Beyond
Algebraic Models of Stringy Spacetime

Tristan Hübsch

@ ICERM: 2026.10.01

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w/Per Berglund

🙏: *Mikiya Masuda*

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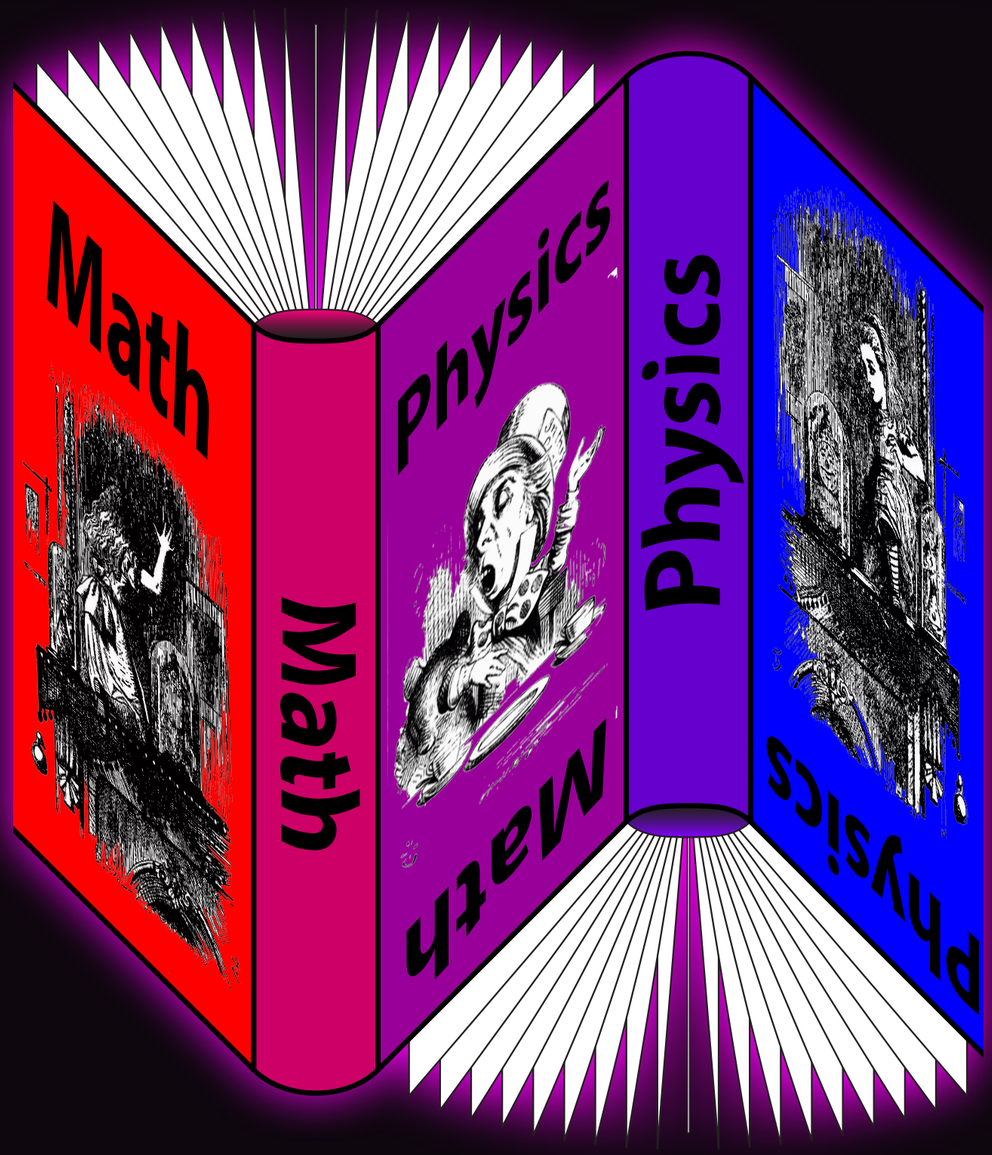
Beyond Algebraic Models (BAM)

Playbill

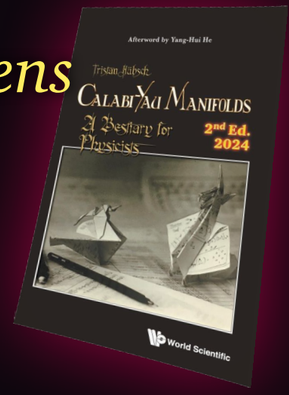
A Decade Ago
A Family of Scrolls

and Their Offspring
...in the Mirror

* "It doesn't matter what it's called,
* ...as long as it has substance."
— S.-T. Yau



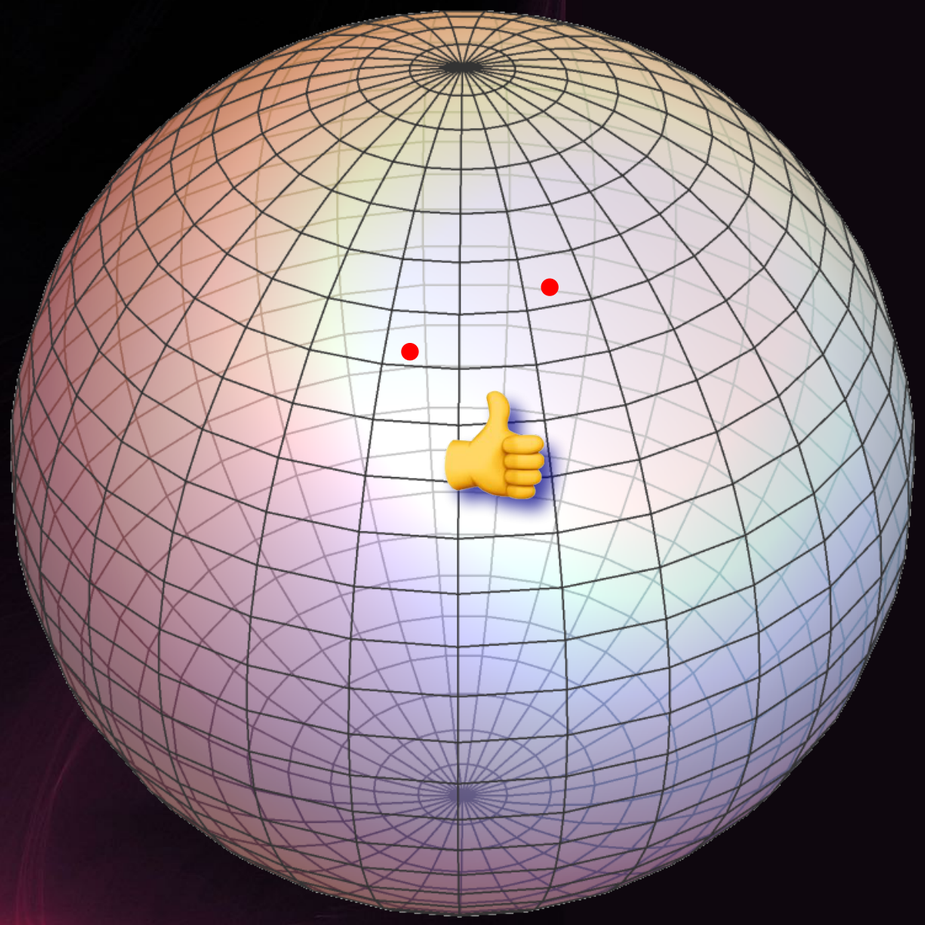
Spiritus Movens



CY*n*-folds in non-Fano Spaces

How hard can it be? CY ⊂ some "nice enough" Ambient space

- Reduce to 0 dimensions: $\mathbb{P}^4[5] \to \mathbb{P}^3[4] \to \mathbb{P}^2[3] \to \mathbb{P}^1[2]$



