Degenerations of Hilbert schemes of points

Joint work with L. Halle, Z. Zhang

ICERM, 26 October, 2026



Leibniz
Universität
Hannover



L. Halle
K. Hulek
Z. Zhang

0. Introduction

1. Expanded degenerations

2. The GIT construction

3. The construction for generalized Kummer varieties

0. Introduction



L. Halle
K. Hulek
Z. Zhang

0. Introduction

1. Expanded degenerations

2. The GIT construction

3. The construction for generalized Kummer varieties

The aim of this talk is to discuss explicit degenerations of irreducible symplectic holomorphic manifolds (IHSM), aka hyperkähler manifolds (HK). We will mostly concentrate on generalized Kummer varieties (but build on degenerations of Hilbert schemes of points on surfaces).

Motivation

- One wants to work in families
- Compactifications of moduli spaces
- Possible candidates for new examples of IHSM (speculative).

For surfaces of Kodaira dimension 0, there is the fundamenal work of Kulikov and Persson-Pinkham.

For example for $K3$ surfaces one has

Theorem (Kulikov, Pinkham-Persson)

Given a degeneration of $K3$ surfaces, one can, after base change and birational modifications of the total space, assume that $\pi : X \to C$ is a Kulikov model, *i.e. the total space X is smooth with $K_X = \mathcal{O}_X$ and the central fibre is one of the following*

(I) *X_0 is smooth,*

(II) *X_0 is a chain*

$$X_0 = Y_0 \cup Y_1 \cup \ldots \cup Y_\ell$$

with Y_0 and Y_ℓ rational surfaces , $(Y_j)_{j=1,\ldots,\ell-1}$ elliptic ruled surfaces and $D_{j,j+1} = Y_j \cap Y_{j+1}$ elliptic curves.

(III) *X_0 consists of smooth rational surfaces which intersect in anti-canonical cycles. The dual complex is homeomorphic to a sphere S^2.*



L. Halle
K. Hulek
Z. Zhang

0. Introduction

1. Expanded degenerations

2. The GIT construction

3. The construction for generalized Kummer varieties



L. Halle
K. Hulek
Z. Zhang

0. Introduction

1. Expanded degenerations

2. The GIT construction

3. The construction for generalized Kummer varieties

Remark

- The difference between type (II) and type (III) is the (non)-existence of triple points in the central fibre.
- These two types can also be distinguished by the nilpotency of the monodromy.
- This distinction also applies to higher dimension.

Existing Work

- Li and Wu have a very general theory of degenerations of Hilbert schemes (not only of Hilbert schemes of points).
- Nagai has produced an explicit Kulikov type model for type II degenerations of degree 2 Hilbert schemes on $K3$ surfaces.
- Together with Martin Gulbrandsen and Lars Halle, later joined by Ziyu Zhang, we developed a GIT approach to degenerations of Hilbert schemes of points on surfaces (in particular $K3$ surfaces) and studied the properties of these degenerations.
- Nagai has developed his theory further. He also proved that the degenerations he constructed are the same as ours.
- Calla Tschanz, later together with Quaasi Shafi, constructed type III degenerations.
- There is a very recent paper by Patrick Kennedy-Hunt on degenerations of generalized Kummer varieties (using log-geometry and tropical methods).



L. Halle
K. Hulek
Z. Zhang

0. Introduction
1. Expanded degenerations
2. The GIT construction
3. The construction for generalized Kummer varieties

Aim of this talk

- Here I want to discuss explicit degenerations of generalized Kummer varieties.

Definition

Let A be an abelian surface and let

$$\mu : \mathrm{Hilb}^n(A) \to A$$

be the summation map. Then the kernel

$$K^{n-1}(A) = \mu^{-1}(0)$$

is the *generalized Kummer variety* of A of dimension $2n-2$.



L. Halle
K. Hulek
Z. Zhang

0. Introduction

1. Expanded degenerations

2. The GIT construction

3. The construction for generalized Kummer varieties

Expanded degenerations



L. Halle
K. Hulek
Z. Zhang

0. Introduction

1. Expanded degenerations

2. The GIT construction

3. The construction for generalized Kummer varieties

Definition

A *simple degeneration* over a smooth curve C is a flat morphism $\pi\colon X \to C$ from a smooth scheme X to C, such that

(1) π is smooth outside the central fibre $X_0 = \pi^{-1}(0)$,

(2) the central fibre X_0 has normal crossing singularities and its singular locus $D \subset X_0$ is smooth.

