Inductive Methods for Counting Number Fields

Brandon Alberts Robert J. Lemke Oliver Jiuya Wang
Melanie Matchett Wood

Harvard University

Counting Extensions of Number Fields

Counting Extensions of Number Fields

$G \subset S_n$ a finite permutation group

A G-**extension** L/k: a degree n field extension whose Galois closure $\widetilde{L}/k$ has Galois group

$$\mathrm{Gal}(\widetilde{L}/k) \cong G$$

as a *permutation group* acting on the n embeddings $L \hookrightarrow \widetilde{L}$.

Counting Extensions of Number Fields

$G \subset S_n$ a finite permutation group
A G-**extension** L/k: a degree n field extension whose Galois closure $\widetilde{L}/k$ has Galois group

$$\mathrm{Gal}(\widetilde{L}/k) \cong G$$

as a *permutation group* acting on the n embeddings $L \hookrightarrow \widetilde{L}$.

Goal: count G-extensions of a number field k, asymptotically by discriminant

$$\#\mathcal{F}_k(G;X) = \#\{L/k : \ \mathrm{Gal}(\widetilde{L}/k) \cong G, \ |\mathrm{disc}(L/k)| \leq X\}$$

Counting Extensions of Number Fields

$G \subset S_n$ a finite permutation group

A G-**extension** L/k: a degree n field extension whose Galois closure $\widetilde{L}/k$ has Galois group

$$\mathrm{Gal}(\widetilde{L}/k) \cong G$$

as a *permutation group* acting on the n embeddings $L \hookrightarrow \widetilde{L}$.

Goal: count G-extensions of a number field k, asymptotically by discriminant

$$\#\mathcal{F}_k(G;X) = \#\{L/k : \ \mathrm{Gal}(\widetilde{L}/k) \cong G, \ |\mathrm{disc}(L/k)| \leq X\}$$

$$a(G) = \min_{g \in G - \{1\}} \mathrm{ind}(g),$$

Counting Extensions of Number Fields

$G \subset S_n$ a finite permutation group

A G-**extension** L/k: a degree n field extension whose Galois closure $\widetilde{L}/k$ has Galois group

$$\mathrm{Gal}(\widetilde{L}/k) \cong G$$

as a *permutation group* acting on the n embeddings $L \hookrightarrow \widetilde{L}$.

Goal: count G-extensions of a number field k, asymptotically by discriminant

$$\#\mathcal{F}_k(G;X) = \#\{L/k : \ \mathrm{Gal}(\widetilde{L}/k) \cong G, \ |\mathrm{disc}(L/k)| \leq X\}$$

$$a(G) = \min_{g \in G - \{1\}} \mathrm{ind}(g), \quad \text{where } \mathrm{ind}(g) = n - \#\{\text{orbits of } g\}$$

Counting Extensions of Number Fields

$G \subset S_n$ a finite permutation group

A G-**extension** L/k: a degree n field extension whose Galois closure $\widetilde{L}/k$ has Galois group

$$\mathrm{Gal}(\widetilde{L}/k) \cong G$$

as a *permutation group* acting on the n embeddings $L \hookrightarrow \widetilde{L}$.

Goal: count G-extensions of a number field k, asymptotically by discriminant

$$\#\mathcal{F}_k(G; X) = \#\{L/k : \ \mathrm{Gal}(\widetilde{L}/k) \cong G, \ |\mathrm{disc}(L/k)| \le X\}$$

$$a(G) = \min_{g \in G - \{1\}} \mathrm{ind}(g), \quad \text{where } \mathrm{ind}(g) = n - \#\{\text{orbits of } g\}$$

Malle's a-Conjecture (2002, 2004)

Counting Extensions of Number Fields

$G \subset S_n$ a finite permutation group

A G-**extension** L/k: a degree n field extension whose Galois closure $\widetilde{L}/k$ has Galois group

$$\mathrm{Gal}(\widetilde{L}/k) \cong G$$

as a *permutation group* acting on the n embeddings $L \hookrightarrow \widetilde{L}$.

Goal: count G-extensions of a number field k, asymptotically by discriminant

$$\#\mathcal{F}_k(G; X) = \#\{L/k : \ \mathrm{Gal}(\widetilde{L}/k) \cong G, \ |\mathrm{disc}(L/k)| \le X\}$$

$$a(G) = \min_{g \in G - \{1\}} \mathrm{ind}(g), \quad \text{where } \mathrm{ind}(g) = n - \#\{\text{orbits of } g\}$$

Malle's *a*-Conjecture (2002, 2004)

There exist positive constants $b, c > 0$ such that

$$\#\mathcal{F}_k(G; X) \sim c\, X^{1/a(G)} (\log X)^{b-1}.$$

Malle's Conjecture

Malle's a-Conjecture (2002, 2004)

There exist positive constants $b, c > 0$ such that

$$\#\mathcal{F}_k(G;X) \sim c\,X^{1/a(G)}(\log X)^{b-1}.$$

Status before ALOWW (2025):

- Proved for $G = S_n$, $n \leq 5$; abelian groups; a handful of other cases including families (detailed history in our paper)

Malle's Conjecture

Malle's a-Conjecture (2002, 2004)

There exist positive constants $b, c > 0$ such that

$$\#\mathcal{F}_k(G; X) \sim c\, X^{1/a(G)} (\log X)^{b-1}.$$

Status before ALOWW (2025):

- Proved for $G = S_n$, $n \leq 5$; abelian groups; a handful of other cases including families (detailed history in our paper)
- This includes at most ~ 237 groups of degree ≤ 23.

