Efficient computations of Yoneda product of Ext groups

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Yoneda Product

Let A be the Steenrod algebra at $p = 2$. Let X, Y, Z be three spectra of finite type.
Recall that in order to compute the composition

$$\pi_* F(Y, Z) \times \pi_* F(X, Y) \to \pi_* F(X, Z),$$

a good place to start with is the Yoneda product of the Adams E_2-terms.

$$\mathrm{Ext}_A(H^* Y, H^* X) \times \mathrm{Ext}_A(H^* Z, H^* Y) \to \mathrm{Ext}_A(H^* Z, H^* X)$$

The Adams differentials obey the Leibniz rule with respect to the Yoneda product.

Yoneda Product

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Proposition (L.)

The bigraded algebra

$$\bigoplus_{t \leq 290} \mathrm{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$$

in internal degrees up to 290 is an algebra with 4453 generators, 37354 basis elements and 420479 (Gröbner and not minimal) relations.

$\mathrm{Ext}(S^0)$

Table: Size of $\mathrm{Ext}(S^0)$

	Generators	**Basis**	**Relations**
$t \leq 261$	2914	23822	231848
$t \leq 290$	4453	37354	420479

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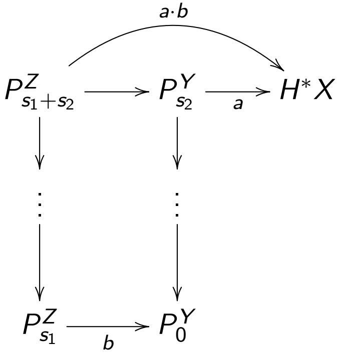
Table: Time consumption for computing Yoneda product of $\mathrm{Ext}(S^0)$

	Old	New	difference
$t \leq 100$	$11.4s$	$0.82s$	14 times faster
$t \leq 150$	$25m$	$6.32s$	239 times faster
$t \leq 250$	$22.7d$	$38m$	860 times faster
$t \leq 290$	-	$9h$	-

Yoneda Product

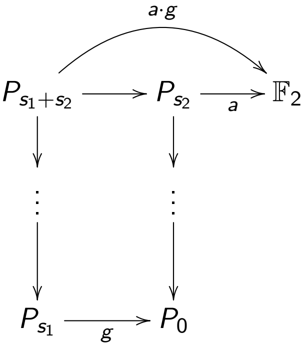
$$\mathrm{Ext}_A^{*,*}(H^*Y, H^*X) \times \mathrm{Ext}_A^{*,*}(H^*Z, H^*Y) \to \mathrm{Ext}_A^{*,*}(H^*Z, H^*X)$$

We compute the product $a \cdot b$ by computing the chain map induced by b.



The classical way method

In order to compute all the Yoneda products for $\mathrm{Ext}(S^0)$, it suffices to compute the chain maps induced by all the generators $g \in \mathrm{Ext}_A^{*,*}(\mathbb{F}_2, \mathbb{F}_2)$.



For example, we can use h_0-chain map to compute the h_0-multiplication in $\mathrm{Ext}(S^0)$. This chain map actually takes a lot of time to compute.

A new method

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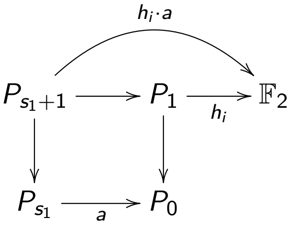

A new method

Assume that $a_1 \leq a_2 \leq \cdots$ are generators of $\mathrm{Ext}(S^0)$ ordered by internal degree and then by Adams filtration.
In order to obtain all the Yoneda products for $\mathrm{Ext}(S^0)$, it suffices to know the $(a_{i_1} a_{i_2} \cdots a_{i_n})$-multiplication on $a_i, i \leq i_1$ for an additive basis $a_{i_1} a_{i_2} \cdots a_{i_n}$ of $\mathrm{Ext}(S^0)$.

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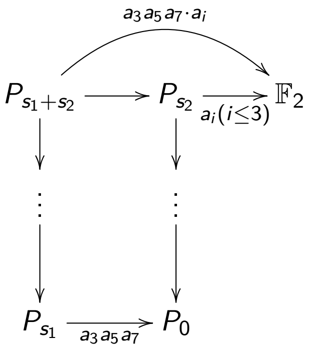

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What is a good choice of an additive basis $a_{i_1} a_{i_2} \cdots a_{i_n}$ for $\mathrm{Ext}(S^0)$?

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For example, if $a_4 a_7 = a_2 a_9$, then we would prefer to use $a_2 a_9$ as a basis element because a_2 is smaller.