The degeneration $\pi\colon X \to C$ is called *strict* if in addition

(3) all components of X_0 are smooth.

Remark

- Condition (2) implies that there are no triple points (type II).
- Condition (3) excludes self-intersections of a component.



L. Halle
K. Hulek
Z. Zhang

0. Introduction

1. Expanded degenerations

2. The GIT construction

3. The construction for generalized Kummer varieties

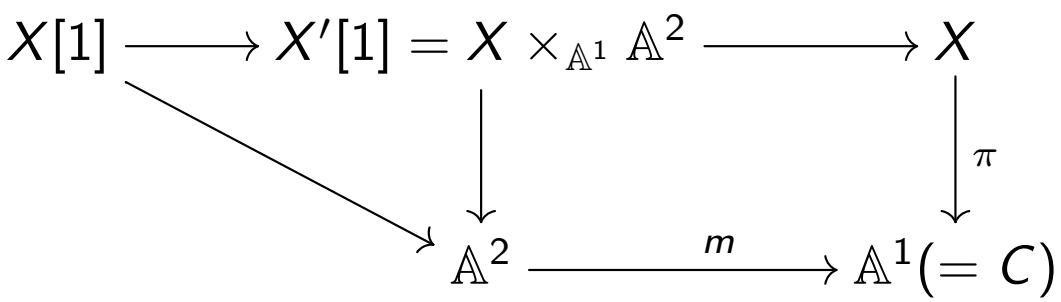
Let $\pi : X \to C$ be a simple degeneration, and let $0 \in C$ be the distinguished base point with fibre X_0. For the moment we assume for simplicity that

$$X_0 = Y_1 \cup Y_2.$$

We further assume $C = \mathbb{A}^1$. We can always reduce to this after choosing a local coordinate t at $0 \in C$.

The concept of *expanded degenerations* was introduced by Li. It can be defined successively as follows.

We set $X[0] = X$. For $n = 1$ we consider



where

$$m(t_1, t_2) = t_1 t_2.$$

Then $X'[1]$ is singular, since it is locally given by $x_1 x_2 = t_1 t_2$. We define $X[1]$ as the blow-up along the Weil divisor $Y_1 \times \{t_2 = 0\}$. This involves a *choice*!

This construction is then repeated inductively.



