

Stability conditions and mirror symmetry for generalized complex tori

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joint with Fabian Haiden

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Towards homological mirror symmetry

What can we test without constructing an equivalence?

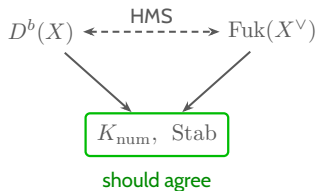
Towards HMS · Stability conditions · Abelian threefolds · Generalized complex tori · Mirror symmetry

HMS is hard; its shadows are computable

$$D^b(X) \simeq \mathrm{Fuk}(X^\vee)$$

Constructing it is **hard**. Instead, compare shared invariants:

- K_{num} , with its Euler pairing
- $\mathrm{Stab} \supset$ Kähler moduli of $X \cong$ complex moduli of $X^\vee$ (expected)



Example: Kontsevich predicted $\mathrm{Stab}(D^b(E^n))$ from the Lagrangians of the mirror torus.

For abelian varieties, the doubled torus is the invariant

$A = (V/\Gamma, I)$, $\widehat{A} = \overline{V}^*/\Gamma^\vee \rightsquigarrow$ the doubled torus

$$T_A = ((V \oplus \overline{V}^*)/(\Gamma \oplus \Gamma^\vee), g, J_A), \quad g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J_A = \begin{pmatrix} I & 0 \\ 0 & -I^t \end{pmatrix}.$$

Theorem (Orlov) $D^b(A) \simeq D^b(A') \iff T_A \cong T_{A'}$ (an isomorphism $A \times \widehat{A} \cong A' \times \widehat{A'}$ preserving g).

Expected (GLO): same doubled torus as a symplectic torus $(Y, B + i\omega) \Rightarrow D^b(A) \simeq \text{Fuk}(Y, B + i\omega)$.

Big takeaway. For abelian varieties, both the stability manifold and mirror symmetry can be read off the doubled torus T_A .

- $K_{\text{num}}(A)_{\mathbb{Q}} = \text{Hodge classes of } T_A$ ($\dim A \leq 3$)
- $\text{Stab}(A) = \text{universal cover of a domain determined by } T_A$ (threefolds: this talk)

Geometry of triangulated categories

What geometric space does a category carry?

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A stability condition: a heart and a central charge

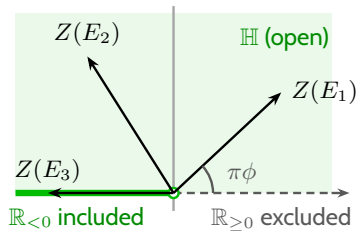
X smooth projective. Two knobs: **the slope and the heart**.

Definition (Bridgeland) $\sigma = (Z, \mathcal{A})$: $\mathcal{A} \subset D^b(X)$ the heart of a bounded t-structure, $Z : K_{\text{num}}(X) \rightarrow \mathbb{C}$ linear, with

- $Z(E) \in \mathbb{H} \cup \mathbb{R}_{<0}$ for $0 \neq E \in \mathcal{A}$
- Harder–Narasimhan filtrations for $\mu_Z = -\text{Re } Z / \text{Im } Z$
- support property: a quadratic form Q with $Q(E) \geq 0$ on semistable E , $Q|_{\ker Z} < 0$

Example: X a curve, $\mathcal{A} = \text{Coh } X$, $Z = -\deg + i \text{ rk}$. In general, at large volume:

$$Z_{B+i\omega}(E) = -\int_X e^{-(B+i\omega)} \text{ch}(E).$$



$Z(E)$: central charge; phase $\phi(E) \in (0, 1]$

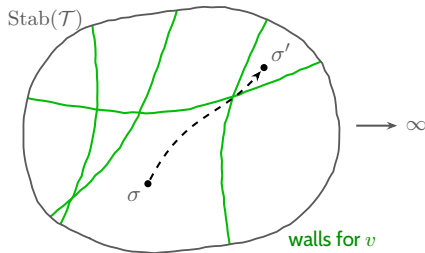
Why study Stab ?

