Lattice Polarizations and K3 Moduli Spaces

based on joint works with:

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(1) To motivate / frame the question : Elliptic Curves / $\mathbb{C}$

- **algebraic eq.** E smooth genus-one curve

$p_0 \in E$, $\exists p_0$ ample, $\varphi_{|3p_0|} : E \to \mathbb{P}^2$ embedding

Weierstrass form: $y^2 = 4x^3 - g_2 x - g_3$

$(g_2, g_3) \in \mathbb{C}^2$, $\Delta = g_2^3 - 27 g_3^2 \neq 0$

Invariants: $[g_2, g_3] \in WP(2,3)$, $J = g_2^3 / \Delta \in \mathbb{C}$

Moduli space:

$$\mathcal{M}_1 = \{ [g_2, g_3] \in WP(2,3) \mid \Delta \neq 0 \}$$

- **Periods** $E = \mathbb{C}/\Lambda$, $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$, $\tau \in \mathbb{C}$ $\mathrm{Im}(\tau) > 0$

  Classifying space : $\mathcal{F}_1 = \Gamma_1 \backslash \mathcal{H}_1$ where

  $$\mathcal{H}_1 = \{ \tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0 \} \quad , \quad \Gamma_1 = SL(2, \mathbb{Z})$$

- the two points of view are related

  $\wp : \mathcal{M}_1 \longmapsto \mathcal{F}_1$ period map , isomorphism

- modular forms : inverse of $\wp$ can be made explicit

  $$[g_2, g_3] = [60 \cdot E_4(\tau) \, , \, 140 \cdot E_6(\tau)]$$

  with E_4, E_6 Eisenstein series.

$$M_*(\Gamma_1, \mathbb{C}) = \mathbb{C}[E_4, E_6]$$

(2) Goal of the talk:

Describe special K3 situations where a similar pattern occurs

- explicit algebraic construction
- explicit coarse moduli space
- classifying space of periods
- connection with appropriately-defined (orthogonal) modular forms

③ Lattice Polarizations (Nikulin, Dolgachev, Alexeev-Engel)

X smooth K3 surface $/\mathbb{C}$, $L_{K3} \simeq H^2(X, \mathbb{Z})$

Fix:
- M even lattice, rank $r \leq 19$, signature $(1, r-1)$
- $M \hookrightarrow L_{K3}$ primitive lattice embedding, $T = M^{\perp}$
- $h \in M_{\mathbb{R}}$ very-irrational, $(h,h) > 0$

Definition

A M-polarized K3 surface is a pair (X, i) such that:

(i) X is smooth K3 surface

(ii) $i : M \hookrightarrow NS(X)$ is primitive lattice embedding

(iii) $i(h)$ is quasi-ample (big and nef)

Two M-polarized K3's (X_1, i_1) and (X_2, i_2) are said to be isomorphic if there is an isomorphism $\varphi : X_1 \to X_2$ such that $\varphi^* \circ i_2 = i_1$.

④ Torelli Theory (Dolgachev, Alexeev-Engel)

Given (X, i), the orthogonal complement $i(M)^{\perp}$ in $H^2(X, \mathbb{Z})$ inherits a K3-type Hodge structure of weight two.

Period domain: $\Omega = \{ [w] \in \mathbb{P}'(T_{\mathbb{C}}) \mid (w,w) = 0, (w, \bar{w}) > 0 \}$

Properties of Ω:

- open variety, $\dim = 20 - r$
- two connected components, pick one, denote it by Ω^+
- Ω^+ is Hermition symmetric domain of type IV_{20-r}
- important arithmetic groups $\widetilde{O}(T) < O^+(T) < O(T)$

$O^+(T)$ = spinor kernel, preserves Ω^+

$\widetilde{O}(T) = \{ \delta \in O^+(T) \mid \forall x \in T^\vee, \delta(x) - x \in S \}$

Theorem (Torelli)

(i) $\exists\ \mathcal{M}_M$ coarse moduli space for M-polarized K3 surfaces

(ii) $\exists$ period map $\wp : \mathcal{M}_M \longrightarrow \mathcal{F}_M = \tilde{O}(T)\backslash \Omega^+$

(iii) $\wp$ is an isomorphism of quasi-proj. varieties

Space of interest :

$$\mathcal{F}_M = \tilde{O}(T)\backslash \Omega^+$$

- $\dim = 20 - r$
- bounded symmetric domain factored by a discrete arithmetic group

(5) Orthogonal Modular Forms

Define $\mathcal{H} = \{ z \in T_{\mathbb{C}} \mid [z] \in \Omega^+ \}$ affine cone over Ω^+

$\Gamma \leq O^+(T)$

Definition

A modular form of weight k and level Γ is a holomorphic map $f : \mathcal{H} \to \mathbb{C}$ satisfying:

(i) $f(\gamma \cdot z) = f(z)$ for $\gamma \in \Gamma$

(ii) $f(\lambda z) = \lambda^{-k} f(z)$ for $\lambda \in \mathbb{C}^*$

Graded ring: $M_*(\Gamma) = \bigoplus_{k \geq 0} M_k(\Gamma)$

$M_*(\Gamma)$ is well-understood, for selected T, due to work by Woitila and Wang-Williams.