Malle's Conjecture

Malle's a-Conjecture (2002, 2004)

There exist positive constants $b, c > 0$ such that

$$\#\mathcal{F}_k(G; X) \sim c\, X^{1/a(G)} (\log X)^{b-1}.$$

Status before ALOWW (2025):

- Proved for $G = S_n$, $n \leq 5$; abelian groups; a handful of other cases including families (detailed history in our paper)
- This includes at most ~ 237 groups of degree ≤ 23.

For **nilpotent** groups:

Malle's Conjecture

Malle's *a*-Conjecture (2002, 2004)

There exist positive constants $b, c > 0$ such that

$$\#\mathcal{F}_k(G; X) \sim c\, X^{1/a(G)}(\log X)^{b-1}.$$

Status before ALOWW (2025):

- Proved for $G = S_n$, $n \leq 5$; abelian groups; a handful of other cases including families (detailed history in our paper)
- This includes at most ~ 237 groups of degree ≤ 23.

For **nilpotent** groups:

- Previously proven for: ≤ 1538 of the 2,739,294 nilpotent transitive groups of degree ≤ 32

Malle's Conjecture

Malle's a-Conjecture (2002, 2004)

There exist positive constants $b, c > 0$ such that

$$\#\mathcal{F}_k(G; X) \sim c\, X^{1/a(G)} (\log X)^{b-1}.$$

Status before ALOWW (2025):

- Proved for $G = S_n$, $n \leq 5$; abelian groups; a handful of other cases including families (detailed history in our paper)
- This includes at most ~ 237 groups of degree ≤ 23.

For **nilpotent** groups:

- Previously proven for: ≤ 1538 of the 2,739,294 nilpotent transitive groups of degree ≤ 32
- ALOWW: at least 2,686,926 of the 2,739,294 nilpotent transitive groups of degree ≤ 32

Good and bad news...

Good news:

Corollary (Alberts–Lemke Oliver–Wang–Wood)

Good and bad news...

Good news:

Corollary (Alberts–Lemke Oliver–Wang–Wood)

Malle's *a*-Conjecture holds for at least 1665 transitive permutation groups of degree ≤ 23 over $\mathbb{Q}$, including at least 339 non-nilpotent groups.

Good and bad news...

Good news:

Corollary (Alberts–Lemke Oliver–Wang–Wood)

Malle's a-Conjecture holds for at least 1665 transitive permutation groups of degree ≤ 23 over $\mathbb{Q}$, including at least 339 non-nilpotent groups.

Bad news:

Good and bad news...

Good news:

Corollary (Alberts–Lemke Oliver–Wang–Wood)

Malle's *a*-Conjecture holds for at least 1665 transitive permutation groups of degree ≤ 23 over $\mathbb{Q}$, including at least 339 non-nilpotent groups.

Bad news:

- It is harder to say which groups we know Malle's *a*-conjecture for

Good and bad news...

Good news:

Corollary (Alberts–Lemke Oliver–Wang–Wood)

Malle's *a*-Conjecture holds for at least 1665 transitive permutation groups of degree ≤ 23 over $\mathbb{Q}$, including at least 339 non-nilpotent groups.

Bad news:

- It is harder to say which groups we know Malle's *a*-conjecture for

Good news:

Good and bad news...

Good news:

Corollary (Alberts–Lemke Oliver–Wang–Wood)

Malle's *a*-Conjecture holds for at least 1665 transitive permutation groups of degree ≤ 23 over $\mathbb{Q}$, including at least 339 non-nilpotent groups.

Bad news:

- It is harder to say which groups we know Malle's *a*-conjecture for

Good news:

- We're going to work on getting it into the LMFDB!

The key idea: inductive counting

The key idea: inductive counting

Approach: fix a normal subgroup $T \trianglelefteq G$ and *stratify* G-extensions by their associated G/T-extension

The key idea: inductive counting

Approach: fix a normal subgroup $T \trianglelefteq G$ and *stratify* G-extensions by their associated G/T-extension

For $q : G \to G/T$ the quotient map, each G-extension L/k determines a G/T-extension M/k (the fixed field $\widetilde{L}^T$)

The key idea: inductive counting

Approach: fix a normal subgroup $T \trianglelefteq G$ and *stratify* G-extensions by their associated G/T-extension

For $q : G \to G/T$ the quotient map, each G-extension L/k determines a G/T-extension M/k (the fixed field $\widetilde{L}^T$)

Strategy:

The key idea: inductive counting

Approach: fix a normal subgroup $T \trianglelefteq G$ and *stratify* G-extensions by their associated G/T-extension

For $q : G \to G/T$ the quotient map, each G-extension L/k determines a G/T-extension M/k (the fixed field $\widetilde{L}^T$)

Strategy:

1. For each G/T-extension M/k, count the number of G-extensions with $\widetilde{L}^T = M$

The key idea: inductive counting

Approach: fix a normal subgroup $T \trianglelefteq G$ and *stratify* G-extensions by their associated G/T-extension