Problems

- existence: is $\text{Stab}(\mathcal{T}) \neq \emptyset$?
- geometry of $\text{Stab}(\mathcal{T})$: is it contractible?
- moduli spaces $M_\sigma(v)$: σ selects families of objects

Applications

- BPS/DT invariants and wall-crossing (Bridgeland; Kontsevich–Soibelman)
- wall-crossing of moduli: $M_\sigma(v) \dashrightarrow M_{\sigma'}(v)$ (Bayer–Macrì)
- asymptotics $\rightsquigarrow$ semiorthogonal decompositions (Halpern-Leistner)
- contractibility $\rightsquigarrow \text{Aut}(\mathcal{T})$ (Bayer–Bridgeland)



Central charges are local coordinates

Theorem (Bridgeland '07) $\mathrm{Stab}(X)$ is a complex manifold of dimension $\mathrm{rk} K_{\mathrm{num}}(X)$, and

$$Z : \mathrm{Stab}(X) \longrightarrow \mathrm{Hom}(K_{\mathrm{num}}(X), \mathbb{C})$$

is a local homeomorphism. $\widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ (acting on $\mathbb{C}$) and $\mathrm{Aut}(D^b(X))$ act, commuting.

Question: which central charges Z occur? In examples, Stab covers an explicit domain:

- **curves:** $\mathbb{P}^1$: $\mathrm{Stab} \cong \mathbb{C}^2$ (Okada); $g \geq 1$: $\mathrm{Stab} \cong \widetilde{\mathrm{GL}}_2^+(\mathbb{R})$ (Bridgeland, Macrì)
- **surfaces:** K3: a component covers a period domain (Bridgeland); abelian surfaces: Stab is connected (Bridgeland; Huybrechts–Macrì–Stellari); $\mathbb{P}^2$ (Li)
- **threefolds:** existence via Bogomolov–Gieseker type inequalities (Bayer–Macrì–Toda; BMS; Oberdieck–Piyaratne–Toda)

The stability manifold of abelian threefolds

Can we describe the space explicitly?

Towards HMS · Stability conditions · **Abelian threefolds** · Generalized complex tori · Mirror symmetry

Before: which central charges? The twisted cubic

X abelian threefold, H ample. $\text{Stab}_H(X)$: the σ whose charge factors through v_H :

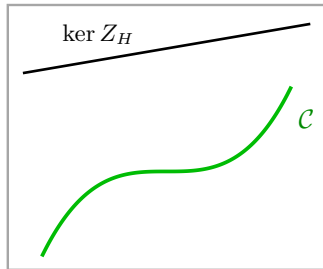
$$\begin{array}{ccc} K_{\text{num}}(X) & \xrightarrow{v_H} & \Lambda_H \cong \mathbb{Q}^4 \\ & \searrow Z & \downarrow Z_H \\ & & \mathbb{C} \end{array}$$

$$v_H(E) = (H^3 \text{ch}_0, H^2 \text{ch}_1, H \text{ch}_2, \text{ch}_3)$$

Key fact: line bundles are stable for every σ (Fu–Li–Zhao), so $Z_H \neq 0$ on the **twisted cubic cone**

$$\mathcal{C} = \{v_H(e^{tH})\} = \left\{H^3\left(1, t, \frac{t^2}{2}, \frac{t^3}{6}\right)\right\} \subset \Lambda_H.$$

Question: which Z_H with $\ker Z_H \cap \mathcal{C} = 0$ are central charges?



schematically, in $\mathbb{P}(\Lambda_H)$: the line $\ker Z_H$ must miss the twisted cubic

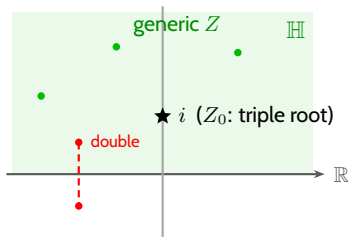
Before: the answer of Bayer–Macrì–Stellari

Base point: the large-volume charge at $B + i\omega = iH$,

$$Z_0(E) = - \int_X e^{-iH} \text{ch}(E).$$

Theorem (Bayer–Macrì–Stellari '16) The universal cover of the Z_0 -component of $\{Z_H : \ker Z_H \cap \mathcal{C} = 0\}$ is a component of $\text{Stab}_H(X)$.

In roots: $p_Z(t) := Z(e^{tH})$ is a cubic. $\ker Z_H \cap \mathcal{C} = 0$ iff p_Z has degree 3 and no real root; the Z_0 -component: **all three roots in $\mathbb{H}$** .



one root below $\mathbb{R}$: excluded

$$p_{Z_0}(t) = -\frac{H^3}{6}(t - i)^3: \text{triple root at } i$$

Theorem A: the obvious obstruction is the only one

Now the full lattice $N(A) = K_{\text{num}}(A) \otimes \mathbb{R} \cong \mathbb{R}^{2+2\rho}$, ρ the Picard rank. $\mathcal{C}_A$: the closure of the rays $\mathbb{R}_{>0} e^D$, $D \in \text{NS}(A)_{\mathbb{R}}$ (semi-homogeneous classes; for $\rho = 1$ this is $\mathcal{C}$).