Theorem (Wang-Williams)

For lattices T listed in the left-side column below, the graded ring $M_*(O^+(T))$ is freely generated by a set of modular forms with weights tabulated in the right-side columns.

T	weights
$H^{\oplus 2}$	4 6 12
$H^{\oplus 2} \oplus A_1(-1)$	4 6 10 12
$H^{\oplus 2} \oplus A_1(-1)^{\oplus 2}$	4 6 8 10 12
$H^{\oplus 2} \oplus A_1(-1)^{\oplus 3}$	4 6 6 8 10 12
$H^{\oplus 2} \oplus A_1(-1)^{\oplus 4}$	4 4 6 6 8 10 12

Here: H - rank 2 hyperbolic

(6) $M = H \oplus E_8(-1) \oplus E_8(-1) \oplus A_1(-1)$, $r = 8$ (Kondo) $T = H \oplus A_1$

$\mathcal{F}_M = SL(2,\mathbb{Z}) \backslash \mathcal{H}_1$ same classifying space as the one for elliptic curves

Theorem

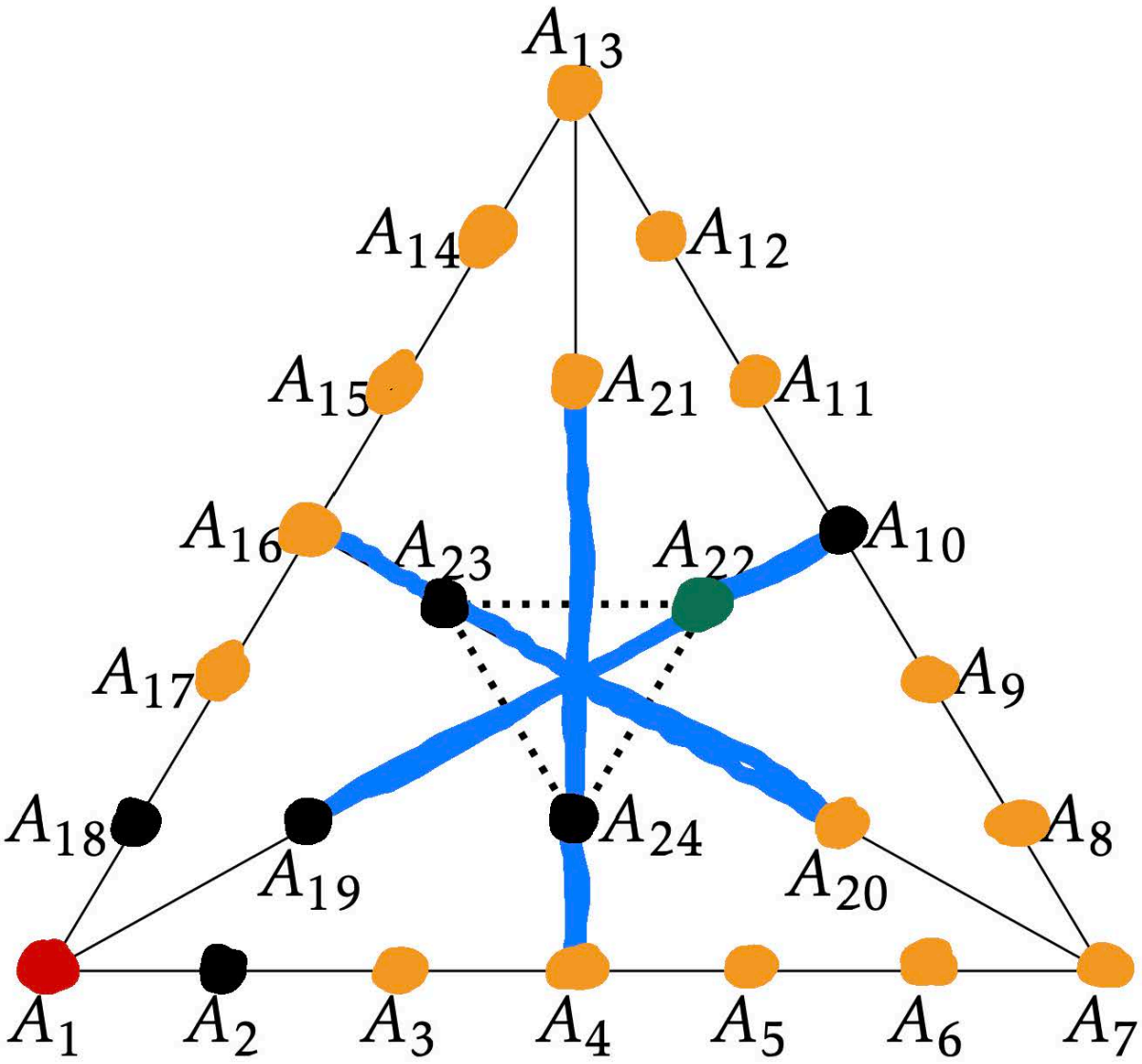
Let $E = \{ y^2 = 4x^3 - g_2 x - g_3 \}$ with $\Delta = g_2^3 - 27 g_3^2 \neq 0$. Consider:

$$x^2 z w - 4x^3 z + 3 g_2^2 x z w^2 + 27 g_3^2 z w^3 - \frac{1}{2}\left(\Delta^2 z^2 w^2 + w^4\right) = 0 \qquad (*)$$

quartic surface in $\mathbb{P}^3(x, y, z, w)$

The minimal resolution of (*) is a K3 surface $X(g_2, g_3)$ endowed with a canonical $H \oplus E_8(-1) \oplus E_8(-1) \oplus A_1(-1)$ polarization.

Generic case : $X(g_2, g_3)$ carries a 24-curve configuration of smooth rational curves



2-intersect

6-intersects

$H \oplus E_8(-1) \oplus E_8(-1) \oplus A_1(-1)$

- All $H \oplus E_8(-1) \oplus E_8(-1) \oplus A_1(-1)$ - polarized K3's can be realized as $X(g_2, g_3)$
- $X(g_2, g_3) \simeq X(\tilde{g}_2, \tilde{g}_3) \iff \exists\, t \in \mathbb{C}^*,\ \tilde{g}_2 = t^2 g_2,\ \tilde{g}_3 = t^3 g_3$
- $\mathcal{M} = \{ [g_2, g_3] \in W(2,3) \mid \Delta \neq 0 \}$

  is a coarse moduli space for

  $H \oplus E_8(-1) \oplus E_8(-1) \oplus A_1(-1)$ - polarized K3 surfaces.

(7) $\boxed{M = H \oplus E_8(-1) \oplus E_8(-1)}$ $\quad r = 18$, $\quad T = H \oplus H$

$$\mathcal{F}_M = \underbrace{\left[SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})\right] \rtimes \mathbb{Z}/2\mathbb{Z} \;\backslash\; \mathcal{H}_1 \times \mathcal{H}_1}_{\text{Hilbert Modular Surface}}$$

Theorem

Let $(a,b,d) \in \mathbb{C}^3$, with $d \neq 0$. Consider the quartic in $\mathbb{P}^3$:

$$y^2zw - 4x^3z + 3axzw^2 + bzw^3 - \frac{1}{2}(dz^2w^2 + w^4) = 0$$

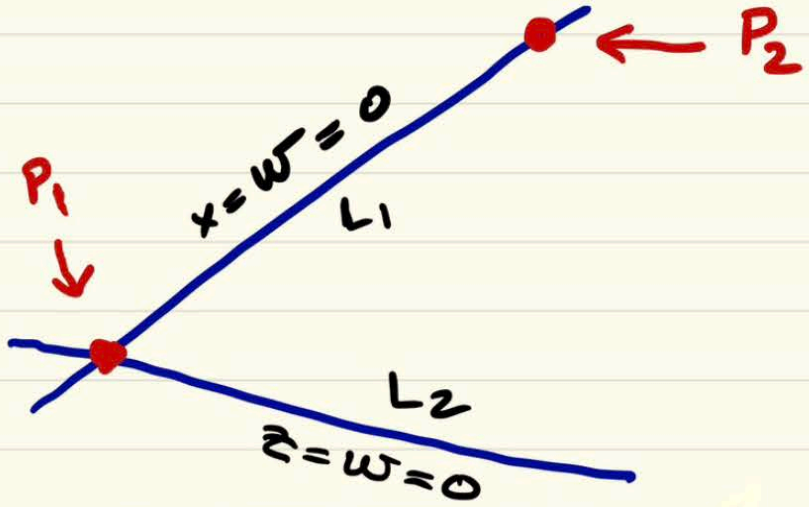
Minimal resolution of the above quartic is a K3 surface $X(a,b,d)$ carrying a canonical $H \oplus E_8(-1) \oplus E_8(-1)$ polarization.

