Moduli of Calabi–Yau 3 folds and Mirror Symmetry.

$X \& Y$ mirror polarized CY3s

w/ UMass grad student Yizhao Wang

complex geometry of X $\sim$ symplectic geometry of Y, and vice versa.

Kontsevich '94 "Homological Mirror Symmetry"

$D^b Coh\, X \simeq \mathcal{F}(Y)$ Fukaya category

Ob : Lagrangian submanifolds

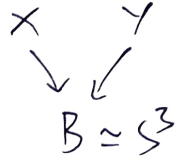
Mor : Floer cohomology.

$\Rightarrow \quad \pi_1(M_Y) \longrightarrow Auteq(D^b(Coh\, X)) / [2]$

M_Y = complex moduli of Y.

– monodromy acts via symplectomorphisms.

Strominger-Yau-Zaslow '96.



dual special Lagrangian torus fibrations
(near "large complex structure limit")

$X \to B$ obtained via Gromov-Hausdorff collapse for Ricci flat metric

Topological SYZ fiber is Clemens vanishing cycle $\left(|x_1| = \dots = |x_4| = |t|^{1/4}\right) \subset (x_1 \cdots x_4 = t)$

Hodge theory: LCSL $\iff$ limit MHS is Hodge-Tate (all type (p,p))

$\iff$ degeneration is maximally unipotent
AND Intermediate Jacobian is maximally degenerating (no compact part)

$\iff$ dlt model $X \rightsquigarrow \overline{X} = \cup X_i$ (K trivial + $\simeq$ semistable)
has 0-stratum AND $H^3(X_i) = 0$ $\forall i$.

HMS + SYZ

$\rightsquigarrow$ LCSL's of M_Y $\longleftrightarrow$ $\{X \mid D^b \mathrm{Coh}\, X \simeq \mathcal{F}(Y)\}$ / bir. equiv.

K3 OK (Mukai, Orlov, Shoukei Ma)

Morrison '93, '96.

Near LCSL, $\exists$ Looijenga compactification

$$(p \in \bar{M}_y) \supset (p \in \bar{M}_y) \setminus \partial \bar{M}_y = (V + iC / L) / \Gamma$$

- $L = \mathrm{Pic}\, X \overset{c_1}{=} H^2(X, \mathbb{Z})$
- $V = L \otimes_{\mathbb{Z}} \mathbb{R} = H^2(X, \mathbb{R})$
- $C \subset V$ open convex cone (e.g. Kähler cone = ample cone)
- $V + iC \subset L \otimes_{\mathbb{Z}} \mathbb{C} = H^2(X, \mathbb{C})$ "tube domain" (complexified Kähler cone)
- $V + iC / L \subset L \otimes_{\mathbb{Z}} \mathbb{C}^\times$, open analytic subset of algebraic torus.
- $\Gamma \curvearrowright (C, L)$ discrete group (e.g. $\mathrm{Aut}\, X$)

ASSUME: Γ acts w/ rational polyhedral fundamental domain on rational closure $C^+ := \mathrm{Conv}(\bar{C} \cap L)$

Morrison Cone Conj 1.

($\Leftrightarrow$ finitely many orbits of faces of C^+)

↝ local compactification $\overline{(V+iC/L)/\Gamma}$

boundary strata $\longleftrightarrow$ Γ-orbits of faces of C^+.

$\left(\longleftrightarrow\right.$ birational contractions & fibrations of X / isom

assuming generalized abundance (L not/assumed effective) $\left.\right)$

$$\dim_{\mathbb{C}} S_\tau = \dim_{\mathbb{R}} (\tau^\perp \cap (C^+)^*) \leq \operatorname{codim}_{\mathbb{R}} (\tau \subset C^+)$$

(equal if rational polyhedral.

—//—

Problem: What is C, Γ?

Ex: K3 . $C = \text{"round cone"} = (x^2 > 0,\ x \cdot h > 0) \subset Pic X \otimes \mathbb{R} \simeq \mathbb{R}^{1,18}$

$\Gamma = W \rtimes Aut(X, \Omega) \doteq Aut(Pic X)$ finite index.