Key fact: semi-homogeneous bundles (rational D) are σ -stable for every σ (Fu–Li–Zhao), so $Z(e^D) \neq 0$: the obvious obstruction.

Domain. $\mathcal{U}^+(A)$: the Z_0 -component of $\{Z : N(A) \rightarrow \mathbb{C} : Z(v) \neq 0 \ \forall v \in \mathcal{C}_A \setminus 0\}$.

Theorem A (Haiden–S.) For every abelian threefold A , $Z : \text{Stab}(A) \longrightarrow \mathcal{U}^+(A)$ is a universal cover, and $\text{Stab}(A)$ is contractible.

Two problems with this definition:

- (1) it only sees classes spanned by the e^D , which beyond threefolds need not span the Hodge classes (e.g. abelian fourfolds of Weil type);
- (2) it uses divisors on A , so it is not manifestly invariant under derived equivalence.

Generalized complex tori

Why is this the right description, independent of an ordinary presentation?

Towards HMS · Stability conditions · Abelian threefolds · **Generalized complex tori** · Mirror symmetry

Generalized complex tori

Definition (Gualtieri) A generalized complex torus is $T = (W/\Lambda, g, J)$, with $\Lambda \cong \mathbb{Z}^{2n} \oplus (\mathbb{Z}^{2n})^\vee$ and g as for T_A , $W = \Lambda_{\mathbb{R}}$, and J any complex structure on W preserving g .

Key fact: $h = g + i g(J\cdot, \cdot)$ is Hermitian of **signature** (n, n) on $W_J := (W, J) \cong \mathbb{C}^{2n}$.

- Abelian variety: the doubled torus T_A , $J_A = \begin{pmatrix} I & 0 \\ 0 & -I^t \end{pmatrix}$
- Symplectic torus $(Y, B + i\omega)$: a complex structure $J_{B,\omega}$ on $Y \times \hat{Y}$

Moduli: $\mathcal{B} = \mathrm{U}(n, n) \backslash \mathrm{O}(2n, 2n; \mathbb{R}) / \mathrm{O}(2n, 2n; \mathbb{Z})$, with $\mathrm{O}(2n, 2n; \mathbb{Z})$ the **T-duality group**. For $n \geq 2$, most T are **not** doubled tori.

Numerical classes of a generalized complex torus

Definition. $\text{Hdg}(T)$: the smallest $\mathbb{Q}$ -algebraic group containing the circle $\cos \theta + \sin \theta J$; it acts on $\bigwedge^\bullet W_J$ (the spin representation of (Λ, g)).

$$N(T) := \left(\bigwedge^\bullet W_J \right)^{\text{Hdg}(T)} \subset \bigwedge^n W_J \quad : \text{ the Hodge classes of } T$$

Key fact: for T_A : $\bigwedge^\bullet W_J \cong H^*(A, \mathbb{C})$ (Mukai; D-brane charges), Hdg is the Hodge group of A , and $N(T_A) = \bigoplus_p H^{p,p}(A, \mathbb{Q})$ ($= K_{\text{num}}(A)_{\mathbb{Q}}$ if $\dim A \leq 3$).

Kontsevich: T should be a “noncommutative abelian variety”, with predicted invariants:

Y symplectic torus	T	A abelian variety
$K(\text{Fuk}(Y))$	$N(T)$	$K_{\text{num}}(D^b(A))$
linear Lagrangians	isotropic n -planes	semi-homogeneous sheaves

$\mathcal{U}(T)$: forms that never vanish on isotropic planes

Definition. $\mathcal{U}(n) = \{ \Omega \in (\wedge^n W_J)^\vee : \Omega|_L \neq 0 \text{ for every maximal isotropic } L \}$.

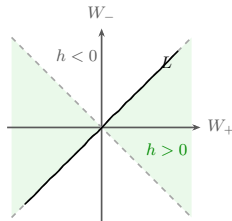
$\mathcal{U}^+(n)$: the component containing the **geometric forms**

$\mathcal{U}_{\text{geom}}$: $h(\nu, \cdot)$, ν a volume element of a positive n -plane.

$\mathcal{U}^+(T)$: the restriction of $\mathcal{U}^+(n)$ to $N(T) \subset \wedge^n W_J$.

