

# Borcherds products and Feynman integrals

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Joint work with Chuck Doran, Eric Pichon-Pharabod, and Pierre Vanhove

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2. Explain what this means for modular parametrizations.
3. Relate the geometry to extensions of mixed Hodge structure.
4. Explain the connection with Feynman integrals.

## Hessians of cubic surfaces

Let  $F(x_1, x_2, x_3, x_4)$  be a homogeneous cubic. Its Hessian is

$$H_F(x) = \det \left( \frac{\partial^2 F}{\partial x_i \partial x_j} \right)_{1 \leq i, j \leq 4}.$$

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A general cubic has a [Sylvester presentation](#)

$$\sum_{i=1}^5 a_i y_i^3 = 0, \quad \sum_{i=1}^5 y_i = 0.$$

# Sylvester Hessians

In suitable coordinates, the family has equation

$$s_3(x) = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4,$$

$$Q_\lambda(x) = (\lambda_1x_1 + \lambda_2x_2 + \lambda_3x_3 + \lambda_4x_4)s_3(x) - \lambda_5x_1x_2x_3x_4 = 0.$$

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Write  $X_\lambda$  for the minimal resolution. The parameters  $\lambda_1, \dots, \lambda_5$  can be viewed as projective parameters on the space of Hessian surfaces, so we may specialize to  $\lambda_5 = 1$ .

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Sylvester Hessians have exactly 10 nodes if and only if

$$\Delta_{\text{nod}} = \prod_{(\varepsilon_1, \dots, \varepsilon_4) \in \{\pm 1\}^4} \left( 1 + \sum_{i=1}^4 \varepsilon_i \sqrt{\lambda_i} \right) \neq 0.$$

# Physics motivation: the three-loop banana integral

On the chart  $x_4 = 1$ , set

$$P_\lambda = (\lambda_1 x + \lambda_2 y + \lambda_3 z + \lambda_4) \left( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + 1 \right) - 1.$$

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Up to physical normalization, the parametric Feynman integral in physical dimension  $D = 2$  for the 3-loop sunset/banana diagram has the form

$$I(\lambda) = \int_\sigma \frac{dx dy dz}{Q_\lambda(x, y, z, 1)} = \int_\sigma \frac{d \log x \wedge d \log y \wedge d \log z}{P_\lambda}, \quad \sigma = \{x, y, z > 0\}.$$

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Here  $\lambda_i = m_i^2/q^2$  are dimensionless parameters.

# The mixed Hodge structure of the integral

After compactifying to  $\mathbb{P}^3$  and resolving the hypersurface and boundary, the integral is a period of

$$\mathcal{H}_\lambda = H^3(P_\Gamma \setminus X_\lambda, B_\Gamma \setminus (B_\Gamma \cap X_\lambda); \mathbb{Q}).$$

Here  $P_\Gamma$  is the ambient toric compactification and  $B_\Gamma$  its boundary.

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The K3 contribution to  $\mathrm{Gr}_4^W \mathcal{H}_\lambda$  is

$$H^2(X_\lambda, \mathbb{Q})_{\mathrm{prim}}(-1) = H^2(X_\lambda, \mathbb{Q})(-1)/\mathbb{Q}(-2)^{16},$$

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## How can we describe the periods of these Hessian K3 surfaces?

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# Rational curves on the Hessian surface

Consider the five planes

$$H_i = \{x_i = 0\} \quad (1 \leq i \leq 4),$$

$$H_5 = \left\{ \sum_{i=1}^4 \lambda_i x_i = 0 \right\}.$$

Their pairwise intersections give ten lines

$\ell_{ij} = H_i \cap H_j$  on the quartic.

