$:

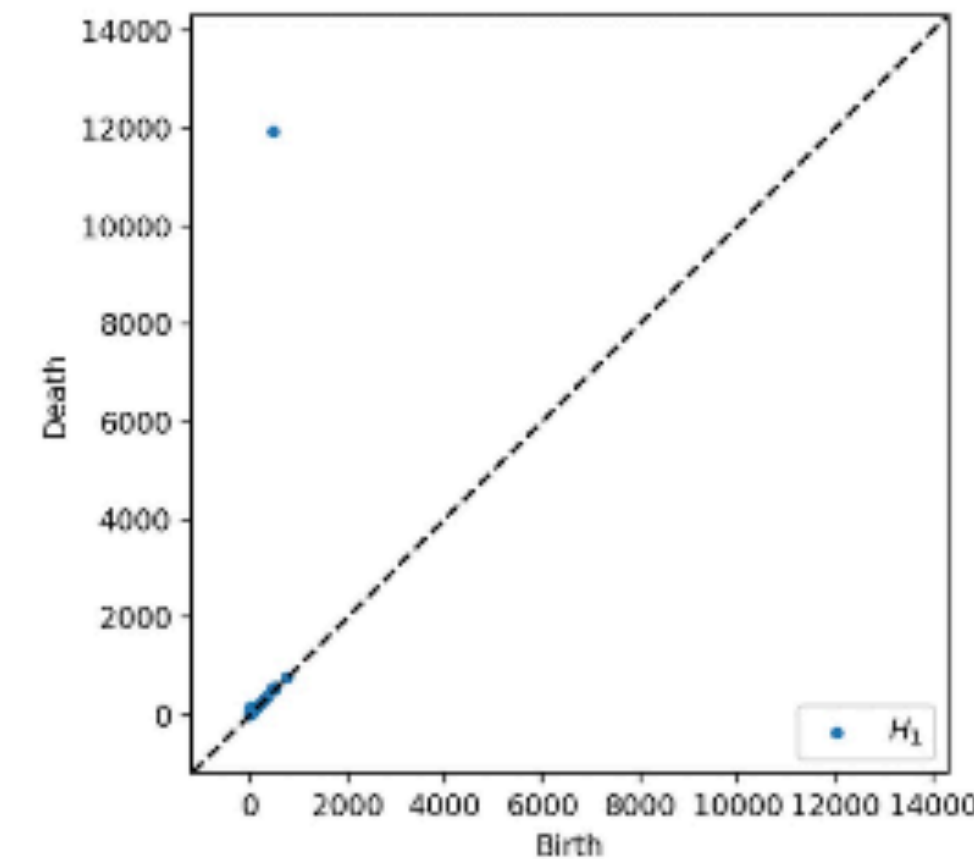
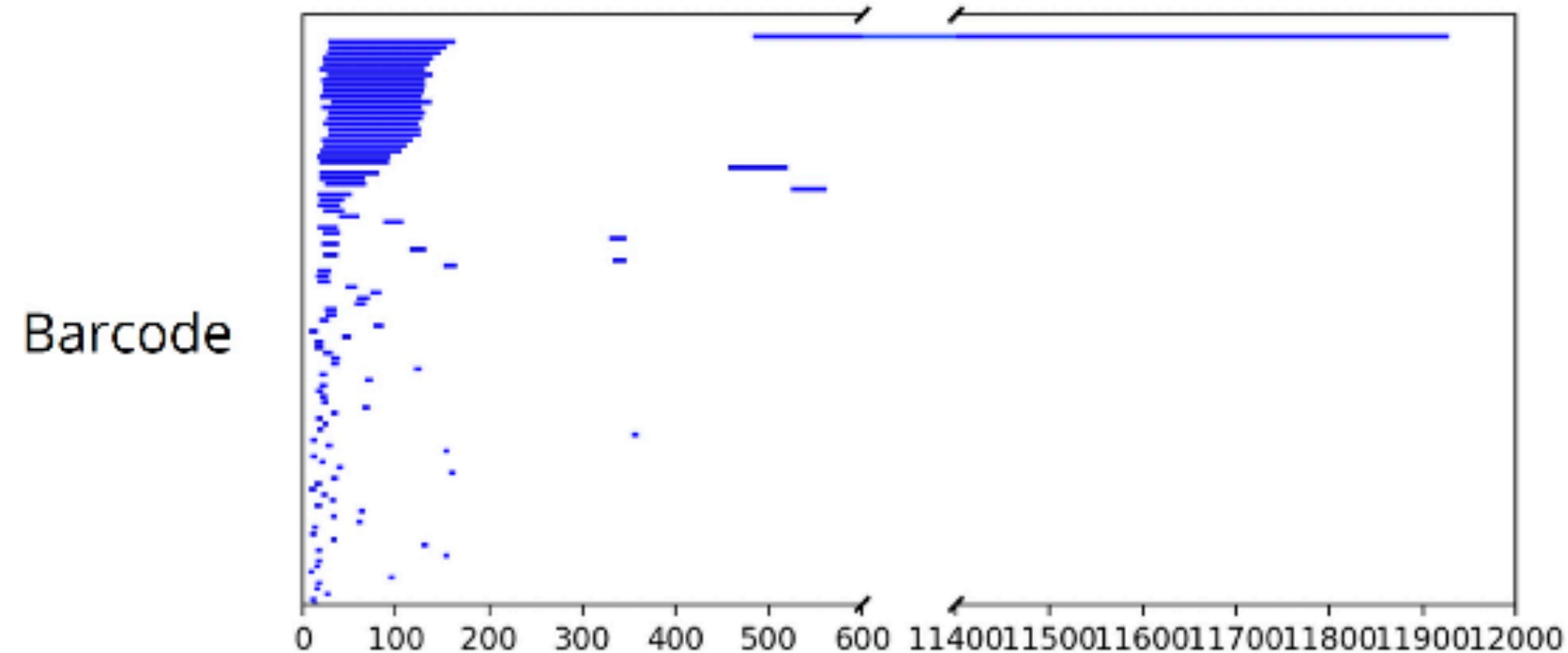
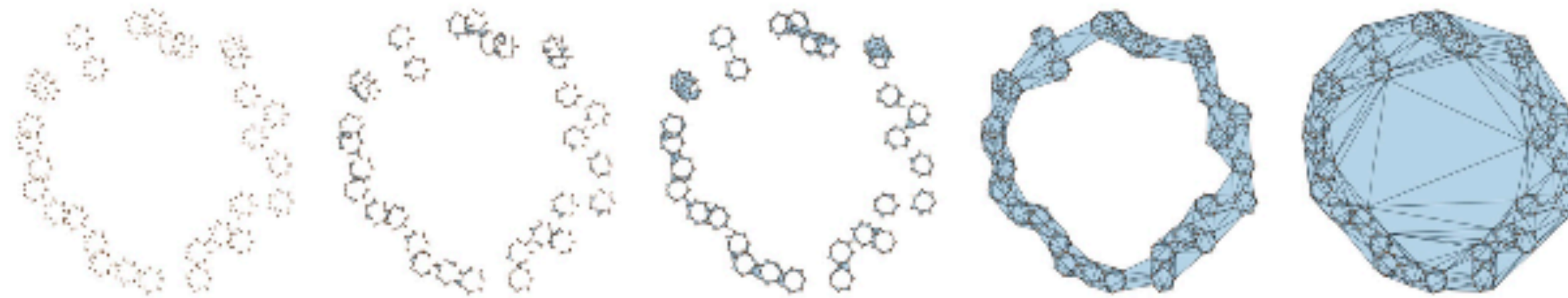
$$F \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} F^2 \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} F = (F \xrightarrow{\text{id}} F \rightarrow 0) \oplus (0 \rightarrow F \xrightarrow{\text{id}} F)$$