For $q : G \to G/T$ the quotient map, each G-extension L/k determines a G/T-extension M/k (the fixed field $\widetilde{L}^T$)

Strategy:

1. For each G/T-extension M/k, count the number of G-extensions with $\widetilde{L}^T = M$
2. Sum over all G/T-extensions M/k with $|\text{disc}(M/k)| \leq X$

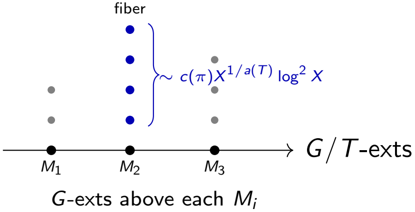
The key idea: inductive counting

Approach: fix a normal subgroup $T \trianglelefteq G$ and *stratify* G-extensions by their associated G/T-extension

For $q : G \to G/T$ the quotient map, each G-extension L/k determines a G/T-extension M/k (the fixed field $\widetilde{L}^T$)

Strategy:

1. For each G/T-extension M/k, count the number of G-extensions with $\widetilde{L}^T = M$
2. Sum over all G/T-extensions M/k with $|\mathrm{disc}(M/k)| \leq X$



For which G and $T \trianglelefteq G$ does this strategy succeed?

For which G and $T \trianglelefteq G$ does this strategy succeed?

Our results are all in the case that G is *concentrated* in T (not balanced), and a single fiber contributes a positive proportion to the count

For which G and $T \trianglelefteq G$ does this strategy succeed?

Our results are all in the case that G is *concentrated* in T (not balanced), and a single fiber contributes a positive proportion to the count

Definition

For which G and $T \trianglelefteq G$ does this strategy succeed?

Our results are all in the case that G is *concentrated* in T (not balanced), and a single fiber contributes a positive proportion to the count

Definition

A transitive permutation group G is **concentrated in** a proper normal subgroup T if T contains all elements of minimal index:

$$\langle g \in G : \mathrm{ind}(g) = a(G) \rangle \neq G.$$

For which G and $T \trianglelefteq G$ does this strategy succeed?

Our results are all in the case that G is *concentrated* in T (not balanced), and a single fiber contributes a positive proportion to the count

Definition

A transitive permutation group G is **concentrated in** a proper normal subgroup T if T contains all elements of minimal index:

$$\langle g \in G : \mathrm{ind}(g) = a(G) \rangle \neq G.$$

Idea: G is concentrated when G-extensions has the lowest power part of their discriminant coming from primes only ramified in the T-extension

For which G and $T \trianglelefteq G$ does this strategy succeed?

Our results are all in the case that G is *concentrated* in T (not balanced), and a single fiber contributes a positive proportion to the count

Definition

A transitive permutation group G is **concentrated in** a proper normal subgroup T if T contains all elements of minimal index:

$$\langle g \in G : \mathrm{ind}(g) = a(G) \rangle \neq G.$$

Idea: G is concentrated when G-extensions has the lowest power part of their discriminant coming from primes only ramified in the T-extension

Fact: Out of 40,238 transitive groups of degree ≤ 31, at least 39,770 ($\approx 99\%$) are concentrated.

For which G and $T \trianglelefteq G$ does this strategy succeed?

Our results are all in the case that G is *concentrated* in T (not balanced), and a single fiber contributes a positive proportion to the count

Definition

A transitive permutation group G is **concentrated in** a proper normal subgroup T if T contains all elements of minimal index:

$$\langle g \in G : \mathrm{ind}(g) = a(G) \rangle \neq G.$$

Idea: G is concentrated when G-extensions has the lowest power part of their discriminant coming from primes only ramified in the T-extension

Fact: Out of 40,238 transitive groups of degree ≤ 31, at least 39,770 ($\approx 99\%$) are concentrated.

Non-concentrated examples: S_n, C_p^r

Counting in the fibers: the Twisted Malle Conjecture

Counting in the fibers: the Twisted Malle Conjecture

For $T \trianglelefteq G$ and $\pi \in \mathrm{Sur}(G_k, G/T)$ corresponding to a G/T-extension M/k:

Counting in the fibers: the Twisted Malle Conjecture

For $T \trianglelefteq G$ and $\pi \in \mathrm{Sur}(G_k, G/T)$ corresponding to a G/T-extension M/k:

Fiber: G-extensions with $\widetilde{L}^T = M \longleftrightarrow q_*^{-1}(\pi) =$ lifts of π to $\mathrm{Sur}(G_k, G)$

Counting in the fibers: the Twisted Malle Conjecture

For $T \trianglelefteq G$ and $\pi \in \mathrm{Sur}(G_k, G/T)$ corresponding to a G/T-extension M/k:

Fiber: G-extensions with $\widetilde{L}^T = M \longleftrightarrow q_*^{-1}(\pi) =$ lifts of π to $\mathrm{Sur}(G_k, G)$

Twisted Malle Conjecture (Alberts 2021)

Counting in the fibers: the Twisted Malle Conjecture

For $T \trianglelefteq G$ and $\pi \in \mathrm{Sur}(G_k, G/T)$ corresponding to a G/T-extension M/k:

Fiber: G-extensions with $\widetilde{L}^T = M \longleftrightarrow q_*^{-1}(\pi) =$ lifts of π to $\mathrm{Sur}(G_k, G)$

Twisted Malle Conjecture (Alberts 2021)

For each fiber π, there exist $a, b, c > 0$ such that

$$\#\{\psi \in q_*^{-1}(\pi) : |\,\mathrm{disc}_G(\psi)| \leq X\} \sim c\, X^{1/a} (\log X)^{b-1}$$

Counting in the fibers: the Twisted Malle Conjecture

For $T \trianglelefteq G$ and $\pi \in \mathrm{Sur}(G_k, G/T)$ corresponding to a G/T-extension M/k:

Fiber: G-extensions with $\widetilde{L}^T = M \longleftrightarrow q_*^{-1}(\pi) =$ lifts of π to $\mathrm{Sur}(G_k, G)$

Twisted Malle Conjecture (Alberts 2021)

For each fiber π, there exist $a, b, c > 0$ such that

$$\#\{\psi \in q_*^{-1}(\pi) : |\,\mathrm{disc}_G(\psi)| \leq X\} \sim c\, X^{1/a}(\log X)^{b-1}$$

Key input (Alberts–O'Dorney 2021): When T is abelian, the Twisted Malle Conjecture holds.

Counting in the fibers: the Twisted Malle Conjecture

For $T \trianglelefteq G$ and $\pi \in \mathrm{Sur}(G_k, G/T)$ corresponding to a G/T-extension M/k:

Fiber: G-extensions with $\widetilde{L}^T = M \longleftrightarrow q_*^{-1}(\pi) =$ lifts of π to $\mathrm{Sur}(G_k, G)$

Twisted Malle Conjecture (Alberts 2021)

For each fiber π, there exist $a, b, c > 0$ such that

$$\#\{\psi \in q_*^{-1}(\pi) : |\operatorname{disc}_G(\psi)| \leq X\} \sim c\, X^{1/a} (\log X)^{b-1}$$

Key input (Alberts–O'Dorney 2021): When T is abelian, the Twisted Malle Conjecture holds.

The number of G-extensions in the fiber over π is asymptotically $\sim c(\pi)\, X^{1/a(T)} (\log X)^{b(k,T(\pi))-1}$, where:

Counting in the fibers: the Twisted Malle Conjecture

For $T \trianglelefteq G$ and $\pi \in \mathrm{Sur}(G_k, G/T)$ corresponding to a G/T-extension M/k:

Fiber: G-extensions with $\widetilde{L}^T = M \longleftrightarrow q_*^{-1}(\pi) =$ lifts of π to $\mathrm{Sur}(G_k, G)$

Twisted Malle Conjecture (Alberts 2021)

For each fiber π, there exist $a, b, c > 0$ such that

$$\#\{\psi \in q_*^{-1}(\pi) : |\,\mathrm{disc}_G(\psi)| \leq X\} \sim c\, X^{1/a}(\log X)^{b-1}$$

Key input (Alberts–O'Dorney 2021): When T is abelian, the Twisted Malle Conjecture holds.

The number of G-extensions in the fiber over π is asymptotically $\sim c(\pi)\, X^{1/a(T)}(\log X)^{b(k,T(\pi))-1}$, where:

- $T(\pi)$ is T with Galois action twisted by π

Counting in the fibers: the Twisted Malle Conjecture

For $T \trianglelefteq G$ and $\pi \in \mathrm{Sur}(G_k, G/T)$ corresponding to a G/T-extension M/k:

Fiber: G-extensions with $\widetilde{L}^T = M \longleftrightarrow q_*^{-1}(\pi) =$ lifts of π to $\mathrm{Sur}(G_k, G)$

Twisted Malle Conjecture (Alberts 2021)

For each fiber π, there exist $a, b, c > 0$ such that

$$\#\{\psi \in q_*^{-1}(\pi) : |\operatorname{disc}_G(\psi)| \leq X\} \sim c\, X^{1/a} (\log X)^{b-1}$$

Key input (Alberts–O'Dorney 2021): When T is abelian, the Twisted Malle Conjecture holds.

The number of G-extensions in the fiber over π is asymptotically $\sim c(\pi)\, X^{1/a(T)} (\log X)^{b(k, T(\pi))-1}$, where:

- $T(\pi)$ is T with Galois action twisted by π
- $c(\pi)$ depends on the $H^1_{ur}(k, T(\pi))$ (unramified cohomology group)

Summing over fibers when T is abelian

Summing over fibers when T is abelian

We want:

$$\#\mathcal{F}_k(G;X) = \sum_{\pi \in q_* \operatorname{Sur}(G_k, G; X)} \#\{\psi \in q_*^{-1}(\pi) : |\operatorname{disc}_G(\psi)| \le X\}$$

Summing over fibers when T is abelian

We want:

$$\#\mathcal{F}_k(G;X) = \sum_{\pi \in q_* \operatorname{Sur}(G_k,G;X)} \#\{\psi \in q_*^{-1}(\pi) : |\operatorname{disc}_G(\psi)| \leq X\}$$