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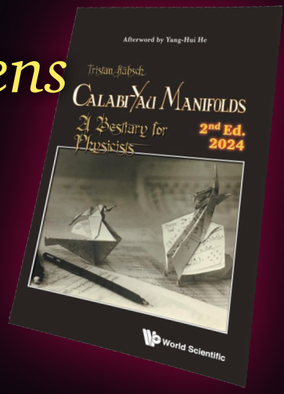

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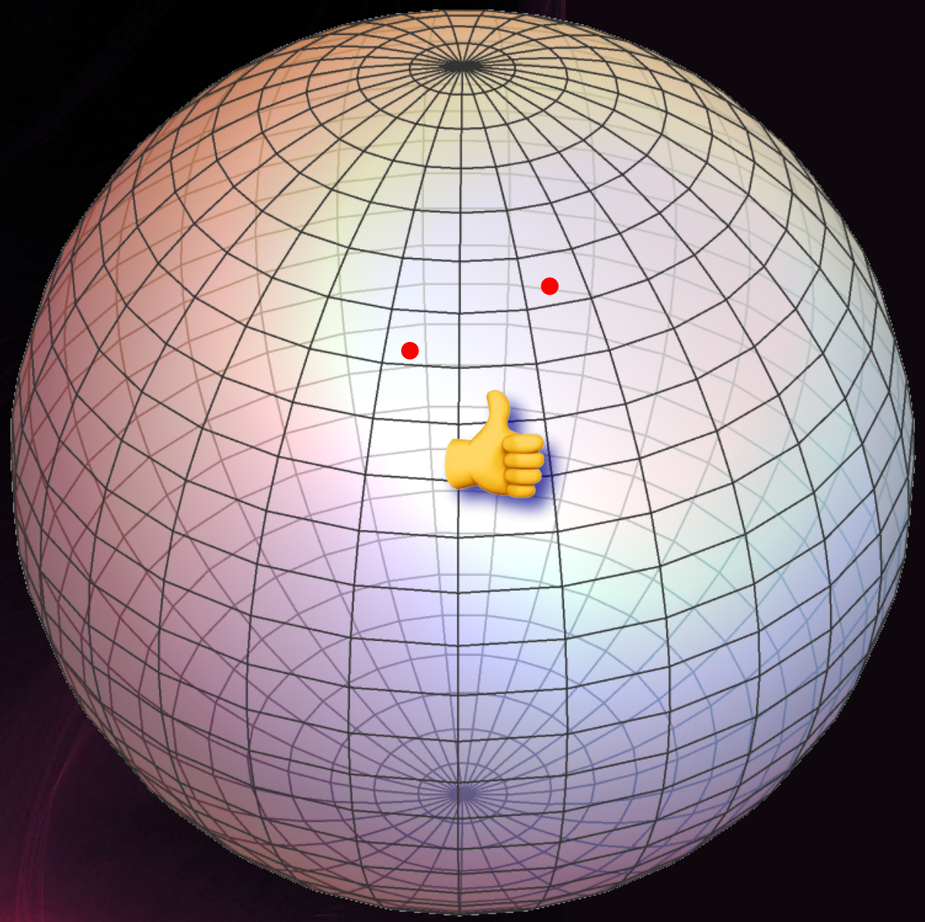

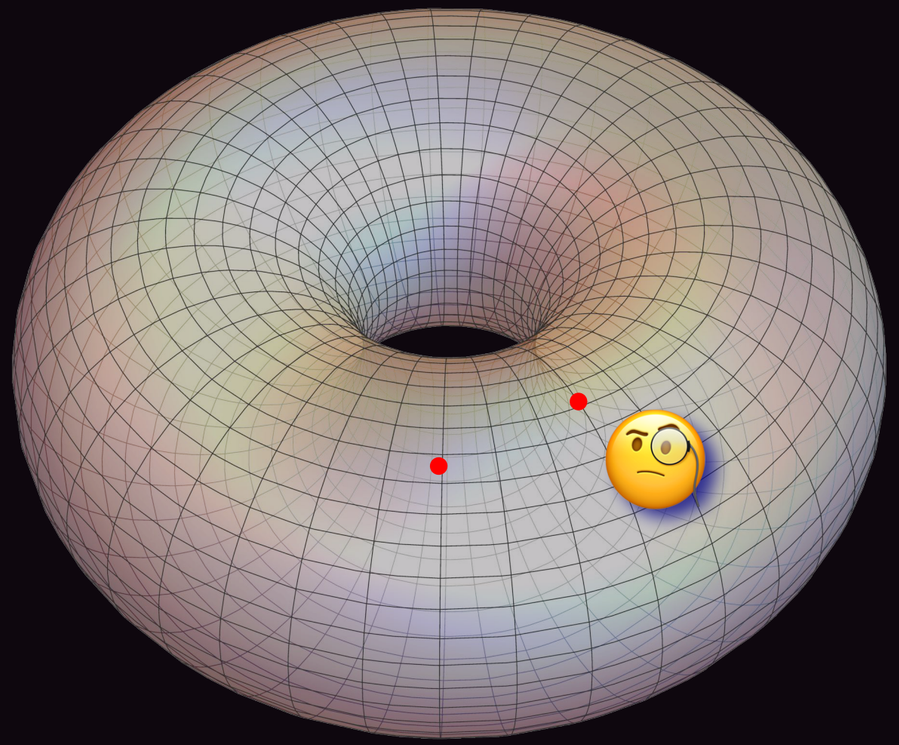

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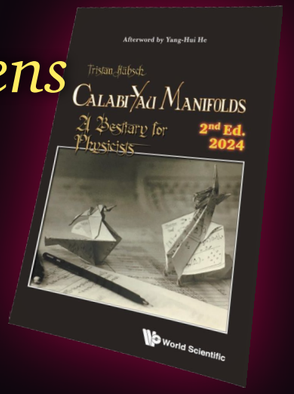