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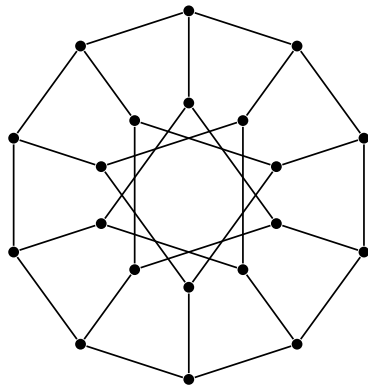
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intersection graph.



$$\#V = 20, \quad \#E = 30.$$

# The polarization lattice

The twenty curve classes span a lattice  $M$  of rank 16. For a general member,

$$\mathrm{NS}(X_\lambda) \cong M, \quad L = M^\perp \cong U \oplus U(2) \oplus A_2(-2).$$

$$\mathrm{sig}(L) = (2, 4), \quad \mathrm{rk} L = 6, \quad 20 - \mathrm{rk} M = 4.$$

The discriminant group is

$$A_L = L^\vee / L \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 3} \oplus \mathbb{Z}/6\mathbb{Z}, \quad |A_L| = |\det L| = 48.$$

# The period space

For  $L = M^\perp$  of signature  $(2, 4)$ , choose one component of

$$\mathcal{D}_L = \{[z] \in \mathbb{P}(L \otimes \mathbb{C}) \mid (z, z) = 0, (z, \bar{z}) > 0\}.$$

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The moduli space of pseudo-ample  $M$ -polarized K3 surfaces is

$$\boxed{\mathcal{M}_L = \Gamma_L \backslash \mathcal{D}_L^+}.$$

A holomorphic 2-form determines its period line  $[\omega] \in \mathcal{D}_L^+$ .

# The Hessian period map

Let  $\Lambda$  be the parameter space of Sylvester Hessians with K3 resolution.

## Theorem (Koike)

With the corresponding marking convention, the period map is an open embedding

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Thus the four affine parameters  $\lambda_i/\lambda_5$  give rational functions on  $\mathcal{M}_L$ .

Their zeros and poles lie outside  $\Pi(\Lambda)$ , since all  $\lambda_i$  are nonzero on  $\Lambda$ .

# The omitted Heegner divisors

A negative vector  $\alpha \in L$  defines a period hyperplane

$$H_\alpha = \{[z] \in \mathcal{D}_L^+ : (z, \alpha) = 0\}. \quad (\text{Noether–Lefschetz divisor})$$

On this locus,  $\alpha$  becomes a class of type  $(1, 1)$ .

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The ten non-Sylvester divisors  $H_{ij}$  come from vectors with

$$\alpha_{ij}^2 = -12, \quad \text{div}(\alpha_{ij}) = 6, \quad 1 \leq i < j \leq 5.$$

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Dardanelli–van Geemen identify these loci with non-Sylvester Hessians. We write

$$D_{\text{ns}} = \bigcup_{i < j} H_{ij} \subset \mathcal{M}_L.$$

# The image of the period map

## Theorem (Doran–Harder–Pichon–Pharabod–Vanhove)

The image of the Hessian period map is

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$$\Pi(\Lambda) = \mathcal{M}_L \setminus D_{\text{ns}}.$$

Thus the ten non-Sylvester divisors are the **entire** omitted locus.

Every other Noether–Lefschetz locus (eg. Eckardt and extra-node loci  $\lambda_i = \lambda_j$  and  $\Delta_{\text{nod}} = 0$  respectively) remain in the image.

## Proof: an elliptic model

The Desargues configuration gives an elliptic fibration with generic singular fibre configuration

$$2I_6 + 2I_2 + 8I_1.$$

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A non-symplectic involution exhibits  $X_\lambda$  as a double cover of a rational elliptic surface with a 2-torsion section branched along two smooth fibres:

$$y^2 = x(x^2 + a(s, t)x + t^3b(s, t)), \quad \deg a = 2, \quad \deg b = 1.$$

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### Proof strategy

Show that all pseudo-ample  $M$ -polarized K3 surfaces admit a realization using this model, then identify explicitly which ones admit a Sylvester presentation.