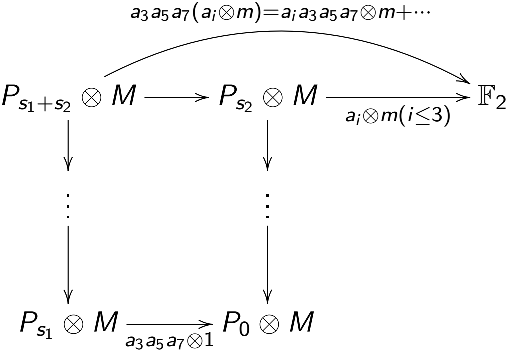
The optimal choice can be obtained by using Gröbner basis of the relations of $\mathrm{Ext}(S^0)$ with respect to the reversed lexicographic order. So that overall a_{i_1} tends to be small for the basis $\{a_{i_1} a_{i_2} \cdots a_{i_n}\}$.

General Yoneda product

For an A-module M, in order to compute $\mathrm{Ext}_A(M, \mathbb{F}_2)$ as an $\mathrm{Ext}(S^0)$-module, we can use the non-minimal resolution

$$\cdots \to P_s \otimes M \to P_{s-1} \otimes M \to \cdots \to P_0 \otimes M \to M \to 0$$

Since A is a Hopf algebra, $- \otimes M : A\mathrm{-mod} \to A\mathrm{-mod}$ is a functor, we can use the following to compute the Yoneda product



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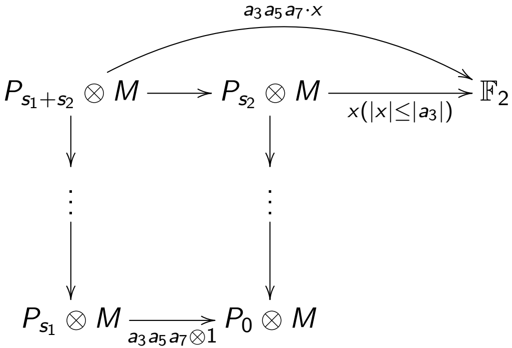

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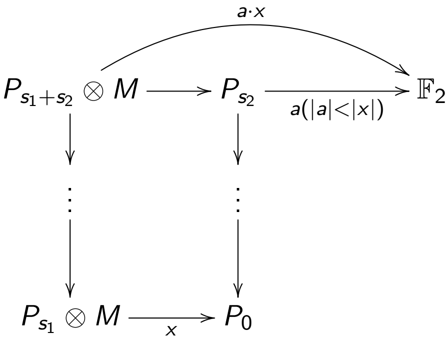

General Yoneda product and tensor product

Note that

- It is simple to obtain $P_\bullet \otimes M \to P_\bullet \otimes N$ for $f : M \to N$.
- $\mathrm{Ext}_A(M, N) \cong \mathrm{Ext}_A(M \otimes DN, \mathbb{F}_2)$.

The computation of the tensor product

$$\mathrm{Ext}_A(M, \mathbb{F}_2) \times \mathrm{Ext}(N, \mathbb{F}_2) \to \mathrm{Ext}(M \otimes N, \mathbb{F}_2).$$

is more primitive than the general Yoneda product

$$\mathrm{Ext}_A(M, N) \times \mathrm{Ext}(N, L) \to \mathrm{Ext}(M, L).$$

Tensor product

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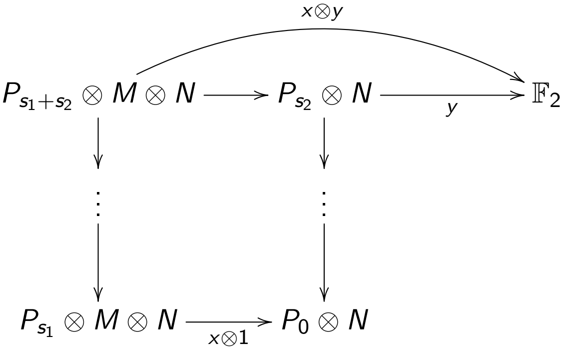

Optimization: use the fact that this tensor product of Ext groups is bilinear over $\mathrm{Ext}(S^0)$.

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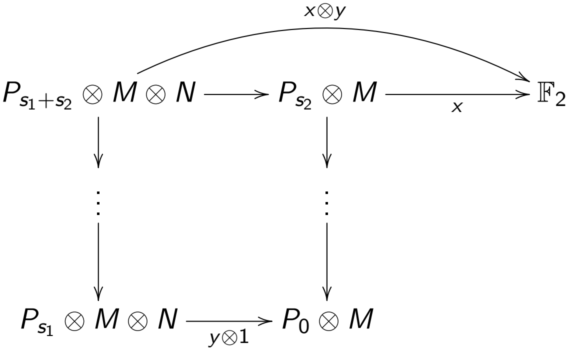

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Further questions

Are there special fast algorithms for the associated ring structure of

$$\mathrm{Ext}_A(M, M)?$$

Thank You!