Key fact: for T_A : $e^D \mapsto$ isotropic planes, and geometric forms $\mapsto$ the charges $-c \int e^{-(B+i\omega)} \text{ch}(\cdot)$; so $Z_0 \in \mathcal{U}^+(T_A)$.

Example ($n = 1$) Isotropic lines in $\mathbb{C}^2$ are the real lines: a form is a real 2×2 matrix, $\mathcal{U}^+(T_E) = \text{GL}_2^+(\mathbb{R})$, covered by $\text{Stab}(E)$. $n = 2$: positive 2-planes (abelian surfaces).



maximal isotropic L : the graph of a unitary map $W_+ \rightarrow W_-$

Theorem B: Stab only sees the doubled torus

Theorem B (Haiden–S.) For every abelian threefold A , $\mathcal{U}^+(A) = \mathcal{U}^+(T_A)$. Hence

$$\text{Stab}(A) \cong \widetilde{\mathcal{U}^+(T_A)}.$$

Key fact: $\mathcal{U}^+(T)$ depends only on $T = (W/\Lambda, g, J)$.

Conjecture (Kontsevich; Haiden–S.) For every abelian variety A , $\text{Stab}(A)$ is the universal cover of $\mathcal{U}^+(T_A)$.

Remark. For $A = E^n$, E without CM, $n \leq 3$, this is **Kontsevich's conjecture**: primitive n -forms on the mirror torus, nonvanishing on its Lagrangian Grassmannian $\text{LGr}(n, 2n)$.

Example: powers of elliptic curves

$A = E^n$. $\mathcal{U}_{\text{geom}}(T_A)$: the charges $-c \int e^{-\beta} \text{ch}(\cdot)$, $c \in \mathbb{C}^*$, $\beta \in \text{NS}(A)_{\mathbb{R}} + i \text{Amp}(A)$.

	E without CM	E with CM
$\mathbb{P}\mathcal{U}_{\text{geom}}(T_A)$	Siegel $\mathbb{H}_n = \text{Sp}(2n, \mathbb{R})/\text{U}(n)$	$\text{U}(n, n)/(\text{U}(n) \times \text{U}(n))$
$\dim_{\mathbb{C}} \mathbb{P}\mathcal{U}_{\text{geom}}$	$n(n+1)/2$	n^2
$\dim_{\mathbb{C}} \text{Stab}(A)^*$	$\binom{2n}{n} - \binom{2n}{n-2}$	$\binom{2n}{n}$
$n = 3$	6 vs 14	9 vs 20

- No CM: $\mathbb{H}_n$ is also the space of complex structures of the mirror, $A_{\Xi} = \mathbb{C}^n/(\mathbb{Z}^n + \Xi\mathbb{Z}^n)$ (GLO)
- CM: $\text{U}(n, n)/(\text{U}(n) \times \text{U}(n))$ is the **Weil-type period domain** — it returns in Mirror symmetry

* proved for $n \leq 3$ ($n = 3$: Theorem A); conjectural for $n \geq 4$. Weil type: abelian $2n$ -folds on which $\mathbb{Q}(\sqrt{-d})$ acts, both embeddings with multiplicity n .

Mirror symmetry

What happens when we exchange the two geometric structures?

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A Kähler class is a second complex structure

So far: $\text{Stab}(A)$ is the universal cover of $\mathcal{U}^+(T_A)$, which depends only on T_A . Now: the second complex structure.

$B + i\omega$ ($B, \omega \in \text{NS}(A)_{\mathbb{R}}$, ω ample) gives a complex structure on $A \times \hat{A}$ commuting with J_A :

$$K_{B+i\omega} = \begin{pmatrix} \omega^{-1}B & -\omega^{-1} \\ \omega + B\omega^{-1}B & -B\omega^{-1} \end{pmatrix}, \quad [J_A, K_{B+i\omega}] = 0.$$

In general: a **polarization** of $T_J = (W/\Lambda, g, J)$ is a complex structure K on (W, g) with

- K commutes with $\text{Hdg}(T_J)$ and $g(J\cdot, K\cdot) > 0$
- plus a numericality condition (automatic for doubled tori)

Example: for A , the polarizations are exactly the $K_{B+i\omega}$.

Mirror symmetry for abelian varieties, and where it fails

Definition (Golyushev–Lunts–Orlov) (A, ω_A) and $(A', \omega_{A'})$ are **mirror** if an isomorphism $A \times \widehat{A} \cong A' \times \widehat{A}'$ preserving g exchanges $J_A \leftrightarrow K_{\omega_{A'}}$ and $K_{\omega_A} \leftrightarrow J_{A'}$.