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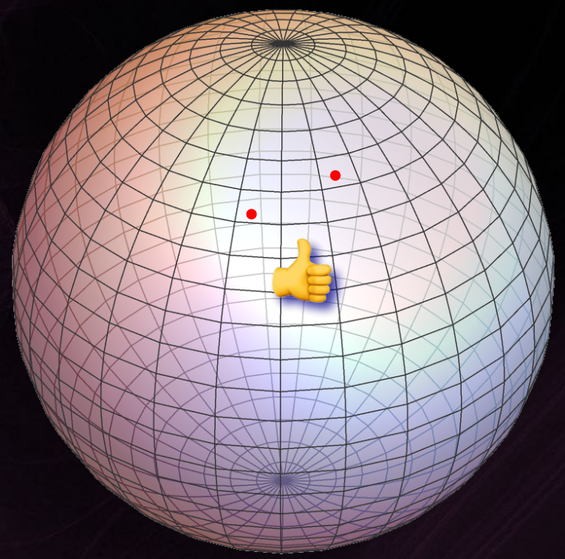
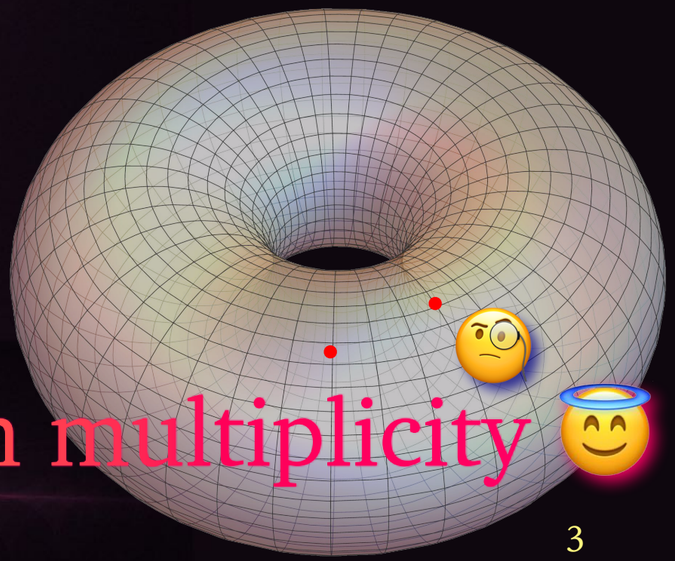
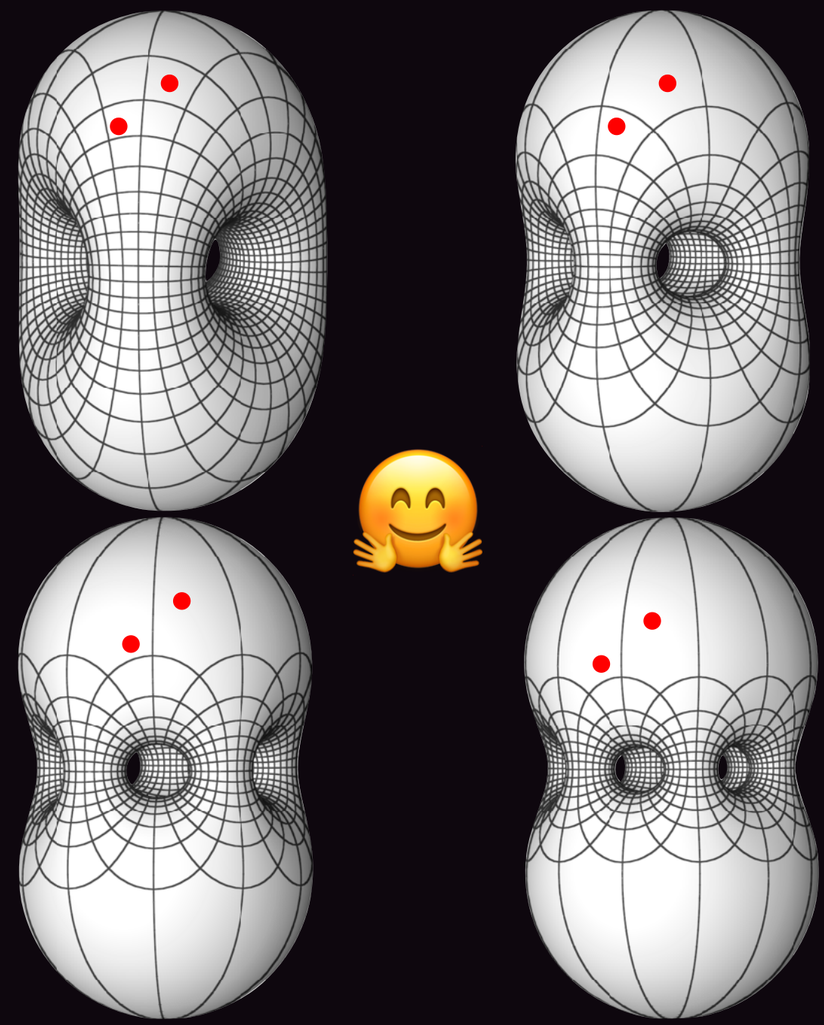


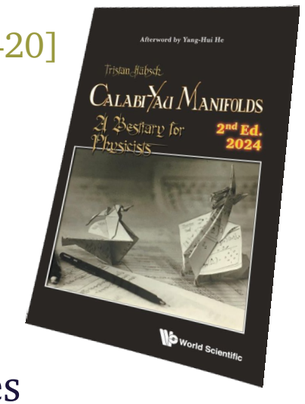
$$\mathcal{X} \in \mathrm{A}[c_1]$$



Counting with multiplicity 😇



[arXiv:1606.07420]



A Decade Ago…

Classical Constructions

- E.g: $$X_m \in \left[\begin{array}{c|c|c} \mathbb{P}^4 & 1 & 4 \\ \mathbb{P}^1 & m & 2-m \end{array}\right]^{(2,86)}_{-168}$$

smooth $\mathbb{R}$ *models*

$b_2 = 2 = h^{1,1}$ dim. space of Kähler classes

$\frac{1}{2} b_3 - 1 = 86 = h^{2,1}$ dim. space of cpx structures

$-168 = \chi = 2(h^{1,1} - h^{2,1})$ the Euler #

- Zero-set of $p(x,y)=0$, $\deg[p] = \binom{1}{m}$ & $q(x,y)=0$, $\deg[q] = \binom{4}{2-m}$
- Generic $\{p=0\} \cap \{q=0\}$ smooth; $\deg_{\mathbb{P}^n}[p] + \deg_{\mathbb{P}^n}[q] = n+1 \Rightarrow \boxed{c_1 = 0}$

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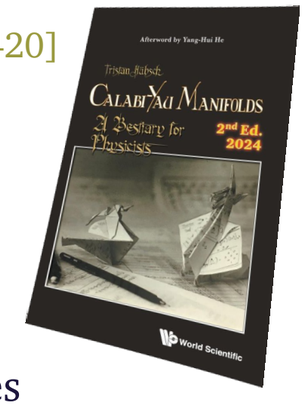

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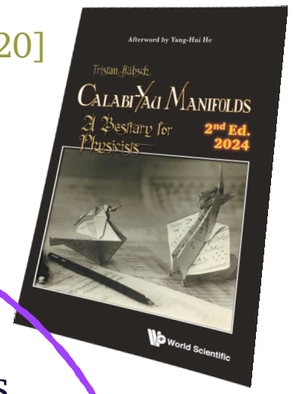

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  - Chern: $c = \dfrac{(1+J_1)^5(1+J_2)^2}{(1+J_1+mJ_2)\,(1+4J_1+(2-m)J_2)}$ for $F_m^{(4)}$ and X_m
  - Then: $C_{4-k}^{(c)}[(aJ_1+bJ_2)^k] = \widetilde{C}_k(4b+ma)$ = *diffeomorphism* invariants
    - e.g., $\int_{F_m} c_1c_2[(aJ_1+bJ_2)] = \int_{X_m} c_2[(aJ_1+bJ_2)]$, but also $\int_{F_m} c_2[(aJ_1+bJ_2)^2]$, …
  - So, $F_m^{(4)} \approx_{\mathbb{R}} F_{m\,(\mathrm{mod}\,4)}^{(4)}$ & $X_m \approx_{\mathbb{R}} X_{m\,(\mathrm{mod}\,4)}$: 4 *diffeomorphism types*

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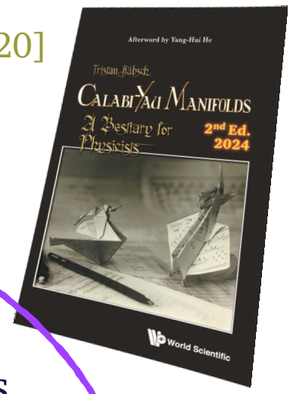