$W = \langle s_\alpha \mid \alpha = [C] \in Pic X,\ C \text{ (-2) curve} \rangle$

$s_\alpha(x) = x + (x \cdot \alpha)\alpha$

$C^+ = W \cdot (Amp X)^+$

Morrison cone conj OK by Ash-Mumford-Rapoport-Tai

HMS: s_α induced by spherical twist in $\mathcal{O}_C(-1)$ (Seidel-Thomas)

$$E \longmapsto \text{Cone}\left(RHom(S, E) \otimes S \xrightarrow{ev} E\right)$$

Mirror to Picard-Lefschetz Monodromy for

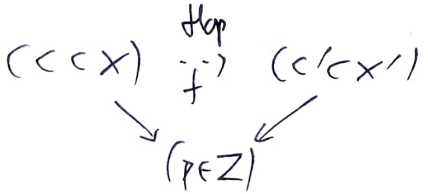
ODP degeneration "generalized Dehn twist"

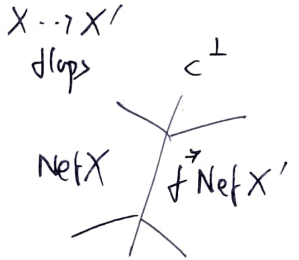
in L = Lagrangian S^2 = vanishing cycle. (Seidel)

$D^b Coh X \ni \mathcal{O}_C(-1) \longmapsto L \in \mathcal{F}(Y)$

CY3.

1. $\mathrm{Nef}\,X \hookrightarrow \mathrm{Mov}\,X = \bigcup f^*\mathrm{Nef}\,X'$ movable cone





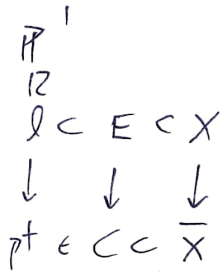
(Witten, Aspinwall-Greene-Morrison.)

$\subset \mathrm{Pic}\,X \otimes \mathbb{R}$.

Faces of $(\mathrm{Mov}\,X)^+$ $\leftrightarrow$ divisorial contractions & fibrations

$X \dashrightarrow X' \longrightarrow Z$ (flops)

2. Wilson



divisorial contraction to curve C, $g = g(C)$

$\rightsquigarrow$ reflection $s(x) = x + (x \cdot \ell)E$ Bourbaki: ℓ root, E coroot

$E \cdot \ell = -2$.

However, must modify Bourbaki formula for $g = 0$:—

For all g, Bourbaki reflection is realized by EZ spherical twist in $\mathcal{O}_E(-1)$ for $E \to C$ (Horja, Szendrői; OR "spherical functor")

BUT for $g=0$ E is rational $\Rightarrow$ $\exists$ full exceptional collection $F_i \in D^b(\mathrm{coh}\, E)$

$\Rightarrow$ pushforwards $S_i \in D^b(\mathrm{coh}\, X$ spherical, $T_{S_i} \curvearrowright D^b(\mathrm{coh}\, X$ supported on E

Modify EZ sph twist by element of this group (in progress).

(Note: For $g>0$ $\nexists$ exceptional objects because $H^1(\mathcal{O}_E) \neq 0$.)

$\mathcal{C}^+ = W \cdot \mathrm{Mov}(X)^+$, W group generated by these reflections associated to divisorial contractions to curves

$\Gamma = W \rtimes \mathrm{Bir}\, X$

Morrison cone conj 2: $\mathrm{Bir}\, X \curvearrowright (\mathrm{Mov}\, X)^+$ w/ RPFD

(+Kawamata) $\Leftrightarrow$ MCC1 + finiteness of Minimal models

Alternatively:

$$W_{g>0} \cdot (\mathrm{Mov} X)^+ = (\mathrm{Mov} X_{gen})^+$$ W. Lutz '24, Wilson.