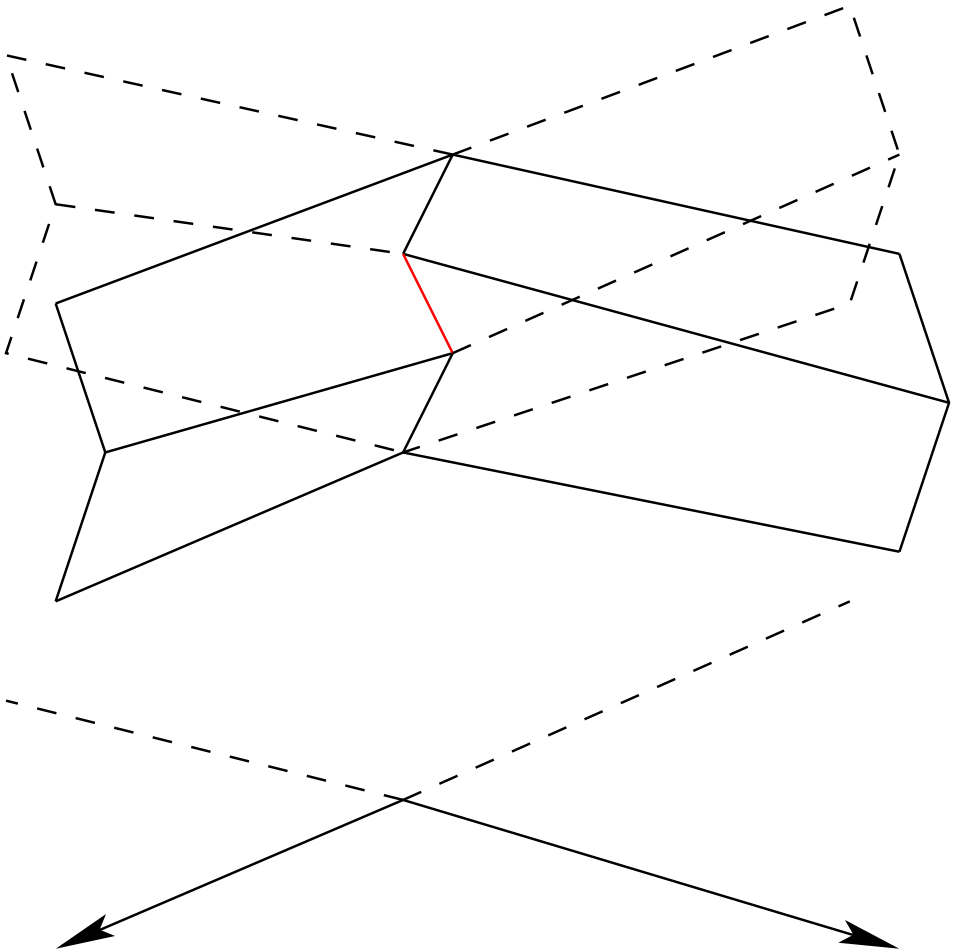
L. Halle
K. Hulek
Z. Zhang

0. Introduction

1. Expanded degenerations

2. The GIT construction

3. The construction for generalized Kummer varieties



Picture of $X[1] \to \mathbb{A}^2$.

L. Halle
K. Hulek
Z. Zhang

0. Introduction

1. Expanded degenerations

2. The GIT construction

3. The construction for generalized Kummer varieties



L. Halle
K. Hulek
Z. Zhang

0. Introduction

1. Expanded degenerations

2. The GIT construction

3. The construction for generalized Kummer varieties

The expanded degenerations $X[n] \to \mathbb{A}^{n+1}$ admit a natural action of a torus. For this we define

$$G[n] = \{(\sigma_1, \dots \sigma_{n+1}) \in \mathbb{G}_m^{n+1} \mid \sigma_1 \dots \sigma_{n+1} = 1\} \cong \mathbb{G}_m^n$$

which acts on $\mathbb{A}^{n+1}$ by

$$(t_1, \dots, t_{n+1}) \mapsto (\sigma_1 t_1, \dots, \sigma_{n+1} t_{n+1}).$$

This action lifts inductively to $X[n]$.
Notice that

$$\mathbb{A}^{n+1}/G[n] \cong \mathbb{A}^1.$$

The above construction depends on choices. Let $\Gamma(X_0)$ be the dual graph of the central fibre X_0. Then the choice of the successive blow-ups is encoded in an *orientation* of the graph $\Gamma(X_0)$.

Example

For $X_0 = Y_1 \cup Y_2$ there are two possible orientation of the dual graph $\Gamma(X_0)$ given by

$$\bullet^{Y_1} \to \bullet^{Y_2} : \text{ blow up along } Y_1 \times \{t_2 = 0\}$$

$$\bullet^{Y_1} \leftarrow \bullet^{Y_2} : \text{ blow up along } Y_2 \times \{t_1 = 0\}.$$

This gives two different small resolutions of $X'[1]$.



L. Halle
K. Hulek
Z. Zhang

0. Introduction

1. Expanded degenerations

2. The GIT construction

3. The construction for generalized Kummer varieties

Properties

Let $\pi : X \to C$ be a simple degeneration.

Proposition

The expanded degeneration $X[n] \to C[n]$ is a morphism of schemes if and only if $\pi : X \to C$ is a strict degeneration, i.e., $\Gamma(X_0)$ contains no loops.

Proposition

The expanded degeneration $X[n] \to C[n]$ is projective if and only if the oriented graph $\Gamma(X_0)$ contains no directed cycles.



L. Halle
K. Hulek
Z. Zhang

0. Introduction

1. Expanded degenerations

2. The GIT construction

3. The construction for generalized Kummer varieties

2. The GIT construction



L. Halle
K. Hulek
Z. Zhang

0. Introduction

1. Expanded degenerations

2. The GIT construction

3. The construction for generalized Kummer varieties

Here we recall our (GHH) GIT construction of degenerations of Hilbert schemes of points, starting with a simple and strict degeneration $\pi : X \to C$ of relative dimension 2. We need one further condition on the graph $\Gamma(X_0)$.

Definition

A graph is called a *bipartite graph* if it admits an orientation such that no vertex is simultaneously a source and a target.