# The non-Sylvester degeneration

On the rational elliptic quotient, the relevant collision is

$$\underbrace{I_6 + I_1 + I_1}_{\text{Euler number 8}} \longrightarrow \underbrace{I_2^*}_{\text{Euler number 8}}$$

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The geometric condition concerns the realization of the distinguished roots by actual irreducible curves, with the prescribed  $M$ -polarization.

## Quick review of Borcherds products

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**Input: a Weil-valued weakly holomorphic  $\mathrm{SL}_2(\mathbb{Z})$  modular form**

$$f(\tau) = \sum_{\gamma, m} c(m, \gamma) q^m \mathbf{e}_\gamma \in M_{-1}^!(\rho_L), \quad q = e^{2\pi i \tau}.$$

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**Output: a meromorphic orthogonal modular form  $\Psi(f)$**

$$\mathrm{div} \Psi(f) = \frac{1}{2} \sum_{\gamma, m < 0} c(m, \gamma) Z(\gamma, m), \quad \mathrm{wt} \Psi(f) = \frac{c(0, 0)}{2}.$$

The  $Z(\gamma, m)$  are Heegner divisors. Near a cusp,  $\Psi(f)$  is an **infinite product**.

# The missing divisors give modular parameters

The image theorem identifies the entire omitted locus

$$U := \Pi(\Lambda) = \mathcal{M}_L \setminus D_{\text{ns}}, \quad D_{\text{ns}} = \bigcup_{i < j} H_{ij}.$$

The parameter ratios are units on  $U$ , so all their zeros and poles lie on these ten Heegner divisors.

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Bruinier's converse applies to  $L = U \oplus U(2) \oplus A_2(-2)$ :

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**Divisors determine parameters up to scalar; Borcherds products give explicit formulas**

# The periods of the Hessian K3 surface

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The residue form is nonvanishing on  $\Lambda$ , so

$$\text{supp}(\text{div } \Phi_\omega) \subseteq D_{\text{ns}} \implies \Phi_\omega = C \Psi(f_\omega), \quad c_\omega(0,0) = 2.$$

## Theorem (Doran–Harder–Pichon–Pharabod–Vanhove)

There is an *explicit* Borchers lift that realizes this modular form obtained by computing the pole orders of  $\omega(z)$  on components of  $D_{\text{ns}}$ .

# Boundary data and explicit constructions

## For parameters and pure periods

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The next step is to understand existence and uniqueness from residues.

# The Feynman integral as an iterated extension

For the three-loop sunset/banana integral in physical dimension  $D = 2$ , let  $V_{\lambda, \mathbb{Q}} = H^2(X_{\lambda}, \mathbb{Q})_{\text{prim}}$  be the primitive Hessian cohomology, orthogonal to  $M$ .

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Dualize the relevant primitive part of the relative cohomology of the integral and twist by  $\mathbb{Q}(-3)$ . The resulting  $\mathcal{H}_{F, \lambda}$  is built in two steps:

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K3 periods describe the pure part. The Feynman integral is a mixed period depending on the **extension classes**.

# Desargues curves and the Enriques involution

The twenty Desargues curves form  $C_{\text{Des}}$ , with graph  $\Gamma_{\text{Des}}$  and  $b_1(\Gamma_{\text{Des}}) = 30 - 20 + 1 = 11$ . Dualizing relative cohomology and twisting gives

$$0 \longrightarrow V_X \longrightarrow E_{\text{Des}} \longrightarrow \mathbb{Q}(-2)^{11} \longrightarrow 0.$$

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The twenty Desargues curves form  $C_{\text{Des}}$ , with graph  $\Gamma_{\text{Des}}$  and  $b_1(\Gamma_{\text{Des}}) = 30 - 20 + 1 = 11$ . Dualizing relative cohomology and twisting gives

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The Enriques involution exchanges the two sets of curves and acts by  $-1$  on  $V_X$ . On the graph,

$$\dim H^1(\Gamma_{\text{Des}}, \mathbb{Q})^+ = 6, \quad \dim H^1(\Gamma_{\text{Des}}, \mathbb{Q})^- = 5.$$

