Fixed loci of symplectic automorphisms of hyperkähler manifolds

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Basics

Definition

An **irreducible hyperkähler manifold** (equivalently, an **irreducible holomorphic symplectic manifold**) M is a compact complex Kähler manifold with $\pi_1(M) = 0$ and $H^{2,0}(M) = \mathbb{C}\sigma$, with σ everywhere non-degenerate.

Remarks.

1. Any holomorphic 2-form σ induces a homomorphism $\tau : \mathcal{T}_X \to \Omega_X$. The form σ is everywhere non-degenerate if and only if τ is bijective.
2. The last condition implies that $\mathcal{K}_X = \mathcal{O}_X$. In particular, $c_1(X) = 0$.
3. From now on, the hyperkähler manifolds we consider are irreducible.

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Examples

1. **K3 surfaces** are the only examples in complex dimension 2.
2. **Standard series of examples** due to Beauville ($n \geq 2$) :
   (i) (Fujiki, $n = 2$.) The **Hilbert schemes** of n points $Hilb^n(S)$, where S is a K3 surface. This is the Douady space parametrizing 0-dimensional subschemes Z of length n. $Hilb^n(S)$ is a smooth connected compact manifold of dimension $2n$ which is the desingularization of $Sym^n(S)$ via the Hilbert-Chow morphism, and $b_2(Hilb^n(S)) = 23$.
   (ii) The **generalized Kummer varieties** $K_n(A)$, where A is an abelian surface. This is the fiber over $0 \in A$ of the natural composition of the Hilbert-Chow morphism and the summation map $Hilb^{n+1}A \to Sym^{n+1}A \to A$ (here $b_2(K_n(A)) = 7$). We also call them of **Kummer n type**.
3. **O'Grady's two exceptional examples** in complex dimensions 10 and 6 with $b_2(OG10) = 24$ and $b_2(OG6) = 8$, respectively (desingularizations of moduli spaces of semistable sheaves).

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Automorphisms and deformations

Definition

Let X be a complex manifold and let G be a subgroup of $\mathrm{Aut}(X)$, the group of automorphisms of X. A **deformation of the pair** (X, G) consists of the following data:

1. *(i)* A flat family $\mathcal{X} \to B$, where B is connected and $\mathcal{X}$ is smooth, and a distinguished point $0 \in B$ such that $\mathcal{X}_0 \cong X$.
2. *(ii)* A faithful action of the group G on $\mathcal{X}$ inducing fiberwise faithful actions of G.

Definition

Two pairs (X, G) and (Y, H) are **deformation equivalent** if (Y, H) is a fiber of a deformation of the pair (X, G).

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Natural and standard pairs

Definition for Hilbert schemes

Let S be a $K3$ surface and let $G \subset \mathrm{Aut}_s(S)$ be a subgroup of the symplectic automorphisms on S. Then G induces a subgroup of the symplectic morphisms on $S^{[n]}$ which we still denote by G. We call the pair $(S^{[n]}, G)$ a **natural pair**. The pair (X, H) is **standard** if it is deformation equivalent to a natural pair.

Definition for generalized Kummer varieties

If A is an abelian surface, the same definitions apply to the generalized Kummer $2n$-fold $K_n(A)$ and symplectic automorphisms preserving $0 \in A$, however notice that the induced action of G on $H^2(K_n(A))$ is not necessarily faithful (while it stays faithful on $K_n(A)$), as there are automorphisms acting trivially on the second cohomology.

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Numerically standard pairs and fixed loci

Definition

Let G be a finite group acting faithfully on a manifold X. Define the **invariant locus** $T_G(X)$ inside $H^2(X,\mathbb{Z})$ to be the fixed locus of the induced action of G on the cohomology. The **co-invariant lattice** $S_G(X)$ is the orthogonal complement $T_G(X)^{\perp}$. The **fixed locus** of G on X is denoted by X^G.

Definition

Let Y be a manifold of $K3^{[n]}$ type or of Kummer n type. A pair (Y,H) is called **numerically standard** if the representation of H on $H^2(Y,\mathbb{Z})$ is isomorphic to that of a standard pair (X,H), up to conjugation by the monodromy group.

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Teichmüller space

Definition

Let X be a compact complex manifold and let $\mathrm{Diff}^0(X)$ be the connected component of its diffeomorphism group containing the identity. Denote by Comp the space of complex structures on X, equipped with the structure of a Fréchet manifold. The **Teichmüller space** of X is the quotient $\mathrm{Teich} := \mathrm{Comp} / \mathrm{Diff}^0(X)$.

Remark/ Definition

The Teichmüller space is finite-dimensional for Calabi-Yau or hyperkähler manifolds X. Let $\mathrm{Diff}^+(X)$ be the group of orientable diffeomorphisms of a complex manifold X. The **mapping class group** $\Gamma := \mathrm{Diff}^+(X)/\mathrm{Diff}^0(X)$ acts on Teich.