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- …so, $m = 0,\ 1,\ 2, \cdots$

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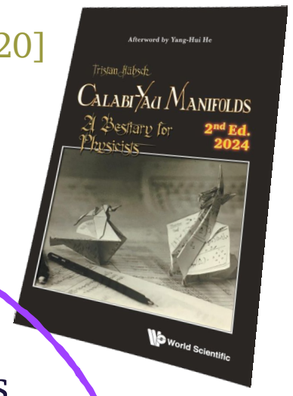

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$$q(x,y) \overset{?}{\sim} \frac{q_0(x)}{y_0} + \frac{q_1(x)}{y_1}$$

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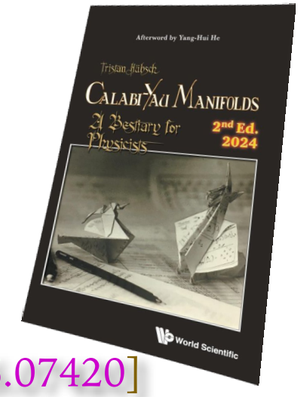
- …so, $m = 0, 1, 2, \cdots 3 \Rightarrow \deg[q] = \binom{4}{-1}$?!

A Decade Ago…

Why Haven't We Thought of This Before?



[AAGGL:1507.03235 + BH:1606.07420]

- $\deg[q] = \binom{4}{-1}$ holomorphic sections?!
  - Not *everywhere* on $\mathbb{P}^4\times\mathbb{P}^1$ — (simple poles)
  - but yes on $F_3^{(4)}\subset\mathbb{P}^4\times\mathbb{P}^1$ — $\geqslant 105$ of 'em!

$$X_m \in \left[\begin{array}{c||c} \mathbb{P}^4 & 1 \\ \mathbb{P}^1 & m \end{array}\middle| \begin{array}{c} 4 \\ 2-m \end{array}\right]^{(2,86)}_{-168}$$

for $m=3$

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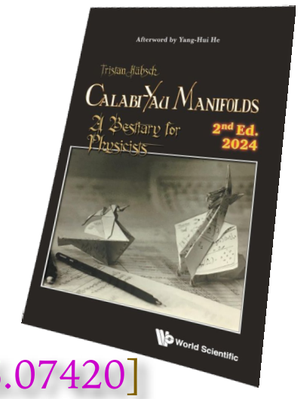

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- How? On $F_3^{(4)}$, $q(x,y)\simeq q(x,y)+\lambda\cdot p(x,y)$ ← *equivalence class!*

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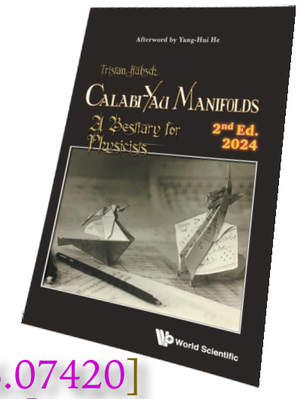

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Two more choices, $\left(\frac{x_0y_0^2}{y_1^3}-\frac{x_1}{y_0}\right)$, $\left(\frac{x_0}{y_1}-\frac{x_1y_1^2}{y_0^3}\right)$

- How? On $F_3^{(4)}$, $q(x,y)\simeq q(x,y)+\lambda\cdot p(x,y)$ ← *equivalence class!* ⇒ $3\times 35=105$ sections
  - [Hirzebruch, 1951] ⇒ $p=x_0y_0^3+x_1y_1^3$

& $q=c(x)\left[\left(\frac{x_0y_0}{y_1^2}-\frac{x_1y_1}{y_0^2}\right)\bmod p_0(x,y)\right]$ $\deg[c]=\binom{3}{0}$

- So, $q_0=\left[q(x,y)+\frac{\lambda\,c(x)}{(y_0y_1)^2}p(x,y)\right]_{\lambda\to-1}=c(x)\left(-2\frac{x_1y_1}{y_0^2}\right)$ in the $y_0\neq 0$ chart
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- [GvG, 1708.00517] *scheme-th.* "*generalized* complete intersections"

Reverse-engineered: Mayer-Vietoris sequence & "patching" of the two charts

Wu-Yang magnetic monopole

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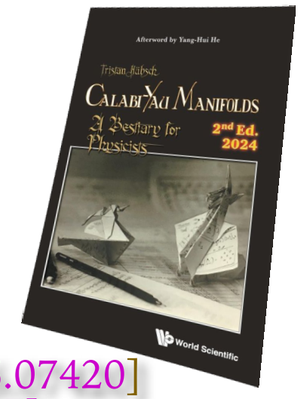

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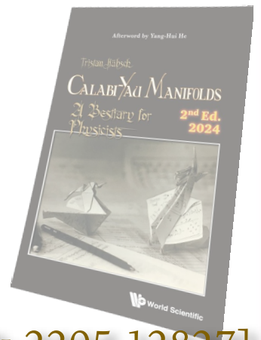
2026: Borisov: no schemes needed—rank-2 v.bdl & codim-2 0-locus

A Family of Scrolls

...in well-tempered counterpoint



[BH:1606.07420, 1611.10300 & 2205.12827]
+more

- For $\left\{ 0 \overset{!}{=} \underbrace{x_0 {y_0}^m + x_1 {y_1}^m}_{:= p_0(x,y)} + \sum_{a=2}^{n} \sum_{\ell=1}^{m-1} \epsilon_{a,\ell}\, x_a\, {y_0}^{m-\ell} {y_1}^{\ell} \right\} = F^{(n)}_{m;\epsilon} \in \left[\begin{matrix} \mathbb{P}^n \\ \mathbb{P}^1 \end{matrix} \middle\| \begin{matrix} 1 \\ m \end{matrix} \right]$

even $p_0(x,y)$ is transverse, $p_0^{-1}(0)$ is smooth

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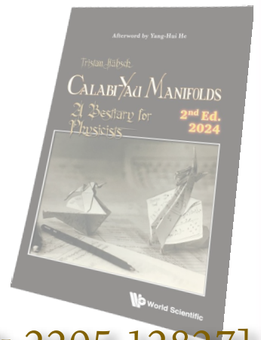
$p_0^{-1}(0) \cap \mathfrak{s}^{-1}(0)$ **is smooth**
$\mathrm{d}p_0(x,y) \wedge \mathrm{d}\mathfrak{s}(x,y) \neq 0$