Inose quartic

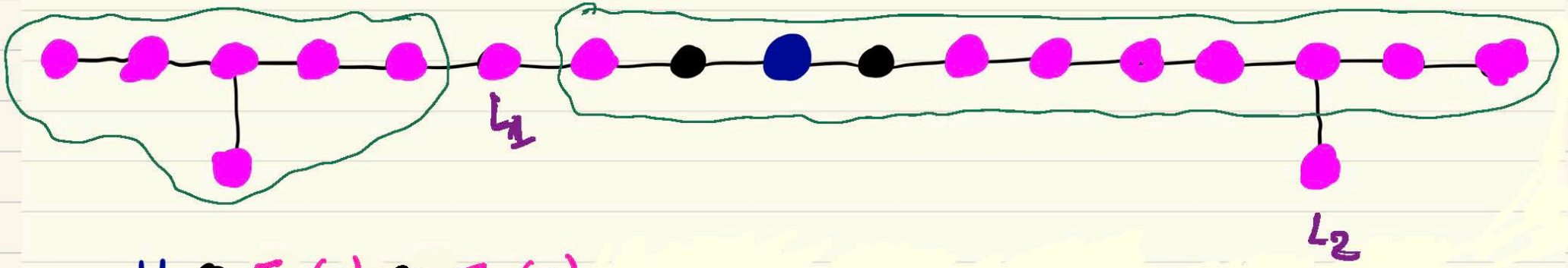
Singular Points
(generic case)

$P_1 : [0\ 1\ 0\ 0] \leftarrow A_{11}$

$P_2 : [0\ 0\ 1\ 0] \leftarrow E_6$



Generic case : 19-curve configuration of rational curves



$H \oplus E_8(-1) \oplus E_8(-1)$

Theorem [C-Doran]

(a) All $H \oplus E_8 \oplus E_8$ - polarized K3 surfaces, up to izomorphism, can be realized as $X(a,b,d)$

(b) $X(a,b,d) \simeq X(\tilde{a},\tilde{b},\tilde{d})$ if and only if

$$\tilde{a} = t^2 \cdot a \,, \quad \tilde{b} = t^3 \cdot b \,, \quad \tilde{d} = t^6 \cdot d$$

for some $t \in \mathbb{C}^*$

Obs: $\mathcal{M}_{H \oplus E_8 \oplus E_8} \overset{\text{def}}{=} \{ [a,b,d] \in WP(2,3,6) \mid d \neq 0 \}$

is a coarse moduli space for $H \oplus E_8 \oplus E_8$ - polarized K3 surfaces.

Recall: $\mathcal{F}_{H \oplus E_8 \oplus E_8} = \Gamma \backslash \mathcal{H}_1 \times \mathcal{H}_1$

$$\Gamma = [SL(2,\mathbb{Z}) \times SL(2,\mathbb{Z})] \rtimes \mathbb{Z}/2\mathbb{Z}$$

Theorem [C-Doran] Inverse period map is given by:

$$[a,b,d] = [g_2(\tau_1) g_2(\tau_2), 27\, g_3(\tau_1) g_3(\tau_2), \Delta(\tau_1)\Delta(\tau_2)]$$

where:

$$g_2(\tau) = 60 \cdot E_4(\tau), \quad g_3(\tau) = 140 \cdot E_6(\tau), \quad \Delta(\tau) = g_2^3(\tau) - 27 \cdot g_3^2(\tau)$$

- $M_*(O^+(T)) = \mathbb{C}[E_4 \otimes E_4, E_6 \otimes E_6, \Delta \otimes \Delta]$

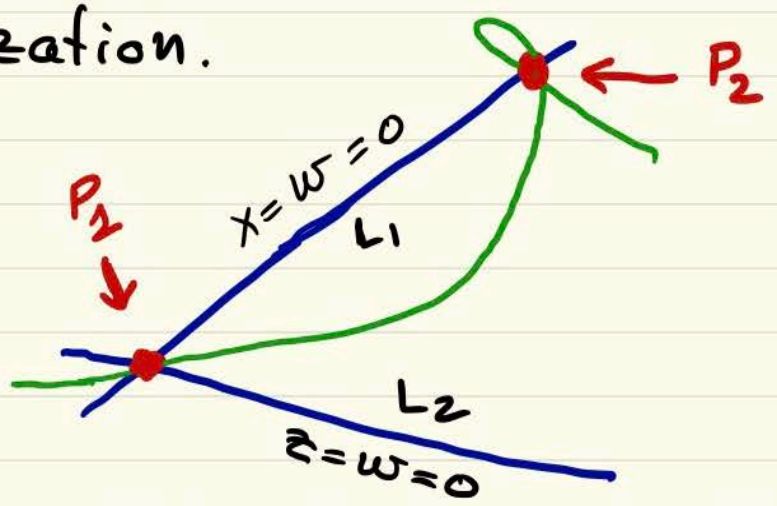
(8) $\boxed{M = H \oplus E_8(-1) \oplus E_7(-1)}$ $\quad r = 17, \quad T = H \oplus H \oplus A_1(-1)$