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## Theorem (in progress)

The anti-invariant Desargues classes span

$$\text{Ext}_{\text{top}}^1(\mathbb{Q}(-2), \mathbb{V}) \cong \mathbb{Q}^5.$$

# Residues, monodromy, and the first extension

## Theorem (in progress)

Let  $U = \mathcal{M}_L \setminus D_{\text{ns}} \cong \Lambda$ . An admissible rational extension of  $\mathbb{Q}(-2)$  by  $\mathbb{V}$  on  $U$  is uniquely determined, up to extension equivalence, by its ten boundary monodromy coefficients  $b_{ij}$ .

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Localization detects the underlying extension class:

$$0 \longrightarrow H^1(U, \mathbb{V}) \xrightarrow{\text{res}} \bigoplus_{i < j} \mathbb{Q} \alpha_{ij}(-1) \xrightarrow{\text{Gys}} H^2(\mathcal{M}_L, \mathbb{V}).$$

The residues must satisfy the Gysin relation.

## Residues and the second extension?

Fix  $0 \rightarrow \mathbb{V} \rightarrow \mathbb{E} \rightarrow \mathbb{Q}(-2)^3 \rightarrow 0$ . An upper extension  $0 \rightarrow \mathbb{E} \rightarrow \mathbb{H} \rightarrow \mathbb{Q}(-3) \rightarrow 0$  has a Kummer quotient

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### Question

For fixed  $\mathbb{E}$ , do the residues  $r_{ij}$  determine the full upper extension?

# Summary and outlook

To apply the extension theory to the three-loop sunset integral, two calculations remain:

- Compute the boundary monodromy of the weight 2 extension classes.
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## What the missing divisors tell us

For the pure periods and parameters, divisor data lead to explicit Borcherds products. For the first extensions, residues determine the classes. For the second extension, something similar might be true, but it remains to be proved.

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## What the missing divisors tell us

For the pure periods and parameters, divisor data lead to explicit Borcherds products. For the first extensions, residues determine the classes. For the second extension, something similar might be true, but it remains to be proved.

The next question is how to turn the residue description into explicit formulas for the mixed periods.

# A Borchers construction for higher extensions?

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Can we extend the Borchers construction to realize extension classes and successive higher extensions explicitly from their residue data?

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## Question

Can we extend the Borchers construction to realize extension classes and successive higher extensions explicitly from their residue data?

We seek, for instance,  $\mathbb{V}$ -valued logarithmic forms with poles on  $D_{\text{ns}}$  representing classes in  $\text{Ext}_{\text{AVHS}}^1(\mathbb{Q}(-2), \mathbb{V})$ , then flat connections whose iterated integrals describe the mixed periods.

# Backup: the Hodge argument for the upper extension

## In progress: conditional on the stated geometric hypotheses

For an admissible variation  $\mathbb{F}$ , one has

$$\begin{aligned} 0 \rightarrow \mathrm{Ext}_{\mathrm{MHS}}^1(\mathbb{Q}, H^0(U, \mathbb{F})) &\rightarrow \mathrm{Ext}_{\mathrm{AVHS}, U}^1(\mathbb{Q}, \mathbb{F}) \\ &\rightarrow \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}, H^1(U, \mathbb{F})) \rightarrow 0. \end{aligned}$$

Apply this to  $\mathbb{F} = \mathbb{V}(3)$ :

$$H^0(U, \mathbb{V}) = 0, \quad H^1(U, \mathbb{V}(3)) = \mathbb{Q}(1)^5 \quad \implies \quad \mathrm{Ext}_{\mathrm{AVHS}, U}^1(\mathbb{Q}(-3), \mathbb{V}) = 0.$$

Thus the map induced by  $\mathbb{E}_5 \rightarrow T_5(-2)$  is injective:

$$\mathrm{Ext}_{\mathrm{AVHS}, U}^1(\mathbb{Q}(-3), \mathbb{E}_5) \hookrightarrow \mathrm{Ext}_{\mathrm{AVHS}, U}^1(\mathbb{Q}(-3), T_5(-2)).$$

A specified full Kummer VMHS class has at most one admissible lift.