[BH:1606.07420, 1611.10300 & 2205.12827]
+more

- For $\left\{ 0 \overset{!}{=} \underbrace{x_0 {y_0}^m + x_1 {y_1}^m}_{:=p_0(x,y)} + \sum_{a=2}^{n} \sum_{\ell=1}^{m-1} c_{a,\ell}\, x_a\, {y_0}^{m-\ell}\, {y_1}^{\ell} \right\} = F^{(n)}_{m;\mathbf{c}} \in \left[\begin{matrix} \mathbb{P}^n \\ \mathbb{P}^1 \end{matrix} \middle\| \begin{matrix} 1 \\ m \end{matrix} \right]$

even $p_0(x,y)$ is transverse, $p_0^{-1}(0)$ is smooth

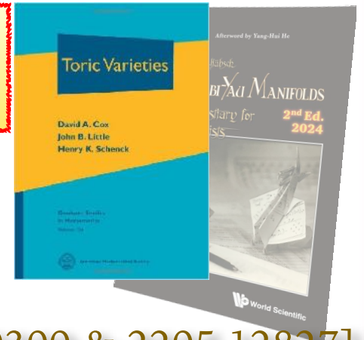
- The central ($\epsilon = 0$) member of the family is a Hirzebruch scroll F_m :
  - Directrix: $\mathfrak{s}(x,y) := \left[\left(\frac{x_0}{{y_1}^m} - \frac{x_1}{{y_0}^m} \right) + \frac{\lambda}{(y_0 y_1)^m} p_0(x,y) \right]$ degree-$\binom{1}{-m}$
- On $F^{(n)}_m$: $p_0(x,y) = x_0 {y_0}^m + x_1 {y_1}^m = 0 \Rightarrow x_0 = -x_1 (y_1/y_0)^m$, @ $y_0 \neq 0$
  - So, $x_1 \to X_1 = \mathfrak{s}(x,y)$ & $(X_i,\ i=2,\cdots) = (x_2,\cdots,x_n; y_0, y_1)$
  - Key: $\det \left[\frac{\partial(p_0(x,y), \mathfrak{s}(x,y), x_2, \cdots; y_0, y_1)}{\partial(x_0, x_1, x_2, \cdots; y_0, y_1)} \right] = \text{const.}$
  - $\mathbb{P}^4 \times \mathbb{P}^1$ bi-degree $\to$ toric $(\mathbb{C}^\times)^2$-action:

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow \mathbb{P}^4$
$-m$	0	0	0	1	1	$\leftarrow \mathbb{P}^1$

A Family of Scrolls

...in well-tempered counterpoint

$p_0^{-1}(0) \cap \mathfrak{s}^{-1}(0)$ **is smooth**
$\mathrm{d}p_0(x,y) \wedge \mathrm{d}\mathfrak{s}(x,y) \neq 0$



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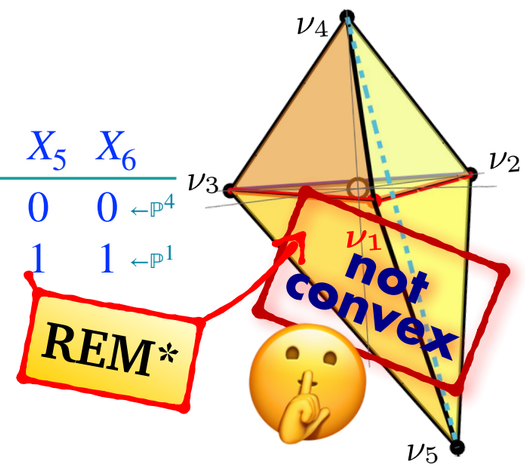
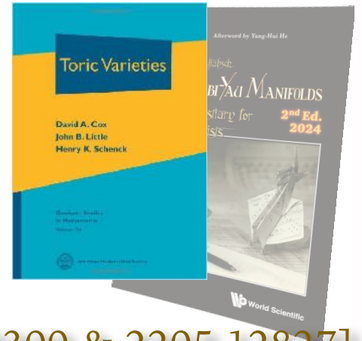


*Reverse-Engineered (Toric) Model

- *What about the rest of the $\epsilon_{a,\ell}$-deformation family?*
  (wherein smooth models are all diffeomorphic to each other)

A Family of Scrolls

...with a meandering motif



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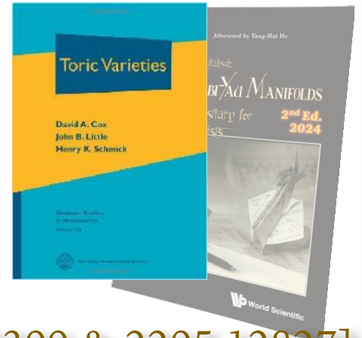
$$\deg=\binom{1}{m-r_0-r_1}:\quad \mathfrak{s}_{\vec{\epsilon}}(x,y;\lambda):=\operatorname*{Flip}_{y_0}\left[\frac{1}{y_0^{r_0}\,y_1^{r_1}}p_{\vec{\epsilon}}(x,y)\right]\ (\mathrm{mod}\ p_{\vec{\epsilon}}(x,y)),\qquad \left[\begin{array}{c|c}\mathbb{P}^n & 1\\ \mathbb{P}^1 & m\end{array}\right]$$

progressively decreasing $r_0+r_1=2m,2m-1,\cdots$, and keeping only Laurent polynomials containing both y_0- and y_1-denominators but no y_0,y_1-mixed ones. The "Flip_{y_i}" operator changes the relative sign of the rational monomials with y_i-denominators. For algebraically independent such sections, restrict to a subset with maximally negative degrees that are not overall (y_0,y_1)-multiples of each other.

A Family of Scrolls

...with a meandering motif



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E.g.: $m=2$

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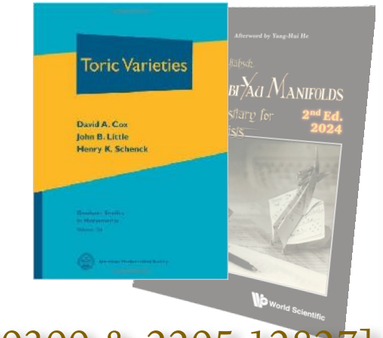
A Family of Scrolls

...with a meandering motif



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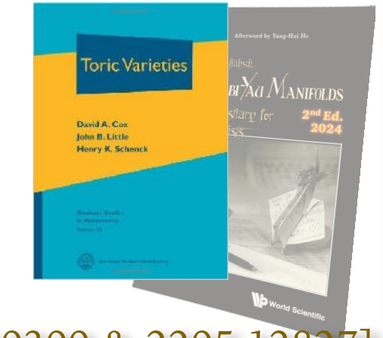
A Family of Scrolls

...with a meandering motif



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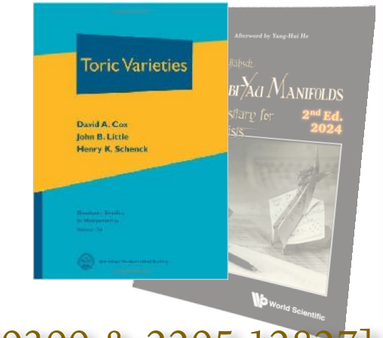
A Family of Scrolls

...with a meandering motif



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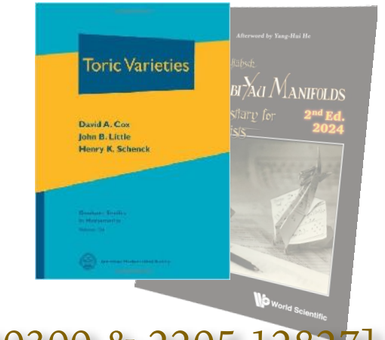
A Family of Scrolls

...with a meandering motif



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A Family of Scrolls

...with a meandering motif

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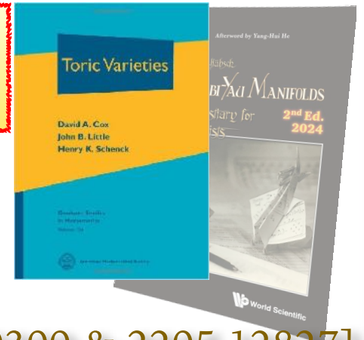