Example

- Bipartite orientation:

$$\bullet \to \bullet \leftarrow \bullet \to \bullet \leftarrow \bullet.$$

- Non-bipartite orientation:

$$\bullet \to \bullet \to \bullet \to \bullet \to \bullet.$$

We shall assume that $\Gamma(X_0)$ is equipped with a bipartite orientation (and it will not matter which one we take). One can, after possibly taking a quadratic base change, and resolving singularities, always assume that one is in this situation.

We start with the *relative Hilbert scheme* of the expanded degeneration $X[n] \to C[n]$:

$$\mathbf{H}^n := \mathrm{Hilb}^n(X[n]/C[n]).$$

Let

$$\begin{array}{ccccc} & & Z^n \subset \mathbf{H}^n \times_{C[n]} X[n] & & \\ & \swarrow^{p} & & \searrow^{q} & \\ \mathbf{H}^n & & & & X[n] \end{array}$$

be the *universal scheme*.



L. Halle
K. Hulek
Z. Zhang

0. Introduction

1. Expanded degenerations

2. The GIT construction

3. The construction for generalized Kummer varieties

Assume that $\pi : X \to C$ is projective. A relatively ample line bundle for π induces a relatively ample line bundle $\mathcal{L}$ for $X[n] \to C[n]$. The line bundle

$$\mathcal{M}_\ell := \det \ p_* \left(q^* \mathcal{L}^{\otimes \ell} |_{Z^n} \right)$$

is relatively ample when $\ell \gg 0$.

Theorem

The line bundle $\mathcal{M}_\ell$ can be equipped with a $G[n]$-linearization.

Remark

- We are now in a position to apply (relative) GIT to the family $\mathbf{H}^n \to C[n]$.



L. Halle
K. Hulek
Z. Zhang

0. Introduction
1. Expanded degenerations
2. The GIT construction
3. The construction for generalized Kummer varieties

We denote the locus of *stable points*

$$\mathbf{H}^n_{\mathrm{GIT}} := \mathbf{H}^n(\mathcal{M}_\ell)^s.$$

Definition

We define the scheme

$$I^n_{X/C} = \mathbf{H}^n_{\mathrm{GIT}} /\!/ G[n] \to C[n]/G[n] = C.$$

Remark

- Outside the origin $0 \in C$, i.e., over $C^* = C \setminus \{0\}$, this coincides with the relative Hilbert scheme $\mathrm{Hilb}^n(X^*/C^*)$. Hence $I^n_{X/C}$ is a degeneration of Hilbert schemes.
- The resulting variety is independent of $\ell \gg 0$.
- Reversing the bipartite orientation gives an isomorphic scheme.



L. Halle
K. Hulek
Z. Zhang

0. Introduction

1. Expanded degenerations

2. The GIT construction

3. The construction for generalized Kummer varieties

Stability condition

One can perform a detailed analysis of the stability condition. The main results are

Theorem

The following holds:

- *A cycle $Z \in \mathbf{H}^n$ is represented by a subscheme $Z \subset X[n]_q$ for some base point $q \in C[n]$. The support of a stable cycle Z is contained in the smooth locus of this fibre.*
- *There is a numerical criterion which determines, in terms of the distribution of Z on the different components of $Z \subset X[n]_q$, whether Z is stable or not.*
- *Stability and semistability coincide.*



L. Halle
K. Hulek
Z. Zhang

0. Introduction

1. Expanded degenerations

2. The GIT construction

3. The construction for generalized Kummer varieties

The construction for generalized Kummer varieties



L. Halle
K. Hulek
Z. Zhang

0. Introduction
1. Expanded degenerations
2. The GIT construction
3. The construction for generalized Kummer varieties

We start with a type II Kulikov model $\mathcal{Y} \to C$ for abelian surfaces. Then the central fibre

$$\mathcal{Y}_0 = G_0 \cup \ldots \cup G_{M-1}$$

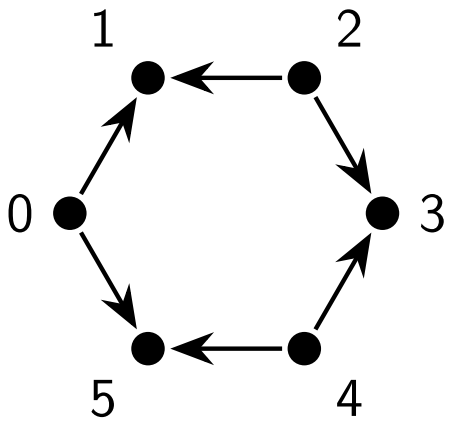
consist of a cycle of M surfaces with the following properties:

- All surfaces G_i are isomorphic to a surface G which is a $\mathbb{P}^1$-bundle over an elliptic curve D.
- The surface G is the compactification of a semi-abelian surface
$$1 \to \mathbb{C}^* \to G^0 \to D \to 1,$$
given by adding a 0-section and an ∞-section.
- The surfaces G_{i-1} and G_i meet transversally along an elliptic curve isomorphic to D:
$$G_{i-1} \cap G_i \cong D.$$

The dual graph is then a circle with M vertices. We want this to be a bipartite graph. This forces M to be even:

$$M = 2N.$$

The picture for $N = 3$ is:





L. Halle
K. Hulek
Z. Zhang

0. Introduction

1. Expanded degenerations

2. The GIT construction

3. The construction for generalized Kummer varieties



L. Halle
K. Hulek
Z. Zhang

0. Introduction

1. Expanded degenerations

2. The GIT construction

3. The construction for generalized Kummer varieties

We can now perform the construction outlined above, using expanded degenerations of the family of abelian surfaces, to obtain the locus of stable points:

$$\mathcal{H}^n = \mathrm{Hilb}^n(\mathcal{Y}[n]/C[n])^{st}.$$

Let

$$\mathcal{Y}^{sm}_{C[n]} \subset \mathcal{Y}[n]$$

be the relatively smooth locus. This has the structure of an abelian group scheme (cf. Néron model).



L. Halle
K. Hulek
Z. Zhang

0. Introduction

1. Expanded degenerations

2. The GIT construction

3. The construction for generalized Kummer varieties

Let $\mathcal{G}$ be an abelian group scheme. Then we can consider the summation morphism

$$\mu : \mathrm{Hilb}^n(\mathcal{G}) \to \mathcal{G}$$

and the *Kummer scheme*

$$\mathrm{Kum}^{n-1}(\mathcal{G}) := \mu^{-1}(0).$$

Applying this to $\mathcal{Y}^{sm}_{C[n]}$ we obtain

$$\mathrm{Kum}^{n-1}(\mathcal{Y}^{sm}_{C[n]}) \subset \mathrm{Hilb}^n(\mathcal{Y}[n]/C[n]).$$

The next step is to take the intersection with the stable locus

$$\mathcal{K}_0^{n-1} := \mathrm{Kum}^{n-1}(\mathcal{Y}^{sm}_{C[n]}) \cap \mathcal{H}^n.$$

Definition

We define

$$\mathcal{K}^{n-1} := \text{the closure of } \mathcal{K}_0^{n-1} \text{in } \mathcal{H}^n.$$

As this is preserved under the action of $G[n]$, and in the stable locus, we can take the GIT quotient to obtain

$$K^{n-1}_{\mathcal{Y}/C} := \mathcal{K}^{n-1} /\!/ G[n].$$

This is the degeneration we finally want.



L. Halle
K. Hulek
Z. Zhang

0. Introduction

1. Expanded degenerations

2. The GIT construction

3. The construction for generalized Kummer varieties



L. Halle
K. Hulek
Z. Zhang

0. Introduction

1. Expanded degenerations

2. The GIT construction

3. The construction for generalized Kummer varieties

Remark

Clearly, outside the special point $0 \in C$ the family $K^{n-1}_{\mathcal{Y}^0/C^0}$ is the family of generalized Kummer varieties associated to the abelian family $\mathcal{Y}^0 \to C^0$.

Proposition

The quotient $K^{n-1}_{\mathcal{Y}/C}$ is integral and flat over C. Moreover, the natural map $K^{n-1}_{\mathcal{Y}/C} \to I^n_{\mathcal{Y}/C}$ is a closed immersion into the Hilbert scheme.



L. Halle
K. Hulek
Z. Zhang

0. Introduction

1. Expanded degenerations

2. The GIT construction

3. The construction for generalized Kummer varieties

Theorem

- *The central fibre of $K^{n-1}_{\mathcal{Y}/C}$ admits a natural stratification. If $n \leq 2N+2$ then we can use this to define a dual complex $\Delta(K^{n-1}_{\mathcal{Y}/C})$.*
- *For $n = 2, 3$ this is a simplicial complex.*
- *For $n = 3$ the complex $\Delta(K^2_{\mathcal{Y}/C})$ is PL-homeomorphic to a 2-simplex.*

Remark

We have developed a tool called *line charts* which is essential for handling how special configurations of points can be smoothed. This allows us to control the combinatorics.