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Period map

Remark/ Definition

By Huybrechts's results, non-separated points in the moduli space of marked hyperkähler manifolds correspond to birational hyperkähler manifolds. Consider the equivalence relation $\sim$ on Teich identifying non-separated points. Let $\mathrm{Teich}_b = \mathrm{Teich}/_\sim$ be the **birational Teichmüller space**.

Definition

Let X be a hyperkähler manifold, and let Teich be its Teichmüller space. Define the **period map** $\mathcal{P} : \mathrm{Teich} \to \mathbb{P}\mathrm{er} \subset \mathbb{P}H^2(X, \mathbb{C})$, sending a given complex structure J to the class of the corresponding line $[H^{2,0}(X, J)] \in \mathbb{P}H^2(X, \mathbb{C})$. The image of $\mathcal{P}$ is an open subset of a quadric, namely, the **period domain** $\mathbb{P}\mathrm{er} := \left\{ I \in \mathbb{P}H^2(X, \mathbb{C}) \;\middle|\; q(I, I) = 0,\ q(I, \bar{I}) > 0 \right\}$.

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Torelli Theorem

Verbitsky's Global Torelli

The period map $\mathcal{P} : \mathrm{Teich}_b^0 \to \mathbb{P}\mathrm{er}$ is an isomorphism on each connected component of Teich_b.

Remark

The period domain $\mathbb{P}\mathrm{er}$ is identified with the following quotient of non-compact Lie groups:

$$\frac{SO(3, b_2 - 3)}{SO(2) \times SO(1, b_2 - 3)}.$$

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Primitive embeddings of lattices

Nikulin's technical result

Let S and N be even lattices of signature (s_+, s_-) resp. (n_+, n_-). Primitive embeddings of S into N are determined by the sets $(H_S, H_N, \gamma, K, \gamma_K)$, where H_S and H_N are subgroups of A_S and A_N, K is an even lattice with signature $(n_+ - s_+, n_- - s_-)$ and discriminant form $-\delta$ where $\delta \cong (q_{A_S} \oplus -q_{A_N})_{|\Gamma_\gamma^\perp/\Gamma_\gamma}$ and $\gamma_K : q_K \to (-\delta)$ is an isometry. Moreover, two such sets $(H_S, H_N, \gamma, K, \gamma_K)$ and $(H'_S, H'_N, \gamma', K', \gamma'_K)$ determine isomorphic sublattices if and only if

- $H_S = \lambda H'_S$, $\lambda \in O(q_S)$,
- $\exists \epsilon \in O(q_{A_N})$ and $\psi \in Isom(K, K')$ such that $\gamma' = \epsilon \circ \gamma$ and $\overline{\epsilon} \circ \gamma_K = \gamma'_K \circ \overline{\psi}$, where $\overline{\epsilon}$ and $\overline{\psi}$ are the isometries induced among discriminant groups.

Connectedness of the moduli space of pairs

Remark

The moduli space of pairs consisting of a hyperkähler manifold of $K3^{[n]}$ type or of Kummer n type together with a symplectic involution is connected. This follows from the following result.

Theorem (K. - Mongardi - Oblomkov)

Let X be a manifold of $K3^{[n]}$ type or of Kummer n type. Let $G \subset \mathrm{Aut}_s(X)$ be a finite group of numerically standard automorphisms. Then (X, G) is a standard pair.

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Idea behind the proof of the previous Theorem

Step 1.

First, we prove that (X, G) is birational to a standard pair (Y, G) by using the Global Torelli theorem, and then we prove that birational pairs can be deformed one into the other while preserving the group action.

Step 2.

Consider the Kuranishi families $\mathcal{X} \to U$ and $\mathcal{Y} \to U$ of local deformations of X and Y. Restrict to the local deformations of pairs (X, G) and (Y, G), which are Kuranishi families over $U^G \subset U$. For very general points $t \in U^G$ of the base, the Kähler cone and the positive cone coincide, and all manifolds birational to $\mathcal{X}_t$ are isomorphic to it, hence $\mathcal{X}_t \cong \mathcal{Y}_t$. This isomorphism is G-equivariant, which implies that the pairs (X, G) and (Y, G) are both deformation equivalent to $(\mathcal{X}_t, G) \cong (\mathcal{Y}_t, G)$, Q.E.D.

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Corollaries

Corollary 1

Let X be a manifold of $K3^{[n]}$ type and let ι be a symplectic involution. Then the fixed locus of ι is a deformation of the fixed locus on $S^{[n]}$ of a symplectic involution coming from the $K3$ surface S.

Corollary 2

Let X be a manifold of Kummer n type and let ι be a symplectic involution. Then the fixed locus of ι is a deformation of the fixed locus on $K_n(A)$ of the -1 involution coming from the Abelian surface A.