Conj. $W \cdot (\mathrm{Mov} X)^+ = \mathrm{Mov}(X, J_{gen})^+$ "symplectic movable cone"

(symplectic flops + $GW_{g=0}$)

cf. Smith-Thomas-Yau, Wilson.

By construction, faces of $C^+ = W \cdot \mathrm{Mov}(X)^+$

$\longleftrightarrow$ divisorial contractions to points & fibrations.

Bakker-Filipazzi-Mauri-Tsimerman '25

(cf. Griffiths '70, Green-Griffiths-Laza-Robles '17)

$\exists\ M_Y \subset \overline{M}_Y^{BB}$ projective compactification. "Baily-Borel".

Geometric meaning: $Y^\times \to \mathbb{D}^\times$ smooth $\rightsquigarrow$

$$\begin{array}{ccc} \mathbb{D}^\times & \longrightarrow & M_Y \\ \cap & & \cap \\ \mathbb{D} & \xrightarrow{\phi} & \overline{M}_Y \end{array}$$

$\phi(0)$ records graded pieces of limit MHS on $H^n(Y_t)$, $n = \dim_{\mathbb{C}} Y\ (=3)$.

limit MHS: WLOG monodromy is unipotent (base change)

$N = \log T$. Weight filtration: $N : W_k \to W_{k-2}$

$N^k : gr^W_{n+k} \xrightarrow{\sim} gr^W_{n-k}$

Hodge filtration (Schmid)

$$F^p = \lim_{\mathrm{Im}\,\tau \to \infty} e^{-\tau N} \cdot F^p(Y_t), \quad t = e^{2\pi i \tau}$$

Large complex structure limit

$Y_t \overset{\text{mirror}}{\sim} (X, \tau \cdot \omega = B_\tau + i\omega_\tau)$ $\quad [\omega] = c_1(L)$, L ample.

"B field". "complexified Kähler form"

$\omega_\tau = (\mathrm{Im}\,\tau)\cdot\omega$, $\mathrm{Im}\,\tau \to \infty$ "large volume limit".

$$\begin{array}{ccc} \mathcal{F}(Y) & \simeq & D^b(\mathrm{Coh}\, X) \\ \circlearrowleft & & \circlearrowleft \\ T & & -\otimes L \end{array} \qquad \overset{\text{K theory}}{\rightsquigarrow} \qquad \begin{array}{ccc} H_3(Y,\mathbb{Q}) & \simeq & H^{ev}(X,\mathbb{Q}) \\ \circlearrowleft & & \circlearrowleft \\ N & & \cup c_1(L) \end{array}$$

Hard Lefschetz

$$\Rightarrow \quad W_{2k} H^3(Y)_{lin} = H_{ev \le 2k}(X, \mathbb{Q})$$

Hodge-Tate ?
$\rightsquigarrow$ interchange of Hodge numbers
$h^{p,q}(X) = h^{n-p,q}(Y)$

Similarly for $L \in (\mathrm{Amp}\, X)^\dagger \subset C^\dagger$ (effective nef cone)

Hard Lefschetz $\Leftrightarrow$ $f: X \to Z$ semismall (de Cataldo - Migliorini)

$n=3$: small contraction OR divisorial contraction to curve.

[WARNING: $\Phi: D^b(\mathrm{Coh}\, X) \xrightarrow{\sim} D^b(\mathrm{Coh}\, X')$ does not respect $\otimes$!]