Example: E without CM: $(E_\tau^n, \Xi) \leftrightarrow (A_\Xi, \tau\Theta)$, $A_\Xi = \mathbb{C}^n / (\mathbb{Z}^n + \Xi\mathbb{Z}^n)$: the Kähler moduli $\mathbb{H}_n$ of one are the complex moduli of the other.

Caution: $A = E_{\text{CM}}^2$, general ω_A : **no** abelian surface is a mirror (GLO) — yet the pair (J_A, K_{ω_A}) exists. (K_{ω_A} has Hodge group $\text{SU}(2, 2)$, irreducible.)

Proposal:

$$(A, \omega_A) \rightsquigarrow (\mathcal{D}, \sigma) \rightsquigarrow ((W/\Lambda, g, J), K)$$

(smooth proper category, (generalized complex torus,
geometric stability condition) polarization)

Mirror symmetry = swap J and K ; allow a noncommutative $\mathcal{D}$ on the other side.

Theorem C: swapping J and K exchanges moduli

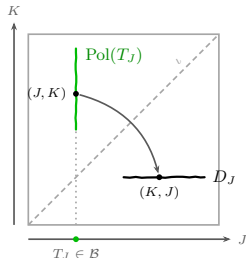
Kontsevich: pairs (J, K) fibre over generalized complex tori,

$$\mathrm{U}(n) \times \mathrm{U}(n) \backslash \mathrm{O}(2n, 2n; \mathbb{R}) / \mathrm{O}(2n, 2n; \mathbb{Z}) \xrightarrow{(J, K) \mapsto J} \mathrm{U}(n, n) \backslash \mathrm{O}(2n, 2n; \mathbb{R}) / \mathrm{O}(2n, 2n; \mathbb{Z}).$$

Theorem C (Haiden–S.) Let $\mathcal{P} = \{(J, K) : K \text{ a polarization of } T_J\}$. The swap $(J, K) \mapsto (K, J)$

- 1) preserves $\mathcal{P}$;
- 2) exchanges fixed J , varying K : $\mathrm{Pol}(T_J)$, the geometric stability parameters,
- 3) with J as polarization: D_J , an algebraic family of polarized abelian $2n$ -folds.

So the swap exchanges **Kähler moduli** with complex moduli at fixed polarization.



The largest mirror families are of Weil type

Theorem C (continued) For $n \geq 2$, $\dim_{\mathbb{C}} D_K \leq n^2$, with equality iff $\sqrt{d} K$ is rational (preserves $\Lambda_{\mathbb{Q}}$) for some $d \in \mathbb{Q}_{>0}$. Then $D_K \cong \mathrm{U}(n, n)/(\mathrm{U}(n) \times \mathrm{U}(n))$ is a **split Weil-type** family for $\mathbb{Q}(\sqrt{-d})$ (split: hyperbolic Hermitian form).

Corollary (CM powers) $A = E^n$, E with CM by F , $n \geq 2$: the n^2 -dimensional space of $B + i\omega$ is exchanged with a split Weil-type family for F . Its very general member is **not** a doubled torus: no abelian mirror exists.

$\Rightarrow$ a **noncommutative** mirror is predicted (conjecture on the next slide).

$n = 2$: four-dimensional families of noncommutative abelian surfaces with quasi-finite period map to these domains (Bayer–Perry–Pertusi–Zhao, announced; cf. Pertusi, arXiv:2602.24016).

Conjecture: noncommutative abelian varieties

Conjecture (Kontsevich; Haiden–S.) Every $(J, K) \in \mathcal{P}$ gives $(\mathcal{D}_{T_J}, \sigma_K)$: a smooth proper n -Calabi–Yau category with a geometric stability condition, in algebraic families over D_K (the J 's with K fixed), with $\text{Stab}(\mathcal{D}_{T_J})$ the universal cover of $\mathcal{U}^+(T_J)$. Mirror symmetry:

$$(\mathcal{D}_{T_J}, \sigma_K) \longleftrightarrow (\mathcal{D}_{T_K}, \sigma_J).$$

Example: $J = J_A$ gives $\mathcal{D}_{T_A} = D^b(A)$, recovering the conjecture on the Theorem B slide; $K = K_{B+i\omega}$ gives the stability condition with charge $-\int e^{-(B+i\omega)} \text{ch}(\cdot)$.

Big takeaway. For abelian varieties, both the stability manifold and mirror symmetry can be read off the doubled torus T_A .