Each term $\sim c(\pi)\, X^{1/a(T)} (\log X)^{b(k,T(\pi))-1}$

Summing over fibers when T is abelian

We want:

$$\#\mathcal{F}_k(G;X) = \sum_{\pi \in q_* \operatorname{Sur}(G_k, G; X)} \#\{\psi \in q_*^{-1}(\pi) : |\operatorname{disc}_G(\psi)| \leq X\}$$

Each term $\sim c(\pi)\, X^{1/a(T)}(\log X)^{b(k,T(\pi))-1}$

The leading coefficient $c(\pi)$ involves $|H^1_{ur}(k, T(\pi))|$

Summing over fibers when T is abelian

We want:

$$\#\mathcal{F}_k(G;X) = \sum_{\pi \in q_* \operatorname{Sur}(G_k, G; X)} \#\{\psi \in q_*^{-1}(\pi) : |\operatorname{disc}_G(\psi)| \leq X\}$$

Each term $\sim c(\pi)\, X^{1/a(T)} (\log X)^{b(k,T(\pi))-1}$

The leading coefficient $c(\pi)$ involves $|H^1_{ur}(k, T(\pi))|$

Two keys questions: (1) Is the sum of constant terms convergent? More precisely is:

Summing over fibers when T is abelian

We want:

$$\#\mathcal{F}_k(G;X) = \sum_{\pi \in q_* \operatorname{Sur}(G_k, G; X)} \#\{\psi \in q_*^{-1}(\pi) : |\operatorname{disc}_G(\psi)| \le X\}$$

Each term $\sim c(\pi)\, X^{1/a(T)}(\log X)^{b(k,T(\pi))-1}$

The leading coefficient $c(\pi)$ involves $|H^1_{ur}(k, T(\pi))|$

Two keys questions: (1) Is the sum of constant terms convergent? More precisely is:

$$\sum_{\pi \in q_* \operatorname{Sur}(G_k, G; X)} |H^1_{ur}(k, T(\pi))| \ll X^{\theta} \quad \text{for some } \theta < 1/a(T)?$$

Summing over fibers when T is abelian

We want:

$$\#\mathcal{F}_k(G;X) = \sum_{\pi \in q_* \operatorname{Sur}(G_k, G; X)} \#\{\psi \in q_*^{-1}(\pi) : |\operatorname{disc}_G(\psi)| \leq X\}$$

Each term $\sim c(\pi)\, X^{1/a(T)} (\log X)^{b(k,T(\pi))-1}$

The leading coefficient $c(\pi)$ involves $|H^1_{ur}(k, T(\pi))|$

Two keys questions: (1) Is the sum of constant terms convergent? More precisely is:

$$\sum_{\pi \in q_* \operatorname{Sur}(G_k, G; X)} |H^1_{ur}(k, T(\pi))| \ll X^\theta \quad \text{for some } \theta < 1/a(T)?$$

and (2) Can we sum the error terms? (we can when T is abelian)

Summing over fibers when T is abelian

We want:

$$\#\mathcal{F}_k(G;X) = \sum_{\pi \in q_* \operatorname{Sur}(G_k, G; X)} \#\{\psi \in q_*^{-1}(\pi) : |\operatorname{disc}_G(\psi)| \leq X\}$$

Each term $\sim c(\pi)\, X^{1/a(T)} (\log X)^{b(k,T(\pi))-1}$

The leading coefficient $c(\pi)$ involves $|H^1_{ur}(k, T(\pi))|$

Two keys questions: (1) Is the sum of constant terms convergent? More precisely is:

$$\sum_{\pi \in q_* \operatorname{Sur}(G_k, G; X)} |H^1_{ur}(k, T(\pi))| \ll X^{\theta} \quad \text{for some } \theta < 1/a(T)?$$

and (2) Can we sum the error terms? (we can when T is abelian)

If yes and yes: we can prove Malle's a-Conjecture for G.

H^1_{ur} and class groups

H^1_{ur} and class groups

$H^1_{ur}(k, T(\pi))$ = group of unramified Galois cohomology classes

H^1_{ur} and class groups

$H^1_{ur}(k, T(\pi))$ = group of unramified Galois cohomology classes

If $T(\pi)$ has trivial Galois action: $H^1_{ur}(k, T(\pi)) = \mathrm{Hom}(\mathrm{Cl}_k, T)$

H^1_{ur} and class groups

$H^1_{ur}(k, T(\pi))$ = group of unramified Galois cohomology classes

If $T(\pi)$ has trivial Galois action: $H^1_{ur}(k, T(\pi)) = \mathrm{Hom}(\mathrm{Cl}_k, T)$

Initial bound (Minkowski):

$$|H^1_{ur}(k, T(\pi))| \ll_{k,\epsilon} |\mathrm{disc}(F(\pi)/\mathbb{Q})|^{d(\hat{T})/2+\epsilon}$$

where $F(\pi)$ is the field of definition of $T(\pi)$.

H^1_{ur} and class groups

$H^1_{ur}(k, T(\pi))$ = group of unramified Galois cohomology classes

If $T(\pi)$ has trivial Galois action: $H^1_{ur}(k, T(\pi)) = \mathrm{Hom}(\mathrm{Cl}_k, T)$

Initial bound (Minkowski):

$$|H^1_{ur}(k, T(\pi))| \ll_{k,\epsilon} |\mathrm{disc}(F(\pi)/\mathbb{Q})|^{d(\hat{T})/2+\epsilon}$$

where $F(\pi)$ is the field of definition of $T(\pi)$.