Theorem [C-Doran]

Let $(a,b,c,d) \in \mathbb{C}^4$ with $(c,d) \neq (0,0)$. Consider the quartic in $\mathbb{P}^3$:

$$y^2zw - 4x^3z + 3axzw^2 + bzw^3 + cxz^2w - \frac{1}{2}(dz^2w^2 + w^4) = 0$$

Singularities are rational double points and the min. resolution is a K3 surface $X(a,b,c,d)$ carrying a canonical $H \oplus E_8(-1) \oplus E_7(-1)$ polarization.

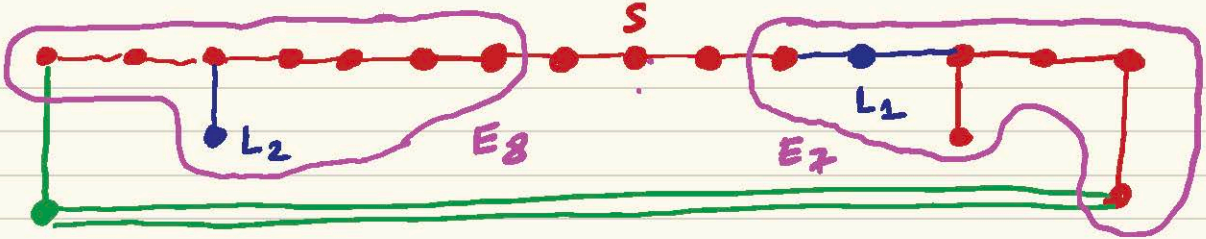


Singular Points
(generic case)

$P_1 : [0\,1\,0\,0] \leftarrow A_{11}$

$P_2 : [0\,0\,1\,0] \leftarrow A_5 \quad (c \neq 0)$

$$2cx - dw = [3ac^2d + 2bc^3 - d^3]\,zw^2 - c^3w^3 + 2c^3y^2z = 0$$



$M = E_8(-1) \oplus E_7(-1) \oplus \langle s \rangle$,



Theorem [c-Doran]

(a) All $H \oplus E_8(-1) \oplus E_7(-1)$-polarized K3 surfaces, up to isomorphism, can be realized as $X(a, b, c, d)$

(b) $X(a, b, c, d) \simeq X(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$ if and only if:

$$\tilde{a} = t^2 \cdot a, \quad \tilde{b} = t^3 \cdot b, \quad \tilde{c} = t^5 \cdot c, \quad \tilde{d} = t^6 \cdot d$$

for some $t \in \mathbb{C}^*$

$$\mathcal{M}_{H\oplus E_8\oplus E_7} \overset{\text{def}}{=} \{\, [a,b,c,d] \in WP(2,3,5,6) \mid (c,d) \neq (0,0) \}$$

is a coarse moduli space for $H\oplus E_8\oplus E_7$ - polarized K3 surfaces.

Siegel Modular Threefold

Lemma $\mathcal{F}_{H\oplus E_8\oplus E_7} = Sp(4,\mathbb{Z}) \backslash \mathcal{H}_2$

$$\mathcal{H}_2 = \left\{ \tau = \begin{pmatrix} x & y \\ y & z \end{pmatrix} \in M(2\times 2,\mathbb{C}) \;\middle|\; \mathrm{Im}(x)\cdot\mathrm{Im}(z) > \mathrm{Im}(y)^2,\ \mathrm{Im}(x)>0 \right\}$$

$$Sp(4,\mathbb{Z}) = \left\{ m = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(4,\mathbb{Z}) \;\middle|\; M^t J M = J \text{ where } J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \right\}$$

$$m\cdot\tau = (A\tau+B)(C\tau+D)^{-1}$$

Theorem [Igusa]

$$M_*(Sp(4,\mathbb{Z}),\mathbb{C}) \simeq \mathbb{C}[\varepsilon_4,\varepsilon_6,C_{10},C_{12},C_{35}] \Big/ C_{35}^2 = P_{70}(\varepsilon_4,\varepsilon_6,C_{10},C_{12})$$

graded ring of modular forms for $Sp(4,\mathbb{Z})$ — Eisenstein Series — Cusp Forms

$$C_{10} = (\cdots)\left[\varepsilon_4\varepsilon_6 - \varepsilon_{10}\right], \quad C_{12} = (\cdots)\left[3^2\cdot 7^2\varepsilon_4^3 + 2\cdot 5^3\cdot\varepsilon_6^2 - 691\cdot\varepsilon_{12}\right]$$

$$M_*(O^+(T)) \simeq \mathbb{C}[\varepsilon_4,\varepsilon_6,C_{10},C_{12}]$$

Theorem [C-Doran, Elkies-Kumar, Malmendier-Morrison]