Dictionary

1. $f: X \longrightarrow Z$ fibration $\overset{\text{Mirror}}{\sim}$ $N^k \neq 0$, $N^{k+1} = 0$

   $\dim Z = k < n = 3$

   $Y \rightsquigarrow \overline{Y} = \cup Y_i$ dlt model

   deepest stratum ("source": Kollár) is Y_k Mirror to general fiber of f

   cf. Laza,

2. $\begin{array}{ccc} X & \longrightarrow & \overline{X} \\ \cup & & \downarrow \\ E & \longrightarrow & p \end{array}$ divisorial contraction $\sim$ maximally unipotent

   $gr_3^W H^3(Y)_{lim} = \oplus H^3(Y_i) \neq 0$

   $h^{2,1} = \#$ components of E.

cf. "local mirror symmetry" (eg. Auroux–Abouzaid–Katzarkov)

E irred, smooth $\Rightarrow$ del Pezzo; $(E \subset X)$ $\overset{\text{Mirror}}{\sim}$ $\begin{array}{c} Y \\ \downarrow \\ \log CY_2 \end{array}$ affine conic bundle

$S > \Delta = C$, $g = 1$.

$H^3(Y) = H^1(C)$

Above: $\exists i.\ (Y_i, D_i)$ obtained from toric pair by blowing up $C \subset$ boundary.

Ex. $\mathbb{P}^2 \simeq E \subset X$ ↓ $\frac{1}{3}(1,1,1) \simeq p \in X$

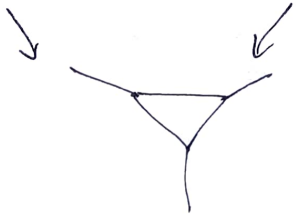
mirror ~

$Y = (uv = x+y+x^{-1}y^{-1}+\lambda) \subset \mathbb{A}^2_{u,v} \times (\mathbb{C}^\times)^2_{x,y}$ ↓ $\check{S} = (\mathbb{C}^\times)^2_{x,y} \to \Delta = \mathbb{C}$

Critical loci of SYZ



toric 1-strata



cf. M. Gross Examples of SLag fibrations

Interior of tube domain

$$\text{Vti}C/L \subset L \otimes \mathbb{C}^\times, \quad L = \text{Pic}\, X.$$

Canonical coordinates $X^\alpha \quad \alpha \in L^* = H_2(X,\mathbb{Z}) = gr_2^W H^3(Y)_{lin}$

(Morrison, Deligne) $X^\alpha = \exp\left(2\pi i \int_\alpha \Omega\right)$ Ω hol. vol form on Y

1. Small contraction.

$$\begin{array}{ccc} X & \longrightarrow & \overline{X} \\ \cup & & \psi \\ U_i = C & \longrightarrow & p \end{array}$$

$C_i \cong \mathbb{P}^1$

Typical: C $(-1,-1)$-curve, $p \in \overline{X}$ ODP,

Mirror $Y \rightsquigarrow \overline{Y}$ smoothing of ODP.

$[C] \in H_2(X,\mathbb{Z}) \longmapsto [L] \in gr_2^W H_3(Y)_{lin}$

L = vanishing cycle = Lagrangian S^3.

Discriminant locus = hypertorus $(X^L = 1)$ $\wedge$ tube

Morrison "Looking glass".

In general (Reid, Pinkham, Friedman) (Small contractions II

$$\begin{array}{ccccc} & & \tilde{S} & & \\ & & \downarrow & & \\ & & \hat{S} & & \\ & \swarrow & \downarrow & \searrow & \\ p \in & S & \subset & X & \\ & \downarrow & & \downarrow f & \\ & \bar{S} & \subset & \bar{X} & \end{array}$$

general hyperplane section S is DuVal.
strict transform $\hat{S}$ dominated by minimal resolution

Chamber decomposition $\operatorname{Mov} X = \bigcup_{g: X \dashrightarrow X'} g^* \operatorname{Nef} X'$ transverse to $f^* \operatorname{Nef} \bar{X}$

is pullback of Weyl hyperplane arrangement in $\operatorname{Pic} \tilde{S}$ under

$$\operatorname{Pic} X \longrightarrow \operatorname{Pic} \hat{S} \hookleftarrow \operatorname{Pic} \tilde{S}.$$

In particular, the chamber decomposition of $\operatorname{Mov} X$ is given by a hyperplane arrangement.