The key technical tool is an inductive bound on $|H^1_{ur}(k, T(\pi))|$:

H^1_{ur} and class groups

$H^1_{ur}(k, T(\pi))$ = group of unramified Galois cohomology classes

If $T(\pi)$ has trivial Galois action: $H^1_{ur}(k, T(\pi)) = \mathrm{Hom}(\mathrm{Cl}_k, T)$

Initial bound (Minkowski):

$$|H^1_{ur}(k, T(\pi))| \ll_{k,\epsilon} |\operatorname{disc}(F(\pi)/\mathbb{Q})|^{d(\hat{T})/2+\epsilon}$$

where $F(\pi)$ is the field of definition of $T(\pi)$.

The key technical tool is an inductive bound on $|H^1_{ur}(k, T(\pi))|$:

Key Lemma (ALOWW)

H^1_{ur} and class groups

$H^1_{ur}(k, T(\pi))$ = group of unramified Galois cohomology classes

If $T(\pi)$ has trivial Galois action: $H^1_{ur}(k, T(\pi)) = \mathrm{Hom}(\mathrm{Cl}_k, T)$

Initial bound (Minkowski):

$$|H^1_{ur}(k, T(\pi))| \ll_{k,\epsilon} |\mathrm{disc}(F(\pi)/\mathbb{Q})|^{d(\hat{T})/2+\epsilon}$$

where $F(\pi)$ is the field of definition of $T(\pi)$.

The key technical tool is an inductive bound on $|H^1_{ur}(k, T(\pi))|$:

Key Lemma (ALOWW)

For any subgroup $M \leq T$, we have

$$|H^1_{ur}(k, T(\pi))| \ll |H^1_{ur}(k, M(\pi))| \cdot |H^1_{ur}(k, (T/M)(\pi))|.$$

H^1_{ur} and class groups

$H^1_{ur}(k, T(\pi))$ = group of unramified Galois cohomology classes

If $T(\pi)$ has trivial Galois action: $H^1_{ur}(k, T(\pi)) = \mathrm{Hom}(\mathrm{Cl}_k, T)$

Initial bound (Minkowski):

$$|H^1_{ur}(k, T(\pi))| \ll_{k,\epsilon} |\mathrm{disc}(F(\pi)/\mathbb{Q})|^{d(\hat{T})/2+\epsilon}$$

where $F(\pi)$ is the field of definition of $T(\pi)$.

The key technical tool is an inductive bound on $|H^1_{ur}(k, T(\pi))|$:

Key Lemma (ALOWW)

For any subgroup $M \leq T$, we have

$$|H^1_{ur}(k, T(\pi))| \ll |H^1_{ur}(k, M(\pi))| \cdot |H^1_{ur}(k, (T/M)(\pi))|.$$

Choose M strategically to break T into simpler pieces:

H^1_{ur} and class groups

$H^1_{ur}(k, T(\pi))$ = group of unramified Galois cohomology classes

If $T(\pi)$ has trivial Galois action: $H^1_{ur}(k, T(\pi)) = \mathrm{Hom}(\mathrm{Cl}_k, T)$

Initial bound (Minkowski):

$$|H^1_{ur}(k, T(\pi))| \ll_{k,\epsilon} |\mathrm{disc}(F(\pi)/\mathbb{Q})|^{d(\hat{T})/2+\epsilon}$$

where $F(\pi)$ is the field of definition of $T(\pi)$.

The key technical tool is an inductive bound on $|H^1_{ur}(k, T(\pi))|$:

Key Lemma (ALOWW)

For any subgroup $M \leq T$, we have

$$|H^1_{ur}(k, T(\pi))| \ll |H^1_{ur}(k, M(\pi))| \cdot |H^1_{ur}(k, (T/M)(\pi))|.$$

Choose M strategically to break T into simpler pieces:

To sum $|H^1_{ur}|$ we need an upper bound on G/T extensions

Application: iterated wreath products

Application: iterated wreath products

Corollary (ALOWW)

Application: iterated wreath products

Corollary (ALOWW)

Let k be a number field, $n_1, \ldots, n_r \geq 2$, and $G = C_{n_1} \wr C_{n_2} \wr \cdots \wr C_{n_r}$. Malle's Conjecture holds for G if any one of the following holds:

Application: iterated wreath products

Corollary (ALOWW)

Let k be a number field, $n_1, \ldots, n_r \geq 2$, and $G = C_{n_1} \wr C_{n_2} \wr \cdots \wr C_{n_r}$. Malle's Conjecture holds for G if any one of the following holds:

- (a) $n_1 = 2$
- (b) $n_2 > 2$
- (c) $n_1, n_2, \ldots, n_r$ are all powers of 2
- (d) $n_1 = 2^d$, $n_2 = 2$, and $(r < 3 \text{ or } 1 \leq d < 6 - 4/n_3)$
- (e) $n_1 = 2^d 3$, $n_2 = 2$, and $(r < 3 \text{ or } 0 \leq d < 13/3 - 4/n_3)$

Application: iterated wreath products

Corollary (ALOWW)