The case $n = 2$



L. Halle
K. Hulek
Z. Zhang

0. Introduction

1. Expanded degenerations

2. The GIT construction

3. The construction for generalized Kummer varieties

We start with a type II Kulikov model of abelian surfaces where the central fibre is a cycle of $2N$ elliptic ruled surfaces. Then $K^1_{\mathcal{Y}/C}$ is a type II Kulikov model of $K3$ surfaces. The dual complex $\Delta(K^1_{\mathcal{Y}/C})$ is an interval subdivided by $N+1$ vertices and containing N edges:



We now describe the three different building blocks of the degeneration.

- The vertices $1, \ldots, N-1$ are elliptic ruled surfaces. These are compactifications of a semi-abelian surfaces G^0. In our context they arise by taking two points on a non-inserted components and its negative (which we assume to be different). This leads to the summation map

$$\mu : G^0 \times G^0 \to G^0$$

and we take the kernel

$$\mu^{-1}(0) \cong G^0.$$

- The vertices $0, N$ are rational surfaces. These arise by taking 2 points on the same non-inserted component and to

$$\mu : \mathrm{Hilb}^2(G^0) \to G^0$$

giving

$$\mu^{-1}(0) \cong \mathrm{Kum}^1(G^0).$$

- Finally, the edges arise by taking 2 points on different inserted components. But now we also have to take the $\mathbb{G}[1]$-action into account, giving the base curve

$$G^0/\mathbb{G}[1] \cong D.$$

we shall now illustrate this in examples.



L. Halle
K. Hulek
Z. Zhang

0. Introduction

1. Expanded degenerations

2. The GIT construction

3. The construction for generalized Kummer varieties

The case $n = 3$

Here we have strata of dimensions 4,3,2.

Dimension 4:

$$G^0 \times G^0, \mathrm{Hilb}^2(G^0), \mathrm{Kum}^2(G^0), G^0 \times G^0/\mu_2$$

$$\mathrm{Hilb}^2(G^0)/\mu_2, G^0 \times G^0/\mu_3, \mathrm{Hilb}^2(G^0)/\mu_3$$

Dimension 3:

$$G^0 \times D, \mathrm{Hilb}^2(G^0)/\mathbb{G}[1]$$

Dimension 2:

$$D \times D$$



L. Halle
K. Hulek
Z. Zhang

0. Introduction

1. Expanded degenerations

2. The GIT construction

3. The construction for generalized Kummer varieties

Dimension 4, narrrow: The easiest case is when we take 3 points on 3 different original components (which have to add up to 0). This leads to

$$\mu : G^0 \times G^0 \times G^0 \to G^0, \mu^{-1}(0) \cong G^0 \times G^0$$

Dimension 4, wide: One can take 1 point on an non-inserted component and 2 points on different inserted components. Because we have inserted components we have a torus $\mathbb{G}[1]$ acting diagonally on the inserted components. This time one has to take

$$\mu : G^0 \times G^0 \times G^0 \to G^0, \quad O = \mu^{-1}(T)$$

where T is the fibre over the origin of the elliptic base curve D.

$$O/\mathbb{G}[1] \cong G^0 \times G^0/\mu_2.$$



L. Halle
K. Hulek
Z. Zhang

0. Introduction

1. Expanded degenerations

2. The GIT construction

3. The construction for generalized Kummer varieties

Dimension 3, wide: We can take 3 points on 2 different inserted components.

$$\mu : G^0 \times \mathrm{Hilb}^2(G^0) \to G^0, \quad O = \mu^{-1}(T)$$

We also have a torus $\mathbb{G}[2]$ acting on the inserted components, leading to

$$O/\mathbb{G}[2] \cong \mathrm{Hilb}^2(G^0)/\mathbb{G}[1].$$

Dimension 2, wide: We take 3 points on different inserted components. On these we have a $\mathbb{G}[3]$ acting. In this case