- Barcode is $\{[1,2], [2,3]\}$

Persistence

Application to data analysis

- **Main idea:** Longer intervals in the barcode are “topological signal”



Persistence diagram

Persistence

Forward inference

- **Main idea:** Longer intervals in the barcode are “topological signal”
- There is a notion of “bottleneck distance” for barcodes
- **Stability Theorem:** For any metric spaces X, Y

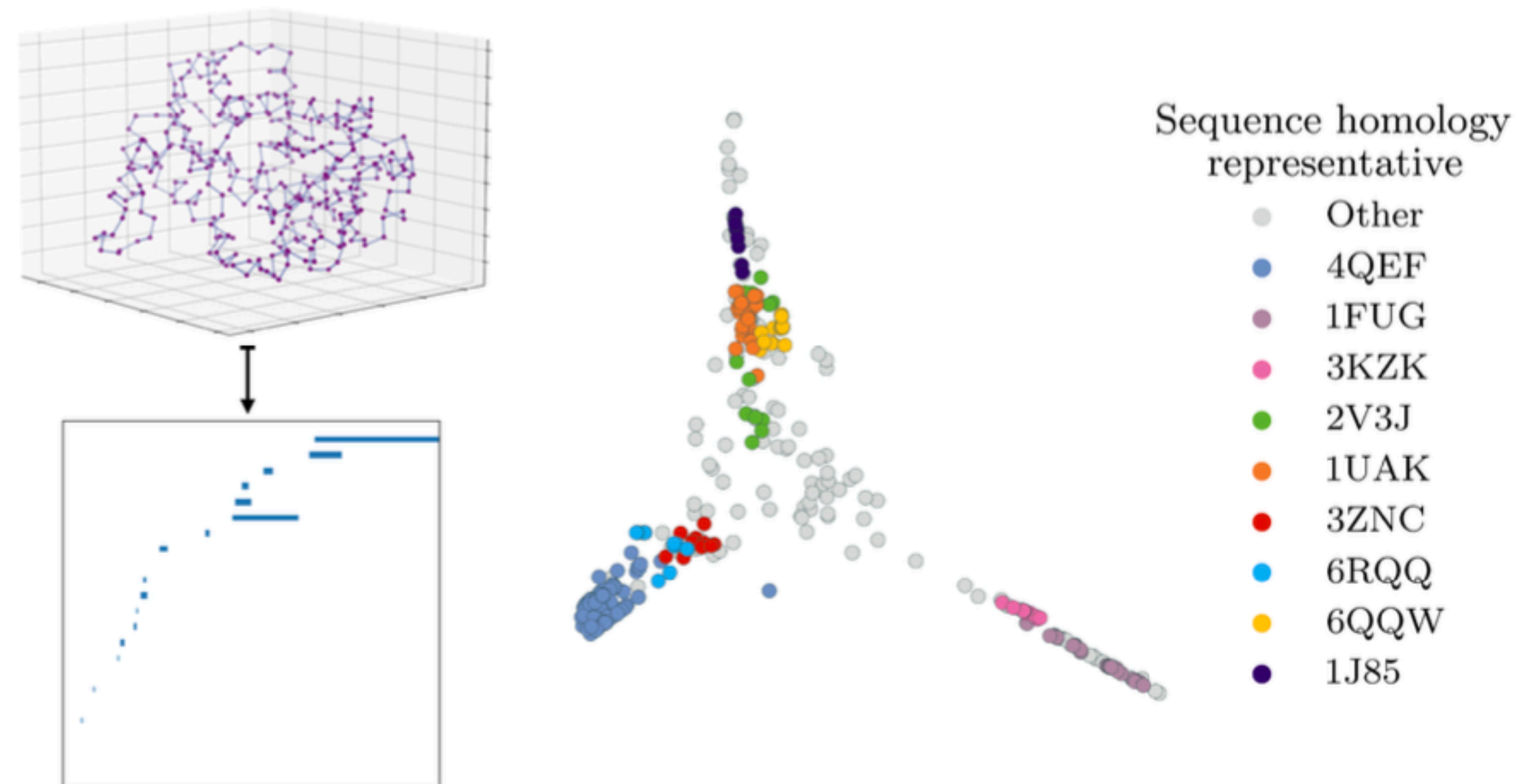
$$d_b(\mathcal{D}(X^{\text{VR}}), \mathcal{D}(Y^{\text{VR}})) \leq 2d_{\text{GH}}(X, Y)$$

- Upshot: Similar geometric inputs have similar barcodes
- This is a very special case of a more general set of theorems

Persistence

Protein example

- **Stability Theorem:** Similar geometric inputs have similar barcodes



3. Inverse problems

Inverse problems

In persistence

- **Stability Theorem**: Similar geometric inputs have similar barcodes
- **Main Question**: What about the other direction?
 - If two spaces have *similar barcodes*, must they be *geometrically similar*?
 - What about if they have *the same* barcode?
 - If not, what can we say about the set of spaces with similar barcodes?

Inverse problems

Some previous work

- **Setup**: X space, $\mathcal{F}$ set of functions $X \rightarrow \mathbb{R}$
- Study subspace of $\mathcal{F}$ so that the sublevel set filtration degree i homology barcodes are $D = [\mathcal{D}_1, \dots, \mathcal{D}_m]$; called $\text{PH}^{-1}(D)$

Inverse problems

Some previous work

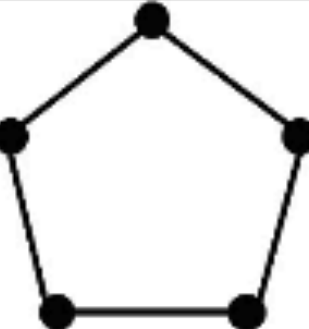
Leygonie & Tillmann (2022). Leygonie & H-P (2021)

Complex K	Barcode D	Fiber
		Polyhedra: [64, 858, 4480, 10860, 13320, 8064, 1920] $\beta_0 = 1, \beta_2 = 1$
		Polyhedra: [3936, 29928, 84516, 113244, 72960, 18240] $\beta_0 = 1, \beta_1 = 1, \beta_2 = 1, \beta_3 = 1$

Cyranka et al (2020)

$X =$ 	$\text{PH}^{-1}(D) \simeq$ 
---	--

Mischaikow & Weibel (2021)

$X =$ 	$\text{PH}^{-1}(D) \simeq$ 
---	--

$X =$ 	$\text{PH}^{-1}(D_\theta) \simeq$ 
---	---

Inverse problems

Specialise to point clouds

- **Our setup:** Point cloud persistence with Čech and V–R filtrations
- $\text{PH}^{-1}(D)$ set of n -point configurations in $\mathbb{R}^d$ with
 - All points distinct
 - $\text{PH}(\mathbf{p}^{\{\check{\mathcal{C}}, \text{VR}\}}(r)) = D$
 - $D = [\mathcal{D}_1, \dots]$ the tuple of degree i homology barcodes

Inverse problems

Point cloud example

- **Our setup:** Point cloud persistence with Čech and V–R filtrations
- V–R example



Inverse problems

Previous point cloud work

- **Our setup:** Point cloud persistence with Čech and V–R filtrations
- Folklore: PH is isometry invariant
- Oudot and Soloman (2018): Ask when $\text{PH}^{-1}(D)$ is empty?
- Kurlin and Smith (2022): Configurations approximately on a line have trivial degree 1 persistent homology.