## Backup: a candidate toric symbol

### In progress

Normalize equations for the five planes by  $\sum_{i=1}^5 y_i = 0$ . They are then determined up to a common scalar. Distinguish 5 and set

$$\kappa_5 = \left\{ \frac{y_1}{y_4}, \frac{y_2}{y_4}, \frac{y_3}{y_4} \right\} \in K_3^M(\mathbb{C}(X)) \otimes \mathbb{Q}.$$

An orientation of the other four indices fixes the sign. The intended lift and regulator are

$$\tilde{\kappa}_5 \in \mathrm{CH}^3(X \setminus C_5, 3)_{\mathbb{Q}}, \quad r(\tilde{\kappa}_5) \in \mathrm{Ext}_{\mathrm{MHS}}^1(\mathbb{Q}(-3), E_5).$$

The open surface supplies the first extension; the regulator supplies the upper one.

The lift, its projected ambiguity, and the Feynman comparison remain to be checked.

## Backup: a logarithmic connection

In local flat frames for  $\mathbb{V}$ , write the extension connection as

$$\nabla = d + \begin{pmatrix} 0 & \Phi \\ 0 & 0 \end{pmatrix}, \quad d\Phi = 0.$$

Near a divisor  $D_i = \{u = 0\}$ , its singular term is

$$\Phi = b_i \alpha_i \frac{du}{u} + \text{holomorphic terms}.$$

Consequently,

$$\text{Res}_{D_i} \Phi = b_i \alpha_i, \quad \oint_{|u|=\varepsilon} \Phi = 2\pi i b_i \alpha_i.$$

Integrating  $\Phi$  supplies the additional mixed periods and their monodromy. In the product case, this leads to multivariable Eichler-type integrals.

## Backup: a modular-curve test case

On a product  $Y_G \times Y_G$ , take  $\mathbb{V} = \mathbb{L} \boxtimes \mathbb{L}$ . Heegner divisors become Hecke correspondences.

For the commuting elliptic period operators  $L_x, L_y$ , a regulator period satisfies

$$L_x R = g, \quad L_y R = h, \quad \boxed{L_y g = L_x h.}$$

Changing  $R$  by  $u$  changes  $(g, h)$  by  $(L_x u, L_y u)$ . The extension data are thus described by

$$\mathcal{A} \xrightarrow{(L_x, L_y)} \mathcal{A}^2 \xrightarrow{(g, h) \mapsto L_y g - L_x h} \mathcal{A},$$

where  $\mathcal{A}$  allows poles along the removed divisor.

This gives a concrete model for constructing logarithmic representatives.

## Backup: Principal parts for the non-Sylvester divisors

For  $H_{ij}$ , let  $\gamma_{ij} = \alpha_{ij}/6 + L \in A_L$ . Then

$$\alpha_{ij}^2 = -12, \quad q_L(\alpha_{ij}/6) = -\frac{1}{6}, \quad \text{ord}(\gamma_{ij}) = 6.$$

Use the symmetrized terms

$$V_{ij} = q^{-1/6}(\mathbf{e}_{\gamma_{ij}} + \mathbf{e}_{-\gamma_{ij}}).$$

If a modular input  $f$  has no other negative terms, then

$$\text{pp}(f) = \sum_{i < j} b_{ij} V_{ij} \quad \implies \quad \text{div } \Psi(f) = \sum_{i < j} b_{ij} H_{ij}.$$

The coefficients  $b_{ij}$  must satisfy modularity constraints. For a weight-zero lift one additionally needs  $c(0,0) = 0$ .