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Use this for identification

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p0 = x0 y0^2 + x1 y1^2; ep[α_] := Table[1/(y0^(α-i) y1^i), {i, 0, α}]; Expand /@ (p0 {ep[5], ep[4], ep[3]})
```

① $\{\{\frac{x_0}{y_0^3}+\frac{x_1 y_1^2}{y_0^5}, \frac{x_0}{y_0^2 y_1}+\frac{x_1 y_1}{y_0^4}, \frac{x_1}{y_0^3}+\frac{x_0}{y_0 y_1^2}, \frac{x_0}{y_1^3}+\frac{x_1}{y_0^2 y_1}, \frac{x_0 y_0}{y_1^4}+\frac{x_1}{y_0 y_1^2}, \frac{x_0 y_0^2}{y_1^5}+\frac{x_1}{y_1^3}\}$, $\cdot y_1$, $\cdot y_0$ ③

② $\{\frac{x_0}{y_0^2}+\frac{x_1 y_1^2}{y_0^4}, \frac{x_0}{y_0 y_1}+\frac{x_1 y_1}{y_0^3}, \frac{x_1}{y_0^2}+\frac{x_0}{y_1^2}, \frac{x_0 y_0}{y_1^3}+\frac{x_1}{y_0 y_1}, \frac{x_0 y_0^2}{y_1^4}+\frac{x_1}{y_1^2}\}, \{\frac{x_0}{y_0}+\frac{x_1 y_1^2}{y_0^3}, \frac{x_0}{y_1}+\frac{x_1 y_1}{y_0^2}, \frac{x_1}{y_0}+\frac{x_0 y_0}{y_1^2}, \frac{x_0 y_0^2}{y_1^3}+\frac{x_1}{y_1}\}\}$

finds $\mathfrak{s}(x,y) = \left[\left(\dfrac{x_0}{{y_1}^2} - \dfrac{x_1}{{y_0}^2}\right) \bmod \left(x_0{y_0}^2 + x_1{y_1}^2\right)\right]$; $\deg = \binom{1}{-2}$, $[\mathfrak{s}^{-1}(0)] = [J_1] - 2[J_2]$.

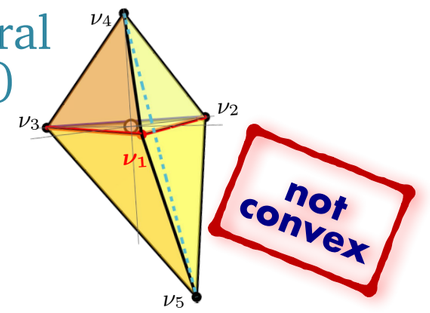
***THE* exceptional section**

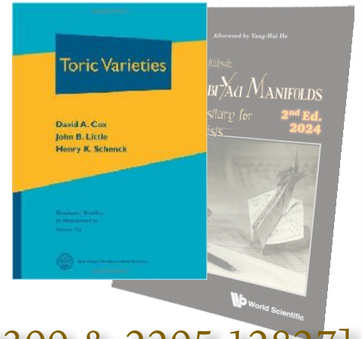
***THE* hallmark exceptional divisor $[S]^2 = -1$ of $F_2^{(2)}$**

A Family of Scrolls

…with cousins joining in…

the central ($\epsilon_{a,\ell}=0$) model



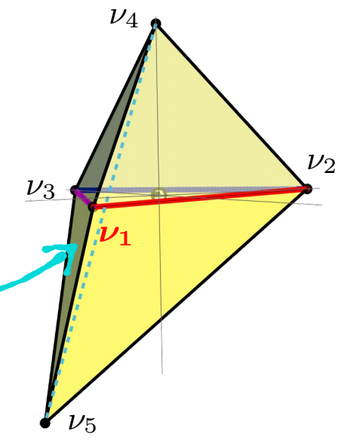


[BH:1606.07420, 1611.10300 & 2205.12827]

+more

- Deform: $p_1(x,y)=x_0{y_0}^5+x_1{y_1}^5+x_2y_0{y_1}^4$ toric $F^{(n)}_{(4,1,0,\ldots)}$
  - Now: $\mathfrak{s}_{1,1}(x,y)=\dfrac{x_0y_0}{{y_1}^5}+\dfrac{x_2}{{y_1}^4}-\dfrac{x_1}{{y_1}^4}$ & $\mathfrak{s}_{1,2}(x,y)=\dfrac{x_0}{y_1}-\dfrac{x_2}{y_0}-\dfrac{x_1y_1^4}{y_0^5}$
  - & $\det\left[\dfrac{\partial(p_1,\mathfrak{s}_{1,1},\mathfrak{s}_{1,2},x_3,\cdots;y_0,y_1)}{\partial(x_0,x_1,x_2,x_3,\cdots;y_0,y_1)}\right]=\text{const.}$

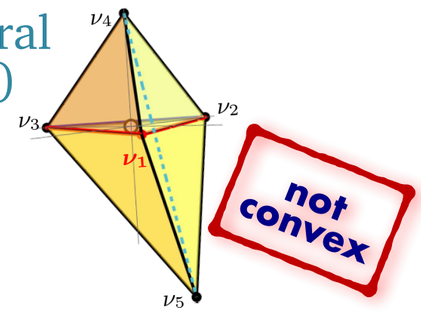
X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow\mathbb{P}^4$
−4	−1	0	0	1	1	$\leftarrow\mathbb{P}^1$

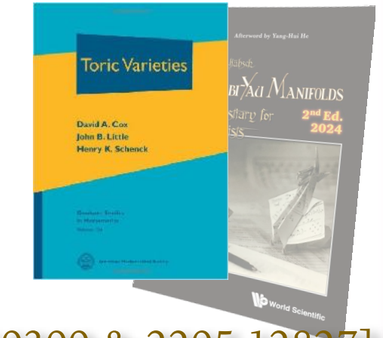


A Family of Scrolls

…with cousins joining in…

the central ($\epsilon_{a,\ell}=0$) model



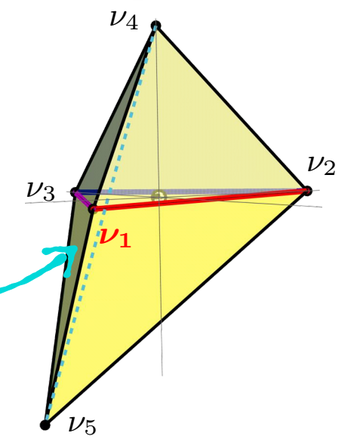


[BH:1606.07420, 1611.10300 & 2205.12827]

+more

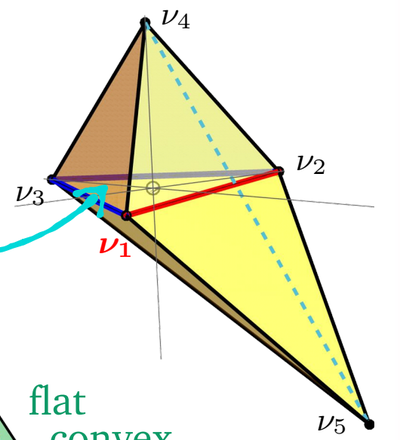