Main questions

- **Our setup:** Point cloud persistence with Čech and V–R filtrations
- Open: homotopy type of $\mathrm{PH}^{-1}(D)$?
 - Universality?
- We look at: dimension of $\mathrm{PH}^{-1}(D)$?
 - It is semi-algebraic, so dimension makes sense
- Goal: combinatorial conditions on $\mathbf{p}$ that
 - bound dimension
 - determine $\mathbf{p}$ up to isometry

Identifiability

- **Our setup:** Point cloud persistence with Čech and V–R filtrations
- **Goal:** combinatorial conditions on $\mathbf{p}$ that *determine $\mathbf{p}$ up to isometry*
- **Definition:** $\mathbf{p}$ is $\{\check{C}, VR\}$ -*identifiable* if, whenever

$$PH(\mathbf{p}^{\{\check{C}, VR\}}(r)) = PH(\mathbf{q}^{\{\check{C}, VR\}}(r)),$$

$\mathbf{p}$ and $\mathbf{q}$ are congruent

- **Goal:** Describe when $\mathbf{p}$ is identifiable
 - Depends on the filtration and the detail of how it works

4. Identifiability from rigidity

Critical graph

Looking more closely at the Vietoris–Rips filtration

- Recall: Rips complex $\mathbf{p}^{\text{VR}}(r)$

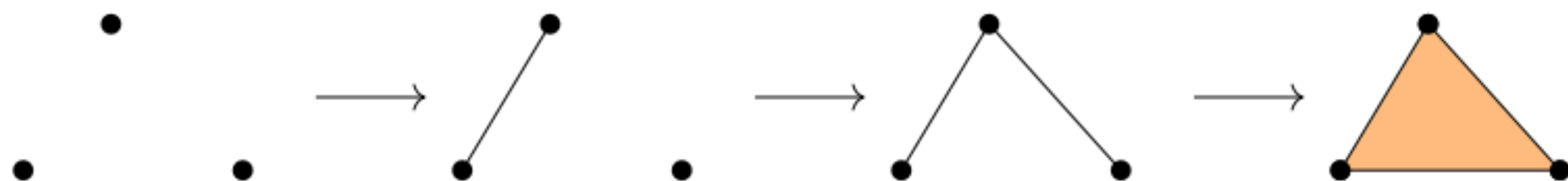
$$\sigma \in \mathbf{p}^{\text{VR}}(r) \quad \Longleftrightarrow \quad \text{diam}(\mathbf{p}_j)_{j \in \sigma} \leq 2r$$

- Changes in the Rips complex happen when r hits edge lengths
- Bounded endpoints of intervals in $\text{PH}^{\text{VR}}(\mathbf{p})$ correspond to the edge additions that change homology
- Definition:** $G_{\mathbf{p}}$ is the *critical graph* of $\mathbf{p}$, consists of the edges corresponding bounded endpoints of barcode intervals