Let k be a number field, $n_1, \ldots, n_r \geq 2$, and $G = C_{n_1} \wr C_{n_2} \wr \cdots \wr C_{n_r}$. Malle's Conjecture holds for G if any one of the following holds:

- (a) $n_1 = 2$
- (b) $n_2 > 2$
- (c) $n_1, n_2, \ldots, n_r$ are all powers of 2
- (d) $n_1 = 2^d$, $n_2 = 2$, and $(r < 3 \text{ or } 1 \leq d < 6 - 4/n_3)$
- (e) $n_1 = 2^d 3$, $n_2 = 2$, and $(r < 3 \text{ or } 0 \leq d < 13/3 - 4/n_3)$

Includes:

Application: iterated wreath products

Corollary (ALOWW)

Let k be a number field, $n_1, \ldots, n_r \geq 2$, and $G = C_{n_1} \wr C_{n_2} \wr \cdots \wr C_{n_r}$. Malle's Conjecture holds for G if any one of the following holds:

a. $n_1 = 2$
b. $n_2 > 2$
c. $n_1, n_2, \ldots, n_r$ are all powers of 2
d. $n_1 = 2^d$, $n_2 = 2$, and ($r < 3$ or $1 \leq d < 6 - 4/n_3$)
e. $n_1 = 2^d 3$, $n_2 = 2$, and ($r < 3$ or $0 \leq d < 13/3 - 4/n_3$)

Includes:

- All Sylow p-subgroups $C_p^{\wr r}$ of S_{p^r}

Application: iterated wreath products

Corollary (ALOWW)

Let k be a number field, $n_1, \ldots, n_r \geq 2$, and $G = C_{n_1} \wr C_{n_2} \wr \cdots \wr C_{n_r}$. Malle's Conjecture holds for G if any one of the following holds:

(a) $n_1 = 2$

(b) $n_2 > 2$

(c) $n_1, n_2, \ldots, n_r$ are all powers of 2

(d) $n_1 = 2^d$, $n_2 = 2$, and $(r < 3 \text{ or } 1 \leq d < 6 - 4/n_3)$

(e) $n_1 = 2^d 3$, $n_2 = 2$, and $(r < 3 \text{ or } 0 \leq d < 13/3 - 4/n_3)$

Includes:

- All Sylow p-subgroups $C_p^{\wr r}$ of S_{p^r}
- Klüners' original counterexample $C_3 \wr C_2$ (in degree 6)

Application: iterated wreath products

Corollary (ALOWW)

Let k be a number field, $n_1, \ldots, n_r \geq 2$, and $G = C_{n_1} \wr C_{n_2} \wr \cdots \wr C_{n_r}$.
Malle's Conjecture holds for G if any one of the following holds:

(a) $n_1 = 2$

(b) $n_2 > 2$

(c) $n_1, n_2, \ldots, n_r$ are all powers of 2

(d) $n_1 = 2^d$, $n_2 = 2$, and $(r < 3 \text{ or } 1 \leq d < 6 - 4/n_3)$

(e) $n_1 = 2^d 3$, $n_2 = 2$, and $(r < 3 \text{ or } 0 \leq d < 13/3 - 4/n_3)$

Includes:

- All Sylow p-subgroups $C_p^{\wr r}$ of S_{p^r}
- Klüners' original counterexample $C_3 \wr C_2$ (in degree 6)
- Many more counterexamples to Malle's Conjecture

The inductive power: compounding new results

The inductive power: compounding new results

The compounding effect: every time we prove Malle's Conjecture for one group, it can serve as input to prove it for more groups.

The inductive power: compounding new results

The compounding effect: every time we prove Malle's Conjecture for one group, it can serve as input to prove it for more groups.

Example:

The inductive power: compounding new results

The compounding effect: every time we prove Malle's Conjecture for one group, it can serve as input to prove it for more groups.

Example:

- Start: Malle's Conjecture known for C_n (abelian)

The inductive power: compounding new results

The compounding effect: every time we prove Malle's Conjecture for one group, it can serve as input to prove it for more groups.

Example:

- Start: Malle's Conjecture known for C_n (abelian)
- Apply our theorem: get $C_n \wr B$ for many B

The inductive power: compounding new results

The compounding effect: every time we prove Malle's Conjecture for one group, it can serve as input to prove it for more groups.

Example:

- Start: Malle's Conjecture known for C_n (abelian)
- Apply our theorem: get $C_n \wr B$ for many B
- Apply again: get $C_{n_1} \wr C_{n_2} \wr B$ for many B

The inductive power: compounding new results

The compounding effect: every time we prove Malle's Conjecture for one group, it can serve as input to prove it for more groups.

Example:

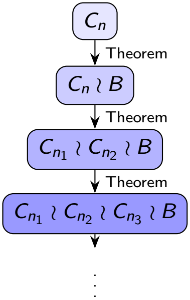
- Start: Malle's Conjecture known for C_n (abelian)
- Apply our theorem: get $C_n \wr B$ for many B
- Apply again: get $C_{n_1} \wr C_{n_2} \wr B$ for many B
- Iterate: infinite families of iterated wreath products

The inductive power: compounding new results

The compounding effect: every time we prove Malle's Conjecture for one group, it can serve as input to prove it for more groups.