Inverse period map for $M = H\oplus E_8(-1)\oplus E_7(-1)$ is

$$[a,b,c,d] = [\varepsilon_4(\tau),\ \varepsilon_6(\tau),\ 2^{12}3^5\cdot C_{10},\ 2^{12}\cdot 3^6\cdot C_{12}]$$

(9) $M = H \oplus E_7(-1) \oplus E_7(-1)$, $r = 16$, $T = H \oplus H \oplus A_1(-1)^{\oplus 2}$

Theorem [C-Malmendier]

Let $(a,b,c,d,e,f) \in \mathbb{C}^6$ with $(c,d) \neq (0,0)$ and $(e,f) \neq (0,0)$

Consider the following quartic surface in $\mathbb{P}^3$:

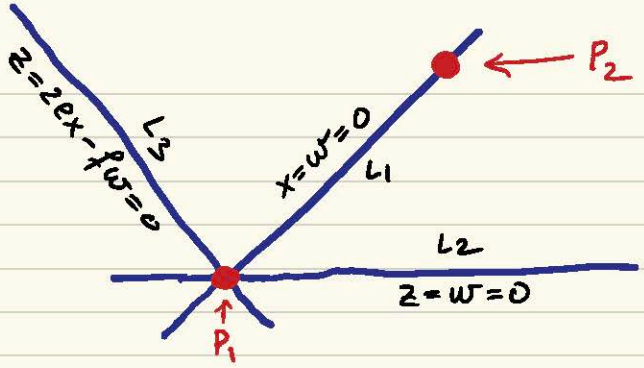
$$y^2zw - 4x^3z + 3axzw^2 + bzw^3 + cxz^2w - \frac{1}{2}(dz^2w^2 + fw^4) + exw^3 = 0$$

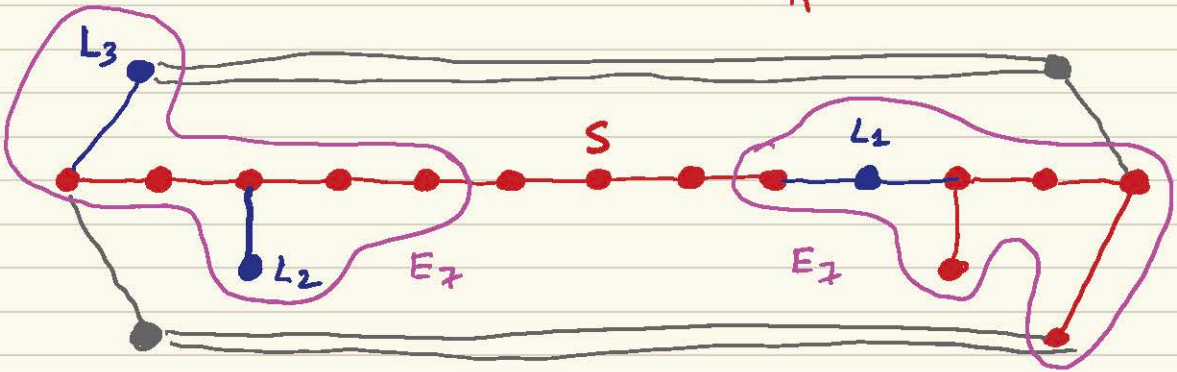
Singularities of above quartic are rational double points and the minimal resolution $X(a,b,c,d,e,f)$ is a K3 surface carrying a canonical $H \oplus E_7(-1) \oplus E_7(-1)$ polarization.

Singular Points
(generic case)

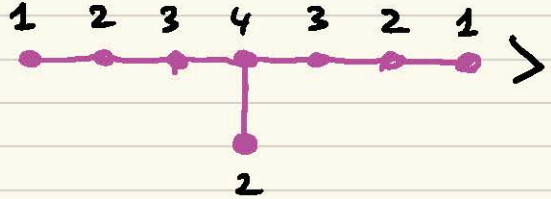
P_1 : [0100] ← A_9

P_2 : [0010] ← A_5





$$M = E_7(-1) \oplus E_7(-1) \oplus \langle S, \rangle$$



Theorem [C- Malmendier]

(a) All $H \oplus E_7(-1) \oplus E_7(-1)$-polarized K3 surfaces, up to isomorphism, can be realized as $X(a,b,c,d,e,f)$