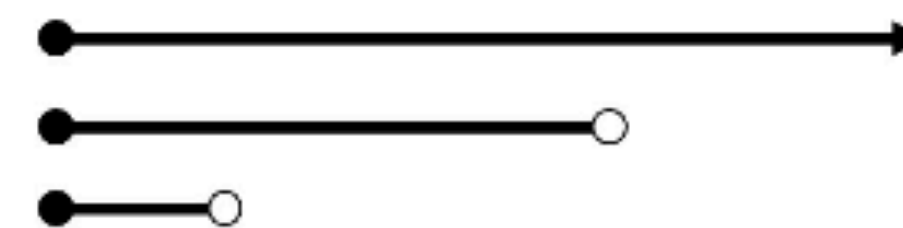
Critical graph

Examples

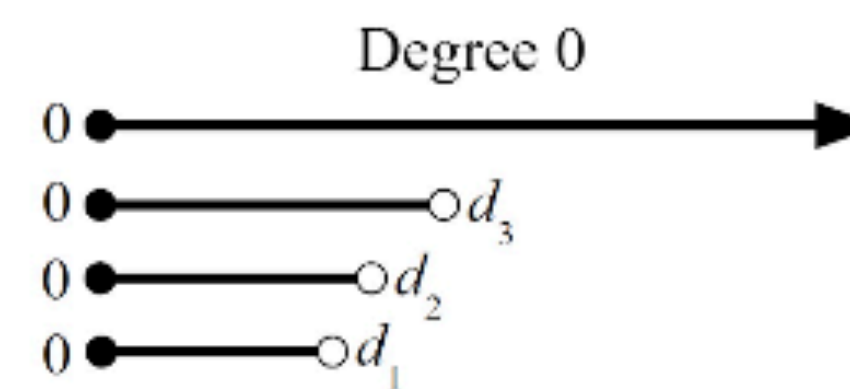
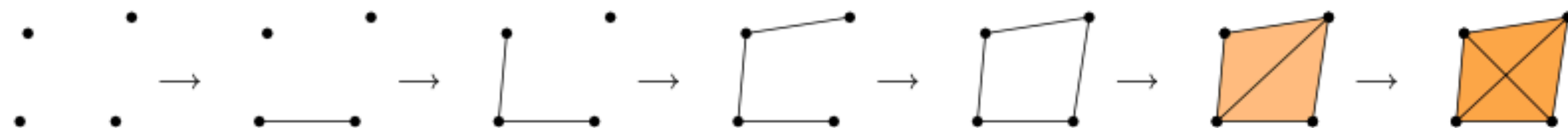
VR filtration



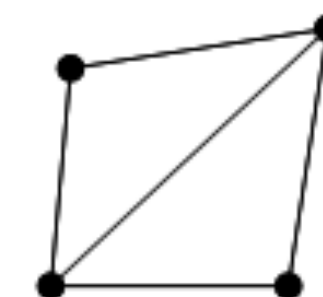
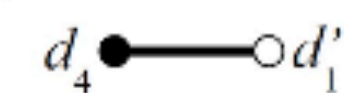
Barcode



G_p



Degree 1



Frameworks

Standard rigidity slides

- $G = (V, E)$ simple, undirected graph
 - $V = [n]$, ordered in the usual way
 - m edges, fix some bijection $E \rightarrow [m]$ to get an ordering
- $\mathbf{p}$ configuration of n points in $\mathbb{R}^d$
- Pair $(G, \mathbf{p})$ is a d -dimensional *framework*

Measurement map

Plus equivalence and congruence

- Measurement map $m_G : (\mathbb{R}^d)^n \rightarrow \mathbb{R}^m$ records square edge lengths

$$m_G(\mathbf{p}) = \left(\|\mathbf{p}_i - \mathbf{p}_j\| \right)_{ij \in E}$$

- $(G, \mathbf{p})$ and $(G, \mathbf{q})$ are *equivalent* if $m_G(\mathbf{p}) = m_G(\mathbf{q})$
 - They have the same edge lengths
- $\mathbf{p}$ and $\mathbf{q}$ are *congruent* if $(K_n, \mathbf{p})$ and $(K_n, \mathbf{q})$
 - Could also say there is $T \in \text{Euc}(d)$ so $\mathbf{q} = (T(\mathbf{p}_i))$

Rigidity and flexibility

And global rigidity

- A d -dimensional $(G, \mathbf{p})$ is *rigid* if there is a nbd $U \ni \mathbf{p}$ so that, for all $\mathbf{q} \in U$, if $(G, \mathbf{p})$ and $(G, \mathbf{q})$ are equivalent, then they are congruent.
- A framework that is not rigid is called *flexible*
- A d -dimensional $(G, \mathbf{p})$ is *globally rigid* if, when $(G, \mathbf{p})$ and $(G, \mathbf{q})$ are equivalent, they are congruent.
 - $(K_n, \mathbf{p})$ always globally rigid

Combinatorial rigidity

And genericity

- A configuration $\mathbf{p}$ is *generic* if its coordinates do not satisfy a non-trivial polynomial with coefficients in $\mathbb{Q}$
 - Examples of non-generic: not in general position, four points on a circle, ...
- **Theorem:** For every dimension d and graph G , either *every* generic $(G, \mathbf{p})$ is (globally) rigid or *no* generic $(G, \mathbf{p})$ is (globally) rigid.
 - Local part [Asimow–Roth '78], global part [Connelly '80s, Gortler–Healy–Thurston '10]

Critical framework

Looking more closely at the Vietoris–Rips filtration

- Recall: Rips complex $\mathbf{p}^{\text{VR}}(r)$

$$\sigma \in \mathbf{p}^{\text{VR}}(r) \quad \Longleftrightarrow \quad \text{diam}(\mathbf{p}_j)_{j \in \sigma} \leq 2r$$

- **Definition:** $G_{\mathbf{p}}$ is the *critical graph* of $\mathbf{p}$, consists of the edges corresponding bounded endpoints of barcode intervals
- **Definition:** $(G_{\mathbf{p}}, \mathbf{p})$ is the *critical framework* of $\mathbf{p}$

First attempt

- **Idea**: If the critical framework $(G_p, \mathbf{p})$ is globally rigid, then $\mathbf{p}$ is VR-identifiable from its barcode
- **Problem**: The association between $\text{PH}^{\text{VR}}(\mathbf{p})$ depends on $\mathbf{p}$
 - Can have $\mathbf{q}$ in $\text{PH}^{\text{VR}}(\mathbf{p})$ with G_p and G_q different
 - Critical frameworks can have the same multi-set of edge lengths without being congruent