$$\mu : G^0 \times G^0 \times G^0 \to G^0, \quad O = \mu^{-1}(T)$$

and

$$O/\mathbb{G}[3] \cong G^0 \times G^0/\mathbb{G}[2] \cong D \times D.$$



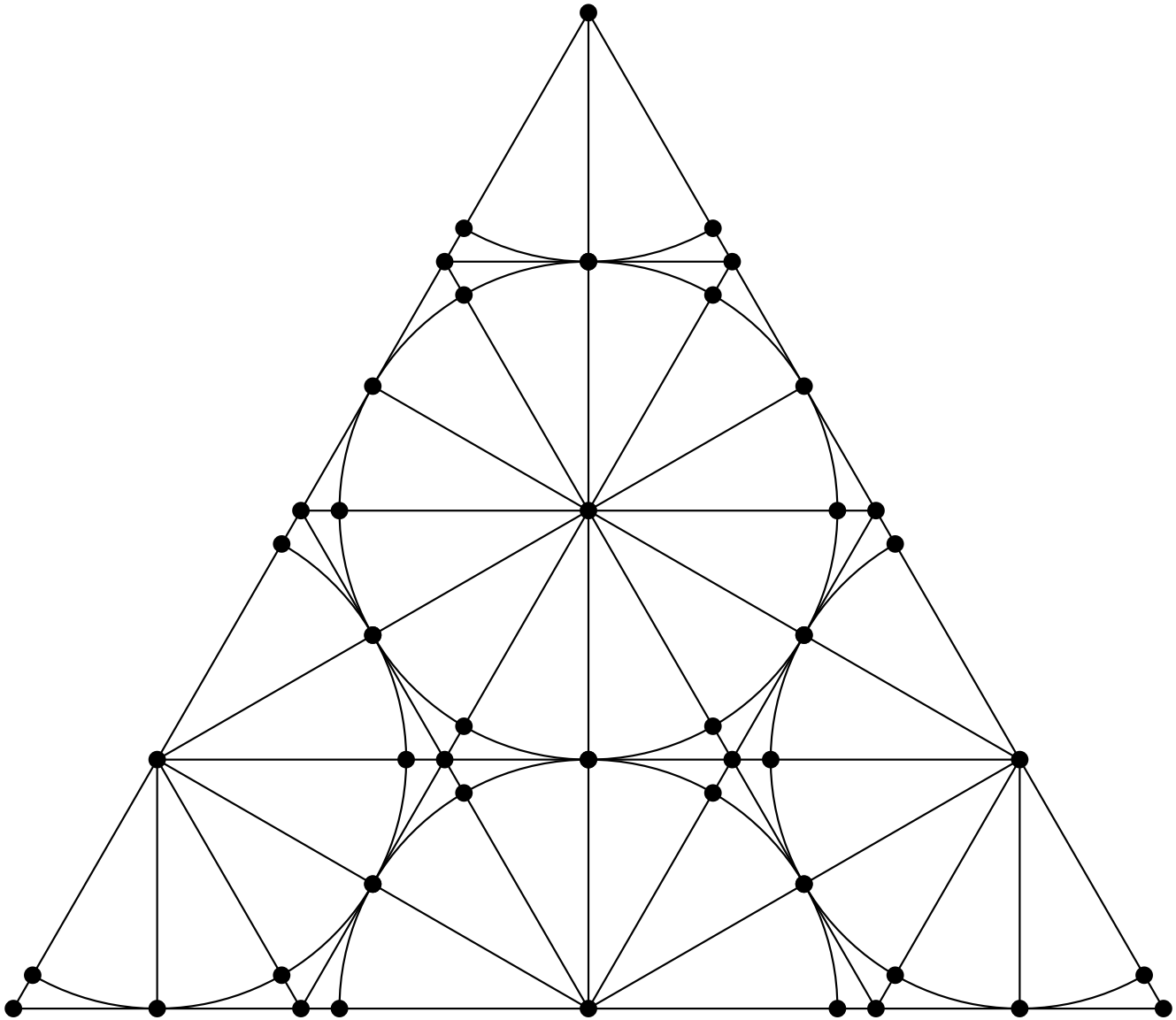
L. Halle
K. Hulek
Z. Zhang

0. Introduction

1. Expanded degenerations

2. The GIT construction

3. The construction for generalized Kummer varieties



The dual complex $\Delta(K^2_{\mathcal{Y}/C})$ for $N = 4$



L. Halle
K. Hulek
Z. Zhang

0. Introduction

1. Expanded degenerations

2. The GIT construction

3. The construction for generalized Kummer varieties



Thank you for your attention



L. Halle
K. Hulek
Z. Zhang

0. Introduction

1. Expanded degenerations

2. The GIT construction

3. The construction for generalized Kummer varieties