(b) $X(a,b,c,d,e,f) \simeq X(\tilde{a},\tilde{b},\tilde{c},\tilde{d},\tilde{e},\tilde{f})$

if and only if:

(i) $(\tilde{a},\tilde{b},\tilde{c},\tilde{d},\tilde{e},\tilde{f}) = (a,b,e,f,c,d)$

OR

(ii) $(\tilde{a},\tilde{b},\tilde{c},\tilde{d},\tilde{e},\tilde{f}) = (t^2 \cdot a, t^3 \cdot b, t^5 \cdot c, t^6 \cdot d, t^{-1} \cdot e, f)$

for some $t \in \mathbb{C}^*$

One defines then the following invariants:

$$K_4 = a$$

$$K_6 = b$$

$$K_8 = ce$$

$$K_{10} = cf + de$$

$$K_{12} = df$$

Theorem [C-Malmendier]

$$\mathcal{M}_{H \oplus E_7(-1) \oplus E_7(-1)} = \left\{ [K_4, K_6, K_8, K_{10}, K_{12}] \in WP(2,3,4,5,6) \;\middle|\; (K_8, K_{10}, K_{12}) \neq (0\,0\,0) \right\}$$

is a coarse moduli space for $H \oplus E_7(-1) \oplus E_7(-1)$ - polarized K3 surfaces.

Period Moduli Space

Matsumoto - Sasaki - Yoshida

$$\mathcal{F}_M = G \backslash D$$

Hermitian Symmetric Domain of Type IV

$$D = \left\{ W \in M(2\times 2, \mathbb{C}) \;\middle|\; i(W^* - W) > 0 \right\}$$

$$W = \begin{pmatrix} x & y \\ z & w \end{pmatrix}, \quad 4 \cdot \mathrm{Im}(x) \cdot \mathrm{Im}(w) > |y - \bar{z}|^2$$

$$G_1 = \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(4, \mathbb{Z}[i]) \;\middle|\; g^* J g = J, \; J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \right\}$$

$$*W \overset{\text{def}}{=} W^t \qquad *g*W = \bar{g}W$$

$$G = G_1 \rtimes \langle * \rangle$$

Theorem [C-Malmendier]

(i) $K_4, K_6, K_8, K_{10}, K_{12}$ are modular forms of weight 4, 6, 8, 10 and 12, respectively, with respect to group G.

(ii) $M_{2k}(G, \mathbb{C}) = \mathbb{C}[K_4, K_6, K_8, K_{10}, K_{12}]$

graded ring of even weight modular forms over G

Going beyond rank 16 ?

Observation (based on rank = 18, 17, 16 examples):

- polarizations M considered are lattices of 2-elementary type, i.e. $\mathcal{D}_M = M^\vee/M \simeq \left(\mathbb{Z}/2\mathbb{Z}\right)^e$
- $|\mathrm{Aut}(X)| < \infty$ [In fact, $\mathrm{Aut}(X) \simeq \left(\mathbb{Z}/2\mathbb{Z}\right)^{\oplus 2}$ for X generic]
- In examples considered, the K3's in question carried a "secondary" Jacobian elliptic fibration that also has a section of order 2.

When does a K3 surface carry such an elliptic fibration?

Nikulin Lattice

rank = 8

$$N = \langle c_1, c_2, \cdots, c_8, \tfrac{1}{2}(c_1 + c_2 + \cdots + c_8) \rangle$$

$$c_i \cdot c_j = -2\delta_{ij}$$

Theorem (Nikulin, Van Geemen, Sarti)

$H \oplus N$-polarization on X $\Longleftrightarrow$ Jacobian elliptic fibration with an order-two section

$\varphi : X \longrightarrow \mathbb{P}^1$ elliptic fibration

S_0 "base" section

S_1 section, has order 2 in $MW(\varphi, S_0)$

$$y^2 = x\left(x^2 + a(t)x + b(t)\right)$$

$S_0 : \; x = \infty$

$S_1 : \; x = 0$

Van Geemen - Sarti form

$$y^2 = x\left(x^2 + a(t)x + b(t)\right)$$ Van Geemen - Sarti form for the "alternate" fibration

Polarization	$a(t)$, $b(t)$	$M_*(O^+(T))$ generators
$H \oplus E_8(-1) \oplus E_8(-1)$	$a(t) = a_4 t + a_6$ $b(t) = b_{12}$	a_4 a_6 b_{12}
$H \oplus E_8(-1) \oplus E_7(-1)$	$a(t) = a_4 t + a_6$ $b(t) = b_{10} t + b_{12}$	a_4 a_6 b_{10} b_{12}
$H \oplus E_7(-1) \oplus E_7(-1)$	$a(t) = a_4 t + a_6$ $b(t) = b_8 t^2 + b_{10} t + b_{12}$	$a_4, a_6, b_8, b_{10}, b_{12}$

Going beyond rank 16 ?