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Let X be a manifold of Kummer n type and let ι be a symplectic involution. Then the fixed locus of ι is a deformation of the fixed locus on $K_n(A)$ of the -1 involution coming from the Abelian surface A.

Illustrative examples: combinatorics

Example 1: even case

Consider $Sym^2(S)$, where S is a $K3$ surface with a symplectic involution i. The involution i induces an involution ι on $Sym^2(S)$. The fixed locus of ι has a component of the form S/i, because locally it consists of unordered pairs of points $\{p, i(p)\}$, where $p \in S$. The rest of the fixed points are isolated unordered pairs of the form $\{p, q\}$, where p and q are fixed points of i on S, with possible repetitions. Nikulin proved that i has 8 fixed points on S. Mongardi proved that the fixed locus of ι on $S^{[2]}$ consists of $28 = \binom{8}{2}$ isolated points and one copy of S. The eight fixed points of type $\{p, p\}$ are contained in the minimal resolution of S/i which is S.

Illustrative examples: combinatorics

Example 2: odd case

Consider $Sym^3(S)$, where S is a $K3$ surface with a symplectic involution i. The involution i induces an involution ι on $Sym^3(S)$. The fixed locus of largest dimension of ι locally looks like $\{p, i(p), q\}$, where $p \in S$ and q is a fixed point of i on S. There are 8 connected components of fixed loci of the form S/i, because there are 8 possibilities for q by Nikulin's result. The rest of the fixed points are isolated of the form $\{p, q, r\}$, where p, q and r are fixed points of i on S. In total, there are $56 = \binom{8}{3}$ points on $S^{[3]}$ corresponding to triples consisting of three different points $\{p, q, r\}$, and all fixed points of the form $\{p, p, q\}$ with $p \neq q$ are contained in the resolution of S/ι. There are eight more isolated fixed points given by schemes fully supported on one point whose reduced scheme structure contains all possible tangent directions.

Theta functions

Definition

The **standard theta function** is:

$$\vartheta(\eta; q) = \sum_{n \in \mathbb{Z}} q^{n^2} \eta^n.$$

Notation

Denote by N_k the following theta-constant $N_k = \Theta(n - 2k)$, where $\Theta(i)$ is the i-th coefficient of the Theta series

$$\Theta(q) = \sum_{i \geq 0} \Theta(i) q^i = \vartheta(q; q^2)^8.$$

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Fixed loci - Hilbert schemes

Theorem 1 (K - Mongardi - Oblomkov)

Let S be a K3 surface and let $\imath$ be a symplectic involution of S. Let $S^{[n]}$ be the Hilbert scheme of n points on S and let us consider the induced action of $\imath$ on $S^{[n]}$. Then all the irreducible components of the locus stabilized by $\imath$ are deformation equivalent to Hilbert schemes of points on a $K3$ surface or isolated points, and their number for each dimension $2k$ is $N_k = \Theta(n-2k)$, where N_k is the constant defined above.

Remark

In particular, the fixed locus Z of largest dimension is the following.
(i) If n is even, then Z consists of one copy of $K3^{[\frac{n}{2}]}$;
(ii) If n is odd, then Z consists of 8 copies of $K3^{[\frac{n-1}{2}]}$.

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Set up for the Kummer case

Remark

All involutions on generalized Kummer manifolds act trivially on the second cohomology, and they can be subdivided into three different geometrical categories:

- Involutions obtained by a translation of a two torsion point.
- Involutions obtained by a sign change composed with a translation by a two torsion point.
- Involutions obtained by a composition of a translation by a point of order at least three with a sign change.

Remark/ Definition

In this paper we deal with the second kind of involutions, and we call them **regular involutions**. All three kinds of involutions above can be recognized by their action on higher cohomology, and for instance, regular involutions act non-trivially on $H^3(X)$.

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Fixed loci - Kummer type

Theorem 2 (K - Mongardi - Oblomkov)

Let $X = K_n(A)$ be a hyperkähler manifold of Kummer n type, and let $\imath \in Aut(A)$ be a regular symplectic involution of the abelian surface A. Then all irreducible components of $X^\imath$ are of $K3^{[k]}$ type, and their number is $N_k = \Theta(n-2k;1)$, where $\Theta(j;1)$ is the j-th coefficient of the Theta series

$$\Theta(q) = \sum_{i \geq 0, \gamma \in A, \gamma^2 = 1} \Theta(i;\gamma) q^i \gamma = \prod_{\gamma \in A, \gamma^2 = 1} \vartheta(\gamma \cdot q; q^2).$$