## Backup: which principal parts are possible?

Start with a finite proposed principal part

$$P = \sum_{\gamma} \sum_{m < 0} a(m, \gamma) q^m \mathbf{e}_{\gamma}$$

having the required exponent congruences and sign symmetry. For a dual cusp form

$$g = \sum_{\gamma} \sum_{n > 0} b_g(n, \gamma) q^n \mathbf{e}_{\gamma}^* \in S_3(\rho_L^{\vee}),$$

the obstruction is the constant term of the pairing:

$$\sum_{\gamma} \sum_{m < 0} a(m, \gamma) b_g(-m, \gamma) = 0 \quad \text{for every } g.$$

These conditions are necessary and sufficient for a weight  $-1$  input with principal part  $P$ . The input is unique; its constant term is then determined.

## Backup: the converse and normalization

### Theorem (Bruinier)

For  $L = U \oplus U(N) \oplus K$ , with  $K$  negative definite and nonzero, a meromorphic modular form with Heegner divisor is a Borcherds product, up to a nonzero constant.

Our lattice is  $U \oplus U(2) \oplus A_2(-2)$ , so

$$\lambda_i/\lambda_5 = c_i \Psi(g_i), \quad \text{wt } \Psi(g_i) = 0.$$

With divisor, weight, and character matched, the quotient is constant:

- It has no divisorial zeros or poles.
- The Baily–Borel boundary has codimension at least two.
- It extends to a regular function on a normal projective variety.

A cusp normalization fixes the constant  $c_i$ .

## Backup: orders along a non-Sylvester degeneration

To approach  $H_{ij}$ , use a one-parameter family

$$\lambda_i = 1 + t, \quad \lambda_j = 1, \quad \lambda_k = t^2 \mu_k \quad (k \notin \{i, j\}).$$

After a change of variables, the limit at  $t = 0$  is a non-Sylvester K3.

Along this path, ratios have directly visible orders:

$$\frac{\lambda_k}{\lambda_j} = t^2 \mu_k, \quad \frac{\lambda_i}{\lambda_j} = 1 + t.$$

These local orders supply the data for the Borchers principal parts.

## Backup: the quartic polarization

For any  $i \in \{1, \dots, 5\}$ , define the standard hyperplane class

$$h = \sum_{j \neq i} [\ell_{ij}] + \sum_{\substack{j < k \\ j, k \neq i}} [r_{ijk}].$$

The relations in  $M$  make this independent of  $i$ . The incidence pairing gives

$$h^2 = 4, \quad h \cdot \ell_{ab} = 1, \quad h \cdot r_{abc} = 0.$$

In the corresponding Sylvester model,  $|h|$  gives the quartic morphism:

- The  $r_{ijk}$  contract to the ten distinguished nodes.
- The  $\ell_{ij}$  map to the ten edges of the pentahedron.
- The five expressions for  $h$  are its five plane sections.

## Backup: discriminant of the elliptic model

For the rational elliptic surface

$$y^2 = x(x^2 + ax + B), \quad B = t^3b(s, t),$$

the Weierstrass discriminant is

$$\Delta_{\text{ell}} = 16B^2(a^2 - 4B) = 16t^6b(s, t)^2(a(s, t)^2 - 4t^3b(s, t)).$$

For general coefficients, the vanishing orders are

| locus                     | $t = 0$ | $b = 0$ | $a^2 - 4t^3b = 0$ |
|---------------------------|---------|---------|-------------------|
| ord $\Delta_{\text{ell}}$ | 6       | 2       | 1 at four points  |
| fibre                     | $I_6$   | $I_2$   | $4I_1$            |

A quadratic base change unramified over these points gives  $2I_6 + 2I_2 + 8I_1$  on the K3 surface.

## Backup: earlier work on Hessian K3 surfaces

Dolgachev–Keum

The automorphism group of a generic Hessian K3 surface.

Dardanelli–van Geemen

Picard lattices and special loci in the Hessian family.

Koike

The period domain as a Hermitian symmetric domain of dimension four.

Kondō

Automorphic forms detecting the loci  $\lambda_i = \lambda_j$ .