- Deform: $p_1(x,y)=x_0y_0^5+x_1y_1^5+x_2y_0y_1^4$ toric $F^{(n)}_{(4,1,0,\ldots)}$
  - Now: $\mathfrak{s}_{1,1}(x,y)=\frac{x_0y_0}{y_1^5}+\frac{x_2}{y_1^4}-\frac{x_1}{y_1^4}$ & $\mathfrak{s}_{1,2}(x,y)=\frac{x_0}{y_1}-\frac{x_2}{y_0}-\frac{x_1y_1^4}{y_0^5}$
  - & $\det\left[\frac{\partial(p_1,\mathfrak{s}_{1,1},\mathfrak{s}_{1,2},x_3,\cdots;y_0,y_1)}{\partial(x_0,x_1,x_2,x_3,\cdots;y_0,y_1)}\right]=\text{const.}$

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow\mathbb{P}^4$
−4	−1	0	0	1	1	$\leftarrow\mathbb{P}^1$



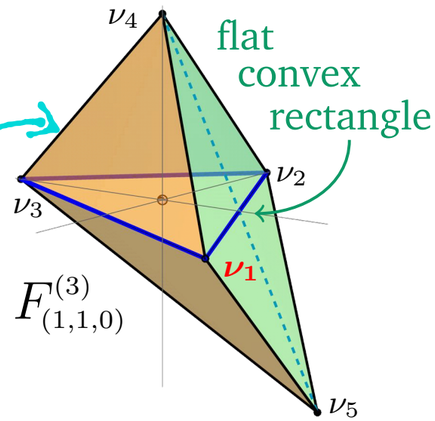
- Deform: $p_2(x,y)=x_0y_0^5+x_1y_1^5+x_2y_0^2y_1^3$ toric $F^{(n)}_{(3,2,0,\ldots)}$
  - Now: $\mathfrak{s}_{2,1}(x,y)=\frac{x_0y_0^2}{y_1^5}+\frac{x_2}{y_1^3}-\frac{x_1}{y_1^3}$ & $\mathfrak{s}_{2,2}(x,y)=\frac{x_0}{y_1^2}-\frac{x_2}{y_0^2}-\frac{x_1y_1^3}{y_0^5}$
  - & $\det\left[\frac{\partial(p_2,\mathfrak{s}_{2,1},\mathfrak{s}_{2,2},x_3,\cdots;y_0,y_1)}{\partial(x_0,x_1,x_2,x_3,\cdots;y_0,y_1)}\right]=\text{const.}$

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow\mathbb{P}^4$
−3	−2	0	0	1	1	$\leftarrow\mathbb{P}^1$



- … and $p_3(x,y)=x_0y_0^5+x_1y_1^5+x_2y_0^2y_1^3+x_3y_0^3y_1^2$
  - → toric $F^{(n)}_{(2,2,1,\cdots)}$

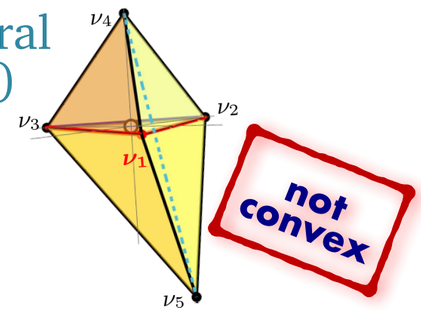
X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow\mathbb{P}^4$
−2	−2	−1	0	1	1	$\leftarrow\mathbb{P}^1$

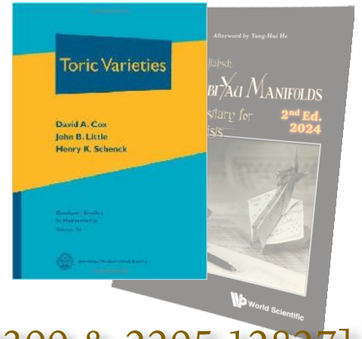


A Family of Scrolls

…with cousins joining in…

the central ($\epsilon_{a,\ell}=0$) model

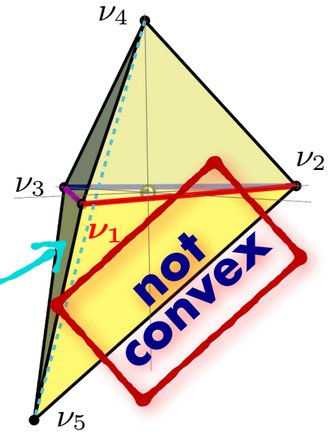




[BH:1606.07420, 1611.10300 & 2205.12827] +more

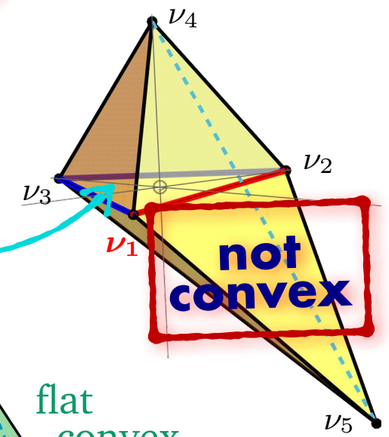
- Deform: $p_1(x,y)=x_0y_0^5+x_1y_1^5+x_2y_0y_1^4$ toric $F^{(n)}_{(4,1,0,\dots)}$
  - Now: $\mathfrak{s}_{1,1}(x,y)=\frac{x_0y_0}{y_1^5}+\frac{x_2}{y_1^4}-\frac{x_1}{y_1^4}$ & $\mathfrak{s}_{1,2}(x,y)=\frac{x_0}{y_1}-\frac{x_2}{y_0}-\frac{x_1y_1^4}{y_0^5}$
  - & $\det\left[\frac{\partial(p_1,\mathfrak{s}_{1,1},\mathfrak{s}_{1,2},x_3,\cdots;y_0,y_1)}{\partial(x_0,x_1,x_2,x_3,\cdots;y_0,y_1)}\right]=\text{const.}$

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow\mathbb{P}^4$
−4	−1	0	0	1	1	$\leftarrow\mathbb{P}^1$



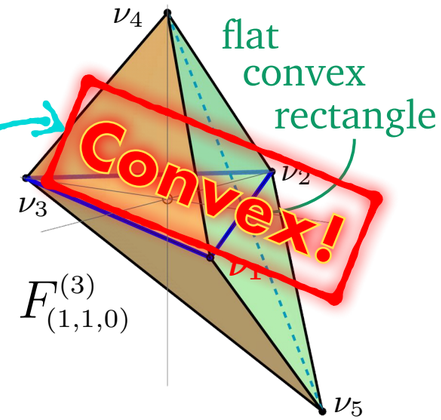
- Deform: $p_2(x,y)=x_0y_0^5+x_1y_1^5+x_2y_0^2y_1^3$ toric $F^{(n)}_{(3,2,0,\dots)}$