Friedman

$$\begin{array}{ccc} \subset & \subset X & <w\ X' \\ \downarrow & \downarrow & \downarrow \\ p & \in \overline{X} & <w\ \overline{X}' \end{array} \qquad (-1,-1)\text{-curves} \quad \downarrow \quad \text{ODPs}$$

Small contractions III

Mirror $\sim$

$$\begin{array}{ccc} Y & \xleftarrow{\sim} & Y' \\ \downarrow & & \downarrow \\ \overline{Y} & \leftarrow & \overline{Y}' \end{array} \qquad \text{ODPs} \qquad (\text{but } [w'] \neq [w])$$

i.e. $\exists$ simultaneous partial resolution of mirror degeneration w/ ODPs.

Ex: $\subset X$ $\quad (0,-2)$-curve

$\downarrow \quad \downarrow$

$p \in \overline{X} \simeq (x^2+y^2+z^2+w^4=0)$

"Pagoda", width 2 (Reid)

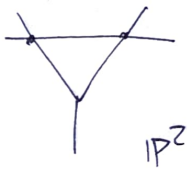
$<w$ $2\times$ODP



$z^2+w^4=0$

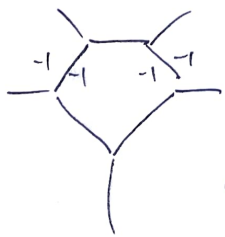
$2\times$ODP

$\overline{Y} = \text{Cone}(dP_7) \leftarrow \overline{Y}' =$



$\mathbb{P}^2$

$\leftarrow$



$dP_7 = Bl^2\,\mathbb{P}^2$

Divisorial contraction to curve

$\mathbb{P}^1 \simeq \ell \subset E \subset X$
$\downarrow \quad \downarrow \quad \downarrow f$
$pt \in C \subset \bar{X}$

mirror $\sim$

$Y \rightsquigarrow \bar{Y}$ over $(x^\ell = 1) \cap$ tube.

$g=1$: $\operatorname{Sing} \bar{Y} = \Gamma$, genus 1 curve mirror to C, transverse A_1.

$g>1$: $\operatorname{Sing} \bar{Y} = \Gamma = \cup \Gamma_i$

"trivalent graph of $\mathbb{P}^1$'s"

w/ $2g-2$ vertices

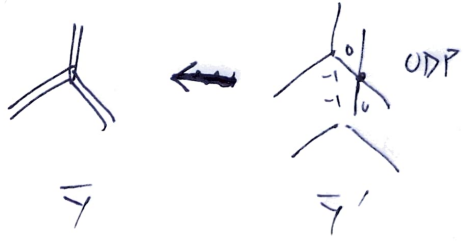
generically transverse A_1

local model $xyz = t^2$ at vertex.

$X \subset\!\!\rightsquigarrow X'$
$\downarrow \quad \downarrow$
$\bar{X} \subset\!\!\rightsquigarrow \bar{X}'$ $2g-2$ ODP's

$\sim$

$Y \dashleftarrow Y'$
$\downarrow \quad \downarrow$
$\bar{Y} \leftarrow \bar{Y}'$



E does not deform,
but $2g-2$ fibers deform to $(-1,-1)$-curves.
(Wilson)

Divisorial contraction to curve. II

$g=0$. (in progress)

$Y \leadsto \bar{Y} = \cup Y_i$. Maximally unipotent / $\exists$ 0-stratum.

Y_1, Y_2 smooth, $Y_1 \cap Y_2$ has A_1

dlt model does not see $\mathrm{gr}_3^W H^3(Y)_{\mathrm{lin}}$ $(h^{2,1}=1)$

In particular, get non-Kulikov degenerations over interior of tube domain.

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Update: Morrison $\neq$ Baily-Borel.

For local $\mathbb{F}_1$ $\begin{pmatrix} E \subset X \\ \downarrow \\ \mathbb{F}_1 \end{pmatrix}$, a curve in the Morrison

compactification of M_Y is contracted to the LCSL of the Baily-Borel (not a 1-stratum)