Example:

- Start: Malle's Conjecture known for C_n (abelian)
- Apply our theorem: get $C_n \wr B$ for many B
- Apply again: get $C_{n_1} \wr C_{n_2} \wr B$ for many B
- Iterate: infinite families of iterated wreath products

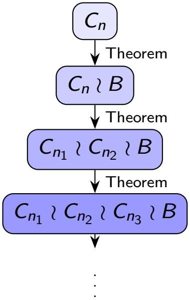


The inductive power: compounding new results

The compounding effect: every time we prove Malle's Conjecture for one group, it can serve as input to prove it for more groups.

Example:

- Start: Malle's Conjecture known for C_n (abelian)
- Apply our theorem: get $C_n \wr B$ for many B
- Apply again: get $C_{n_1} \wr C_{n_2} \wr B$ for many B
- Iterate: infinite families of iterated wreath products



Primitive groups: Our Theorem also handles *primitive* groups G with abelian normal T—no wreath product structure needed.

Main Theorem: S_3-wreath products

Main Theorem: S_3-wreath products

For non-abelian T, we need to prove the Twisted Malle Conjecture first.

Main Theorem: S_3-wreath products

For non-abelian T, we need to prove the Twisted Malle Conjecture first.

Theorem (ALOWW)

Main Theorem: S_3-wreath products

For non-abelian T, we need to prove the Twisted Malle Conjecture first.

Theorem (ALOWW)

Let k be a number field, B a transitive permutation group of degree m (with at least one B-extension of k), and $G = S_3 \wr B$.
Suppose $\theta \geq 0$ satisfies

$$\sum_{F \in \mathcal{F}_k(B;X)} |\operatorname{Cl}_F[2]|^{2/3} \ll_{m,k} X^{\theta}.$$

Main Theorem: S_3-wreath products

For non-abelian T, we need to prove the Twisted Malle Conjecture first.

Theorem (ALOWW)

Let k be a number field, B a transitive permutation group of degree m (with at least one B-extension of k), and $G = S_3 \wr B$.
Suppose $\theta \geq 0$ satisfies

$$\sum_{F \in \mathcal{F}_k(B;X)} |\operatorname{Cl}_F[2]|^{2/3} \ll_{m,k} X^{\theta}.$$

(i) If $\theta < 2$: $\#\mathcal{F}_{3m,k}(G;X) \sim c(k,G)\,X$.

Main Theorem: S_3-wreath products

For non-abelian T, we need to prove the Twisted Malle Conjecture first.

Theorem (ALOWW)

Let k be a number field, B a transitive permutation group of degree m (with at least one B-extension of k), and $G = S_3 \wr B$.
Suppose $\theta \geq 0$ satisfies

$$\sum_{F \in \mathcal{F}_k(B;X)} |\operatorname{Cl}_F[2]|^{2/3} \ll_{m,k} X^{\theta}.$$

1. If $\theta < 2$: $\#\mathcal{F}_{3m,k}(G;X) \sim c(k,G)\,X$.
2. If $\theta \geq 2$: $\#\mathcal{F}_{3m,k}(G;X) \ll_{m,k,\epsilon} X^{(\theta+1)/3+\epsilon}$.

Main Theorem: S_3-wreath products

For non-abelian T, we need to prove the Twisted Malle Conjecture first.

Theorem (ALOWW)

Let k be a number field, B a transitive permutation group of degree m (with at least one B-extension of k), and $G = S_3 \wr B$.
Suppose $\theta \geq 0$ satisfies

$$\sum_{F \in \mathcal{F}_k(B;X)} |\operatorname{Cl}_F[2]|^{2/3} \ll_{m,k} X^{\theta}.$$

1. If $\theta < 2$: $\#\mathcal{F}_{3m,k}(G;X) \sim c(k,G)\,X$.
2. If $\theta \geq 2$: $\#\mathcal{F}_{3m,k}(G;X) \ll_{m,k,\epsilon} X^{(\theta+1)/3+\epsilon}$.

Here $T = S_3^m \trianglelefteq S_3 \wr B$, and we prove new cases of the Twisted Malle Conjecture for this nonabelian T.

Further directions

Further directions

- Does the b proven above agree with Wang's conjecture?

Further directions

- Does the b proven above agree with Wang's conjecture?

Further directions

- Does the b proven above agree with Wang's conjecture?
- **More T.** The method applies with any T for which the Twisted Malle Conjecture is known with explicit π-dependence. What other groups T can we prove this for?

Further directions

- Does the b proven above agree with Wang's conjecture?
- **More T.** The method applies with any T for which the Twisted Malle Conjecture is known with explicit π-dependence. What other groups T can we prove this for?
- **Non-concentrated groups.** Theorem (ii) gives new upper bounds even when we can't prove the full asymptotic. How good can these be?

Further directions

- Does the b proven above agree with Wang's conjecture?
- **More T.** The method applies with any T for which the Twisted Malle Conjecture is known with explicit π-dependence. What other groups T can we prove this for?
- **Non-concentrated groups.** Theorem (ii) gives new upper bounds even when we can't prove the full asymptotic. How good can these be?
- **Local conditions.** Can we adapt the method to count fields with specified splitting behavior at finitely many primes?