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Ideas behind the proofs

Sketch of the ideas behind Theorem 1

From the connectedness of the moduli space of pairs (X, G), where X is of $K3^{[n]}$ type, and G is a finite group of numerically standard automorphisms, we can restrict our attention to the case when $X = S^{[n]}$ and ι comes from a symplectic involution on S (Corollary 1). As the fixed locus of an automorphism deforms smoothly in an equivariant family, our claim would follow if we prove it in this setting. Let us denote by Y the minimal resolution of S/ι. We also prove that all components of the fixed locus are Hilbert schemes over Y. Since ι on X comes from an automorphism of S, it also factors through the Hilbert-Chow morphism, and it is also defined on $Sym^n S$. All irreducible components of the fixed locus of ι are holomorphic symplectic subvarieties of X, and in this case each one of them has even dimension $2m$. Denote the fixed locus of dimension $2m$ by F_{2m}.

Ideas behind the proofs

Sketch of the ideas behind Theorem 1 - continued

By Nikulin's theorem, the symplectic involution on a $K3$ surface has 8 fixed points $f_1, \cdots, f_8$. A fixed point of ι on $Sym^n(S)$ is therefore supported on the fixed points and on pairs of points $(p, \iota(p))$. Therefore, on $Sym^n(S)$ each component F_{2m} is isomorphic to

$$Sym^m S/\iota \times Sym^{n-2m}(f_1 \cup \cdots \cup f_8).$$

Let us denote by U_i some small analytic neighborhoods around f_i. We analyse the structure of the fixed locus in these analytic neighborhoods, using Nakajima quiver varieties and their corresponding Kac-Moody Lie algebras, in order to obtain these formulas.

Ideas behind the proofs

Sketch of the ideas behind Theorem 2

From Corollary 2, we can restrict to the case when $X = K_n(A)$ and ι comes from a the involution -1 on A. Since the involution ι acts as the identity on the classes in $H^2(X, \mathbb{R})$, it preserves the exceptional divisor of $K_n(A)$. Therefore, it descends to an action on $Sym^{n+1}A$ preserving the fiber over 0 of the Albanese map. As in the previous theorem, first we look at this action on $Sym^{n+1}A$, and then let we consider the Albanese fiber of the fixed locus. The irreducible components of the fixed locus of ι are symplectic subvarieties of X, and each one of them has even dimension $2m+2$. Let us label the fixed locus of dimension $2m+2$ by F_{2m} (whose structure in Sym^{n+1} is the same of F_{2m+2} in the previous Theorem). The symplectic involution on the Abelian surface A has 16 fixed points $f_1, \cdots, f_{16}$. Similarly to the previous theorem, we count the number of fixed components of F_{2m} using Nakajima quiver varieties in order to obtain these formulas.

Generalizations of these results

Set-up

Consider a K3 surface S together with a symplectic action of a finite group G on S. The action of G is called **symplectic** if the induced action on $H^{2,0}(S)$ is trivial, i.e., the resolution of the quotient S/G is a K3 surface. Following the works of Nikulin and of Mukai, Xiao gives a combinatorial classification of finite groups G admitting a symplectic action on S together with the number of fixed points of each type (there are 81 such groups). The types of fixed points of the action of G on S correspond to the types of singularities of the quotient S/G. An analogous classification for Abelian surfaces was given by Fujiki.

Symplectic Automorphisms

Theorem 3 (K - Mongardi - Oblomkov)

Let S be a K3 surface and let G be a finite symplectic group of automorphisms of S. Let $S^{[n]}$ be the Hilbert scheme of n points on S, and consider the induced action of G on $S^{[n]}$. Then all the irreducible components of the locus stabilized by G are deformations of Hilbert schemes of points on a $K3$ surface or isolated points, and their number for each dimension $2k$ is $N_k = \Theta_G[n - |G|k]$, where $\Theta_G[i]$ is the i-th coefficients of the Theta series below.

Definition

$\Theta_G(q) = \sum_i \Theta_G[i] q^i =$
$\prod_{i \in Sing(S/G)} \vartheta_{\Delta_i}(q^{d^1(\Delta_i)|G|/|G_i|}, \ldots, q^{d^{r_i}(\Delta_i)|G|/|G_i|}; q^{|G|})$

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Possible restrictions and generalizations

Remark

If we assume that $n-1$ is a prime power, then any numerically standard group in cohomology becomes standard, and therefore we can deform the pair (X, G) to a natural pair $(S^{[n]}, G)$. Hence, the following main result follows from the previous theorem.

Theorem 4 (K - Mongardi - Oblomkov)

Let X be a hyperkähler manifold of $K3^{[n]}$ type such that $n-1$ is a prime power. Let $G \subset Aut(X)$ be a finite group of symplectic automorphisms which is numerically standard in cohomology. Then all irreducible components of X^G are of $K3^{[k]}$ type, and their number is $N_k = \Theta_G[n - |G|k]$.

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Thanks

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