Perhaps look at lattice polarizations M satisfying:

- $M = H \oplus (\text{ADE types})$
- $\exists \; H \oplus N \hookrightarrow M$
- $|\mathrm{Aut}(X)| < \infty$
- $? \leq r$ (in this talk $13 \leq r$)

Classification results (Nikulin, Shimada, Festi-Veniani, Garbagnati-Selgado) allow one to narrow down the list.

List :

$r = 18$ $\quad H \oplus E_8(-1) \oplus E_8(-1)$

$r = 17$ $\quad H \oplus E_8(-1) \oplus E_7(-1)$

$r = 16$ $\quad H \oplus E_7(-1) \oplus E_7(-1)$

$r = 15$ $\quad H \oplus E_8(-1) \oplus D_4(-1) \oplus A_1(-1)$

$r = 14$ $\quad H \oplus E_8(-1) \oplus A_1(-1)^{\oplus 4}$, $H \oplus D_8(-1) \oplus D_4(-1)$

$r = 13$ $\quad H \oplus E_7(-1) \oplus A_1(-1)^{\oplus 4}$

Theorem (C-Malmendier-Williams)

Let M be one of the lattices on the previous list, rank ≥ 14.

(i) A M-polarized K3 surface X carries a canonical Jacobian elliptic fibration $\varphi : X \to \mathbb{P}^1$ with an additional section of order two. Generically, $MW(\varphi) = \mathbb{Z}/2\mathbb{Z}$.

(ii) A van Geemen-Sarti form for the fibration above can be given as $y^2 = x\left(x^2 + a(t)x + b(t)\right)$ where:

(a) For all M, except $H \oplus D_8(-1) \oplus D_4(-1)$:

$$a(t) = t^3 + a_4 t + a_6 \qquad b(t) = \sum_{i=0}^{18-\lambda} b_{2(6-i)} t^i$$

(b) For $M = H \oplus D_8(-1) \oplus D_4(-1)$:

$$a(t) = 2\left[c_2 t^2 + c_6 t + c_{10}\right], \quad b(t) = t^5 + d_8 t^3 + d_{12} t^2 + d_{16} t + d_{20}$$

(iii) Coefficients $\{a_n, b_m\}$ and $\{c_n, b_m\}$ are unique, up to weighted scaling

(iv) $\mathcal{M}_M = \{ [a_n : b_m] \in WP(\text{weights}) \mid \text{conditions} \}$

gives a coarse moduli space for M-polarized k3's.

(v)

M	T
$H \oplus E_8(-1) \oplus E_8(-1)$	$H \oplus H$
$H \oplus E_8(-1) \oplus E_7(-1)$	$H \oplus H \oplus A_1(-1)$
$H \oplus E_7(-1) \oplus E_7(-1)$	$H \oplus H \oplus A_1(-1)^{\oplus 2}$
$H \oplus E_8(-1 \oplus D_4(-1) \oplus A_1(-1)$	$H \oplus H \oplus A_1(-1)^{\oplus 3}$
$H \oplus E_8(-1) \oplus A_1(-1)^{\oplus 4}$	$H \oplus H \oplus A_1(-1)^{\oplus 4}$
$H \oplus D_8(-1) \oplus D_4(-1)$	$H \oplus H(2) \oplus D_4(-1)$

Coefficients $\{a_n, b_m\}$ and $\{c_c, d_m\}$ are modular forms with respect to $O^+(T)$ and generators for the graded ring $M_*(O^+(T))$.

Case $M = H \oplus E_7(-1) \oplus A_1(-1)^{\oplus 4}$ $\quad r = 13 \quad T = H \oplus H \oplus A_1(-1)^{\oplus 5}$

$$y^2 = x\left[x^2 + a(t)x + b(t)\right]$$

$$a(t) = t^3 + a_4 t + a_6$$

$$b(t) = b_2 t^5 + b_4 t^4 + b_6 t^3 + b_8 t^2 + b_{10} t + b_{12}$$

- Domain Ω^+ contains a special divisor C'
  (corresponding to $H \oplus D_8(-1) \oplus D_4(-1)$)

- Coefficients $\{a_n, b_m\}$ are meromorphic modular forms of weights $(2, 4, 4, 6, 6, 8, 10, 12)$ with respect to a subgroup $\Gamma < O^+(T)$

- $\{a_n, b_m\}$ generate the graded ring $M^!_*(\Gamma)$ of meromorphic modular forms with poles along the divisor C'.