Local identifiability

Definition

- **Definition:** $\mathbf{p}$ is locally $\{\check{C}, VR\}$ -*identifiable* if, there is a nbd $U \ni \mathbf{p}$ so that whenever, $\mathbf{q} \in U$ and

$$PH(\mathbf{p}^{\{\check{C}, VR\}}(r)) = PH(\mathbf{q}^{\{\check{C}, VR\}}(r)),$$

$\mathbf{p}$ and $\mathbf{q}$ are congruent

- Informally: same barcode implies congruent

Local identifiability

Result

- **Theorem (Beers–Harrington–Leygoine–Lim–LT ‘24+)**: If $\mathbf{p}$ is generic and $G_{\mathbf{p}}$ is locally d -rigid, then $\mathbf{p}$ is locally VR-identifiable.
- **Ideas:**
 - The map $\mathbf{p} \mapsto \text{PH}(\mathbf{p}^{\text{VR}}(r))$ is semi-algebraic
 - Decompose configuration space into semi-algebraic cells $S_{\leq}$ on which the map is $m_{G_{\mathbf{p}}}(\mathbf{p})$
 - Pick $U \subseteq S_{\leq}$ and use local rigidity

Local identifiability

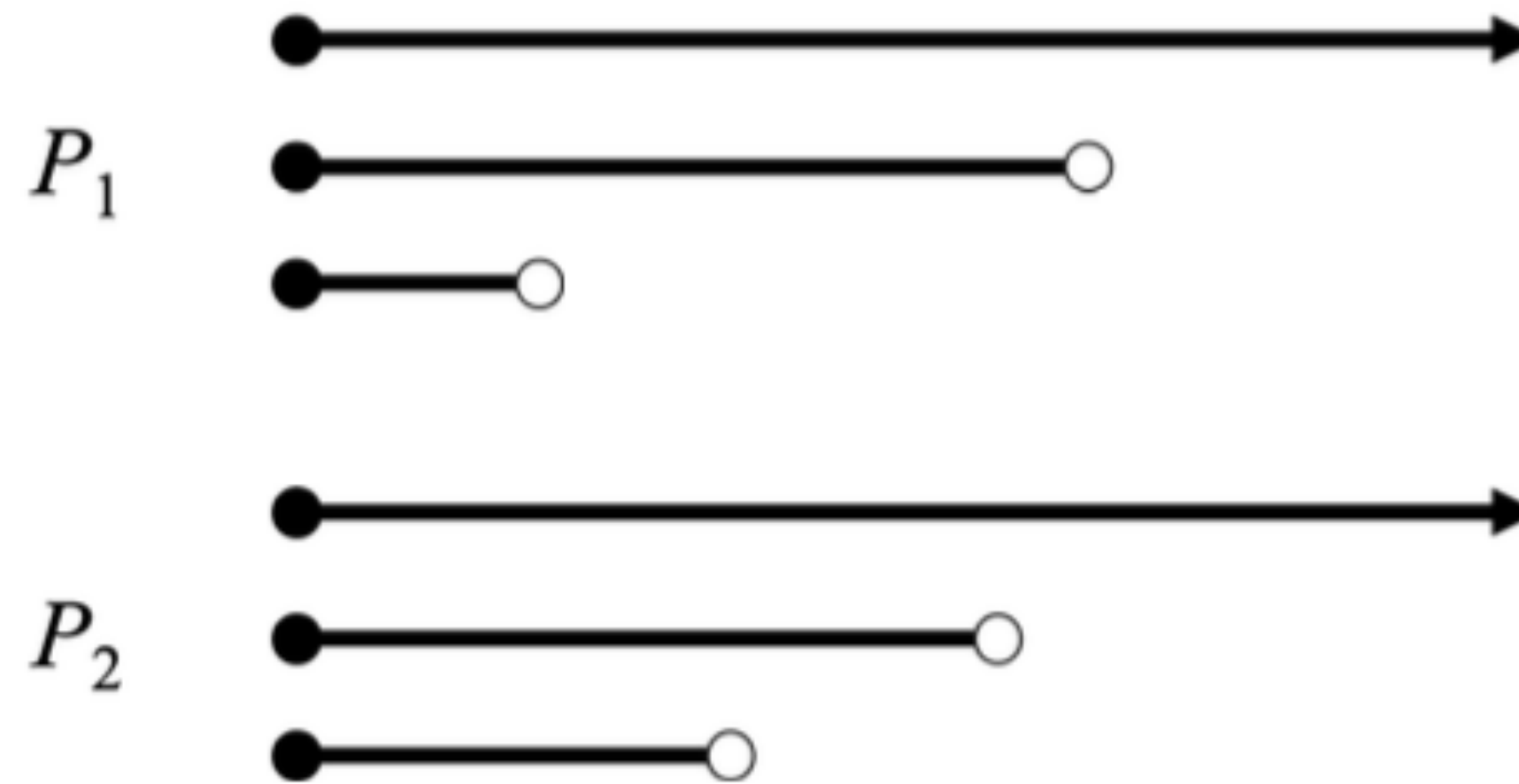
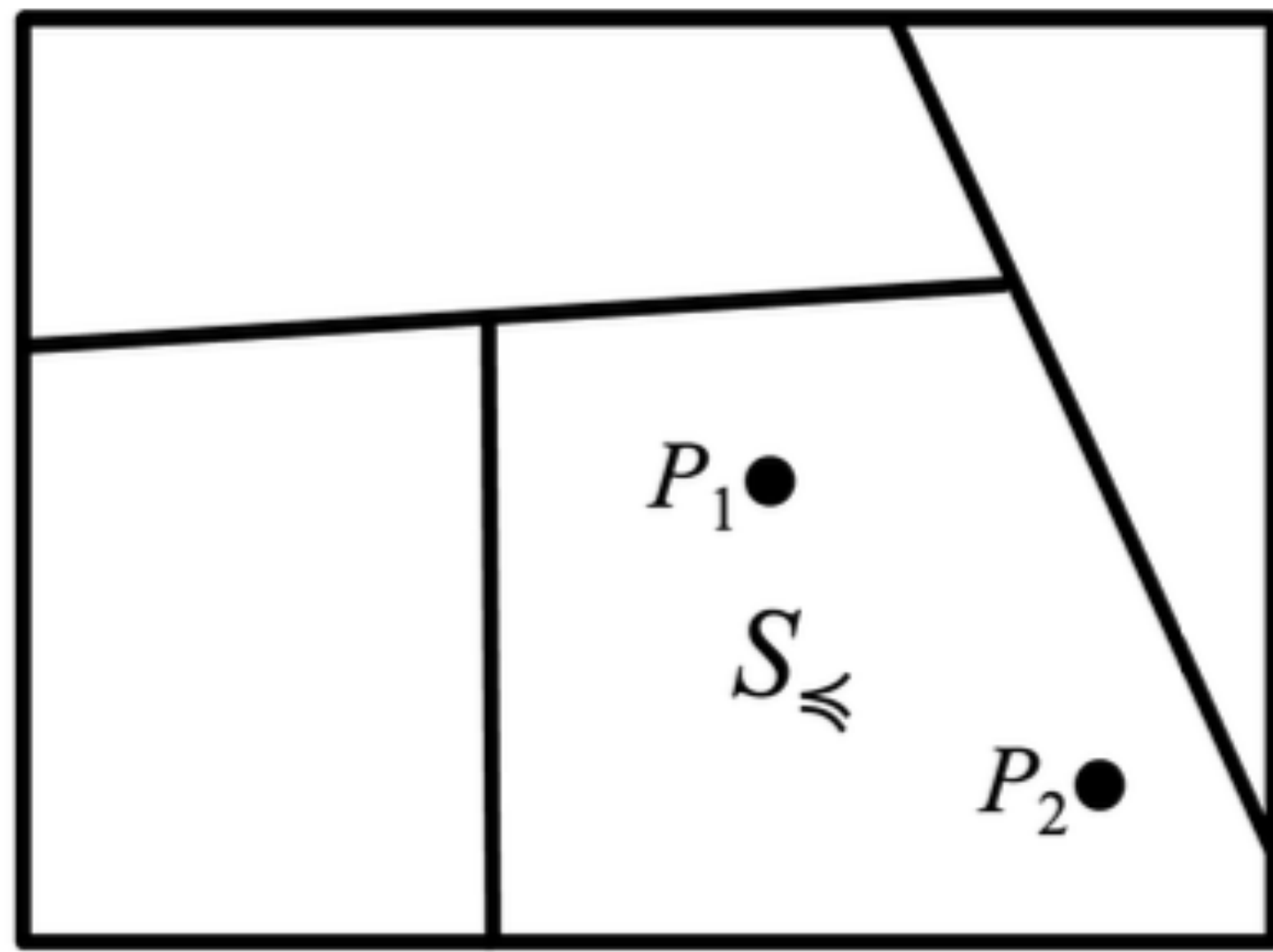
Result

- **Theorem (Beers–Harrington–Leygoine–Lim–LT ‘24+)**: If $\mathbf{p}$ is generic and $G_{\mathbf{p}}$ is locally d -rigid, then $\mathbf{p}$ is locally VR-identifiable.
- **Ideas**:
 - Decompose configuration space into *semi-algebraic* cells $S_{\leq}$ on which the map is $m_{G_{\mathbf{p}}}(\mathbf{p})$
 - Semi-algebraic nature observed by [Carrière, et al. 2022]

Local identifiability

Result

- **Theorem (Beers–Harrington–Leygoine–Lim–LT ‘24+):** If $\mathbf{p}$ is generic and $G_{\mathbf{p}}$ is locally d -rigid, then $\mathbf{p}$ is locally VR-identifiable.



