Automatic differential propagation

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The premise

Spectral sequence $E_2 \implies E_\infty$

$E_{r+1} =$ homology of (E_r, d_r) (bi)graded dga

- Know E_2 (in a range)
- Don't know $d_r : E_r \to E_r$ (and the goal is to determine these)

Assume:

- d_r satisfies the Leibniz rule $d_r(xy) = d_r(x)y \pm xd_r(y)$
- know some starting values $d_r(a_i) = b_i$

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Question

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Basic example: suppose

- a is a permanent cycle $(d_i(a) = 0$ for all $i)$
- $d_r(b) = 0$

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Improvements on a common strategy:

- use **all** multiplications, not just easy-to-keep-track-of ones
- "do better than 1-step deductions" (proof by contradiction, deduction using multiple pages at once)

Prior work: $p = 2$ Adams spectral sequence, d_2 differentials

Theorem (Beauvais-Feisthauer, 2022): Using only

- knowledge of the Adams E_2 page, including product structure, through the 184-stem
- $d_2(h_4) = h_0 h_3^2$ (stem 15)
- $d_2(\Delta^2 d_0^2) = d_0 jm$ (stem 76)
- the Leibniz rule $d_r(xy) = d_r(x)y \pm x d_r(y)$

one can compute >95% of the Adams spectral sequence d_2's up to stem 140.

Prior work: $p = 2$ Adams spectral sequence, all differentials

Theorem (Lin–Wang–Xu, 2024): Using

- Adams E_2 and E_3 pages for hundreds of CW complexes, including
  - product structure
  - comparison maps, and
- three additional Adams differentials
- cofiber sequence techniques
- the "generalized Leibniz rule" (synthetic spectra techniques)

the authors compute most of the Adams spectral sequence differentials needed for their Kervaire invariant one result in stem 126.

Organize this as matrices

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(goal is to find the entries)

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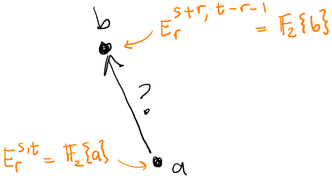

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$d_r^{s,t}$ is a 1×1 matrix

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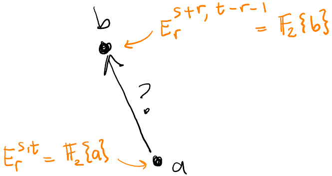

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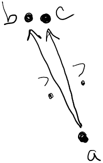
$d_r(a) = x_1 c$
$d_r(b) = x_2 c$
$d_r^{s,t} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$



$d_r(a) = x_1 b + x_2 c$
$d_r^{s,t} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Variables and constraints

Upshot: system of equations representing how to find d_r (given complete knowledge of E_r)

- variables are the x_i's (matrix entries of d_r matrix)
- constraints:
  - Leibniz rule
  - "Prior knowledge differentials"
  - $d_r^2 = 0$

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- constraints:
  - Leibniz rule $\leftarrow$ always linear
  - "Prior knowledge differentials" $\leftarrow$ affine linear (x_i = constant)
  - $d_r^2 = 0$ $\leftarrow$ nonlinear

Forget $d_r^2 = 0$; then this is just linear algebra. Or don't forget it, and use a SAT solver.

More on the linear algebra

Finding the d_r differentials $\iff$ solving $A\mathbf{x} = \mathbf{b}$.

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- One solution $\mathbf{x}$ is one *possible* set of d_r values.
- We want to know what values are *forced* by the constraints.
- All other solutions have the form $\mathbf{x} + \mathbf{y}$ for $\mathbf{y} \in \ker(A)$.
- If all $\mathbf{y}$'s are zero in the i^{th} spot, then x_i is learned for certain.

Interpage propagation

Issue:

partially unknown

$$E_2 \overset{\ker d_2 / \operatorname{im} d_2}{\rightsquigarrow} E_3 \overset{\ker d_3 / \operatorname{im} d_3}{\rightsquigarrow} E_4 \quad \ldots$$

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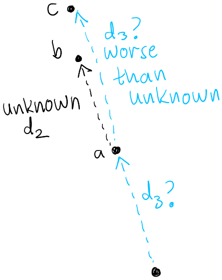

Unclear how to phrase this as one constraint problem

Strategy

1. Initial pass: give up on trying to find d_r's in degrees where d_{r-1} is uncertain
2. Trial and error pass: for each remaining uncertain $d_r(x)$:
   - Try $d_r(x) = 0$, propagate to all higher pages
   - Try $d_r(x) = 1$, propagate to all higher pages

   (This also incorporates the $d^2 = 0$ constraint.)
3. Learned anything? Go back to step 2.

Unstable Adams spectral sequence

$$E_2^{*,*,*} = \mathrm{Ext}_{\mathscr{U}\mathscr{M}}^{*,*}(\Sigma^n \mathbb{F}_2, \mathbb{F}_2) \implies \pi_{s+n}(S^n)$$

Propagation of just the d_2 differentials:

- Start: E_2 page in internal degree ≤ 100 (from sseq)
- 174410 variables (possible differentials)
- 756189 constraints
- Runs for ≈ 45 minutes
- 98% known