Local identifiability

Result

- **Theorem (Beers–Harrington–Leygoine–Lim–LT ‘24+)**: If $\mathbf{p}$ is generic and $G_{\mathbf{p}}$ is locally d -rigid, then $\mathbf{p}$ is locally VR-identifiable.
- **Remarks:**
 - Converse also true
 - If not locally VR-identifiable, can deform framework inside $S_{\leq}$

Identifiability

Main result

- Theorem (Beers–Harrington–Leygoine–Lim–LT ‘24+): If $\mathbf{p}$ is generic and $G_{\mathbf{p}}$ is globally d -rigid, then $\mathbf{p}$ is identifiable.
- Remarks:
 - Converse is open
 - Sufficient conditions for $G_{\mathbf{p}}$ to be globally d -rigid open
 - Infinite families based on grids exist

Generic unlabeled global rigidity

- **Theorem (Gortler–LT–Thurston '19):** Let $G = ([n], E)$ be globally d -rigid and $(G, \mathbf{p})$ generic. If $H = ([n], F)$ and $(H, \mathbf{q})$ has the same multi-set of edge lengths, then G and H are isomorphic and, after relabeling, $\mathbf{q}$ is congruent to $\mathbf{p}$.
- Notably, $\mathbf{q}$ need not be generic
- Case of $G = K_n$ already non-trivial [Boutin–Kemper '04]
- If $\mathbf{q}$ generic [Garamvölgyi–Gortler–Jordán '21] relax hypothesis on H

Identifiability

Main result

- **Theorem (Beers–Harrington–Leygoine–Lim–LT ‘24+)**: If $\mathbf{p}$ is generic and $G_{\mathbf{p}}$ is globally d -rigid, then $\mathbf{p}$ is identifiable.
- **Ideas:**
 - If $\text{PH}^{\text{VR}}(\mathbf{q}(r)) = \text{PH}^{\text{VR}}(\mathbf{p}(r))$, then $m_{G_{\mathbf{p}}}(\mathbf{p})$ and $m_{G_{\mathbf{q}}}(\mathbf{q})$ agree up to permuting coordinates
 - Unlabelled GGR implies that $G_{\mathbf{p}}$ and $G_{\mathbf{q}}$ are isomorphic
 - After relabelling, $\mathbf{p}$ and $\mathbf{q}$ are congruent

5. Bonus results

Critical framework

For Čech-filtration

- Recall: Rips complex $\mathbf{p}^{\text{VR}}(r)$

$$\sigma \in \mathbf{p}^{\check{\mathbf{C}}}(r) \quad \Longleftrightarrow \quad \bigcap_{j \in \sigma} D(\mathbf{p}_j, r) \neq \emptyset$$

- Changes in the nerve complex happen when r passes the circumradius of $\{\mathbf{p}_j : j \in \sigma\}$
- Bounded endpoints of intervals in $\text{PH}^{\text{VR}}(\mathbf{p})$ correspond to simplex insertions that change homology
- **Definition:** $G_{\mathbf{p}}$ is the *critical hypergraph* of $\mathbf{p}$, consisting of simplices corresponding to bounded barcode endpoints
- **Definition:** $(G_{\mathbf{p}}, \mathbf{p})$ is the *critical framework* of $\mathbf{p}$

Circumsphere rigidity

- **Definition:** G_p is the *critical hypergraph* of $\mathbf{p}$
 - Not necessarily a uniform hypergraph
- **Definition:** $(G_p, \mathbf{p})$ is the *critical framework* of $\mathbf{p}$
- Measurement map for circumsphere frameworks measures the circumradius of the edges
 - Defined only for general enough position
- Measurement is a rational map; Asimow–Roth type hold

Local identifiability

Result for Č-filtration

- **Theorem (Beers–Harrington–Leygoine–Lim–LT ‘24+)**: If $\mathbf{p}$ is generic and $G_{\mathbf{p}}$ is locally d -circumsphere rigid, then $\mathbf{p}$ is locally Č-identifiable.
- **Remarks:**
 - Circumsphere rigidity largely undeveloped
 - GGR and unlabelled GGR wide open
 - Rigidity improvements yield identifiability improvements

Fibre dimension bounds

- The map to the full barcode is semi-algebraic
- Can use dimension theory to get upper and lower bounds on fibre dimension
- Upper bound: use spanning trees
- Lower bound (generic $\mathbf{p}$ only): use Jacobian rank at $\mathbf{p}$

6. Summary

What we learned

- Inverse problems in persistent homology relatively new area
- Point cloud persistence closely connected to rigidity and global rigidity
- Identifiability results for VR persistent homology
- Local results for VR and Č
- Dimension bounds in a very general setup
- Lots of interesting open problems

Thanks!