  - Now: $\mathfrak{s}_{2,1}(x,y)=\frac{x_0y_0^2}{y_1^5}+\frac{x_2}{y_1^3}-\frac{x_1}{y_1^3}$ & $\mathfrak{s}_{2,2}(x,y)=\frac{x_0}{y_1^2}-\frac{x_2}{y_0^2}-\frac{x_1y_1^3}{y_0^5}$
  - & $\det\left[\frac{\partial(p_2,\mathfrak{s}_{2,1},\mathfrak{s}_{2,2},x_3,\cdots;y_0,y_1)}{\partial(x_0,x_1,x_2,x_3,\cdots;y_0,y_1)}\right]=\text{const.}$

X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow\mathbb{P}^4$
−3	−2	0	0	1	1	$\leftarrow\mathbb{P}^1$



- … and $p_3(x,y)=x_0y_0^5+x_1y_1^5+x_2y_0^2y_1^3+x_3y_0^3y_1^2$
  - → toric $F^{(n)}_{(2,2,1,\dots)}$ for $n=3$, $F^{(3)}_{(2,2,1)}\approx F^{(3)}_{(1,1,0)}$ [~Veronese] Fano!

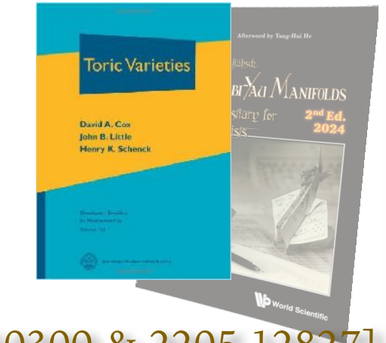
X_1	X_2	X_3	X_4	X_5	X_6	
1	1	1	1	0	0	$\leftarrow\mathbb{P}^4$
−2	−2	−1	0	1	1	$\leftarrow\mathbb{P}^1$



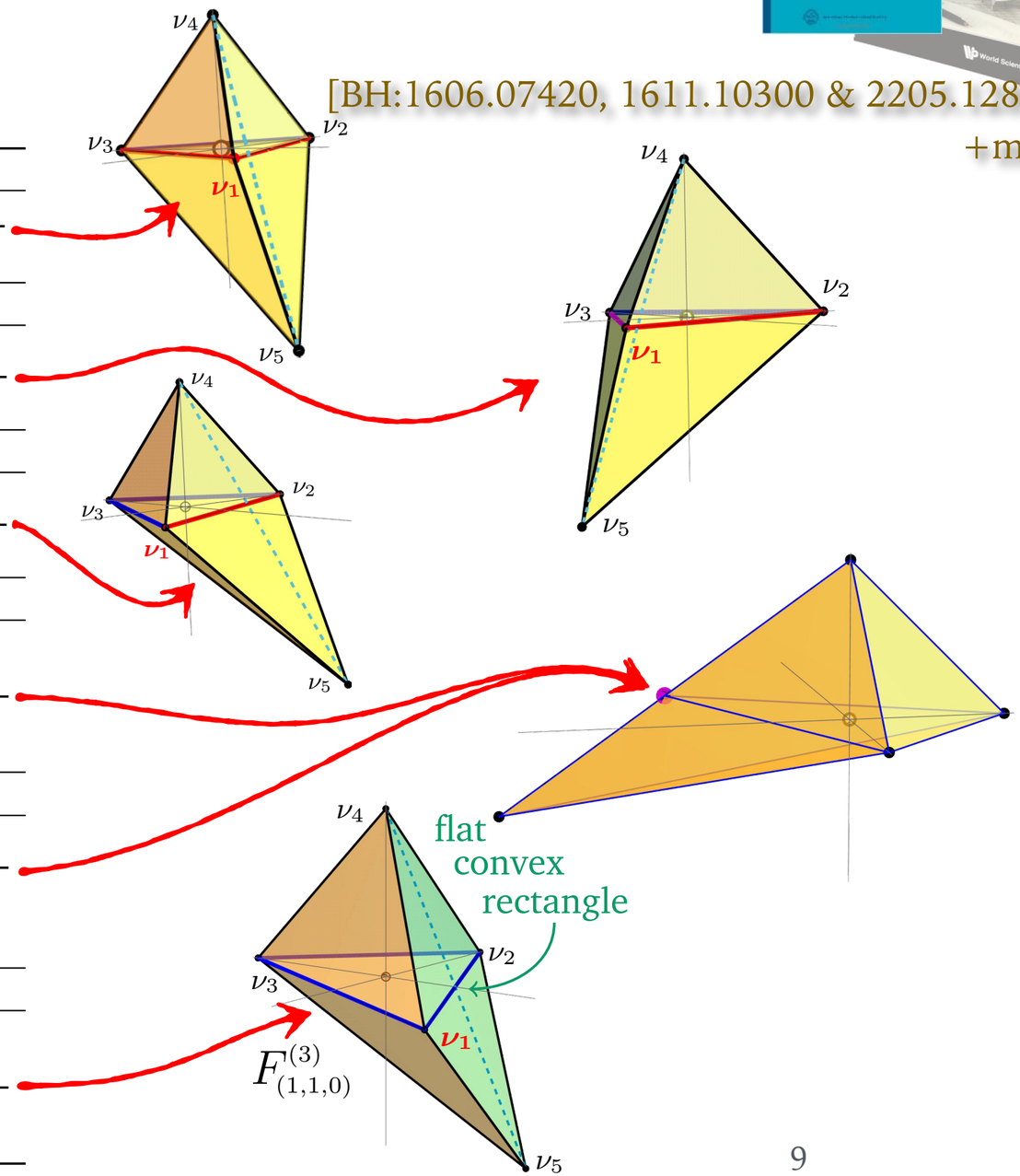
A Family of Scrolls

...in elaborate hierarchy

Deg.	Biprojective Defining Section	Name, GLSM Charges
	$p_0(x,y) = x_0\, y_0^5 + x_1\, y_1^5$	$F_5^{(3)}$
$\binom{1}{-5}$:	$\mathfrak{s}_0(x,y) = \left[\left(\frac{x_0}{y_1^5} - \frac{x_1}{y_0^5}\right)\Big/ p_0(x,y)\right]$	$\begin{array}{ccccc} X_1 & X_2 & X_3 & X_4 & X_5 \\ \hline 1 & 1 & 1 & 0 & 0 \\ -5 & 0 & 0 & 1 & 1 \end{array}$
	$p_1(x,y) = p_0(x,y) + x_2\, y_0^4 y_1$	$F_{4,1}^{(3)}$
$\binom{1}{-2}$:	$\mathfrak{s}_{1a}(x,y) = \left[\left(\frac{x_0\, y_0}{y_1^5} - \frac{x_1}{y_0^4} + \frac{x_2}{y_1^4}\right)\Big/ p_1(x,y)\right]$	$\begin{array}{ccccc} X_1 & X_2 & X_3 & X_4 & X_5 \\ \hline 1 & 1 & 1 & 0 & 0 \\ -4 & -1 & 0 & 1 & 1 \end{array}$
$\binom{1}{-2}$:	$\mathfrak{s}_{1b}(x,y) = \left[\left(\frac{x_0}{y_1} - \frac{x_1\, y_1^4}{y_0^5} - \frac{x_2}{y_0}\right)\Big/ p_1(x,y)\right]$	
	$p_2(x,y) = p_0(x,y) + x_2\, y_0^3\, y_1^2$	$F_{3,2}^{(3)}$
$\binom{1}{-3}$:	$\mathfrak{s}_{2a}(x,y) = \left[\left(\frac{x_0\, y_0^2}{y_1^5} - \frac{x_1}{y_0^3} + \frac{x_2 y_0}{y_1^4} + \frac{x_3}{y_1^3}\right)\Big/ p_3(x,y)\right]$	$\begin{array}{ccccc} X_1 & X_2 & X_3 & X_4 & X_5 \\ \hline 1 & 1 & 1 & 0 & 0 \\ -3 & -2 & 0 & 1 & 1 \end{array}$
$\binom{1}{-1}$:	$\mathfrak{s}_{2b}(x,y) = \left[\left(\frac{x_0}{y_1^2} - \frac{x_1 y_1^3}{y_0^5} - \frac{x_2}{y_0^2}\right)\Big/ p_2(x,y)\right]$	
	$p_3(x,y) = p_0(x,y) + x_2 y_0^4 y_1^1 + x_3 y_0^1 y_1^4$	$F_{3,1,1a}^{(3)} \approx_{\mathbb{R}} F_2^{(3)}$
$\binom{1}{-3}$:	$\mathfrak{s}_{3a}(x,y) = \left[\left(\frac{x_0 y_0}{y_1^4} - \frac{x_1 y_1}{y_0^4} + \frac{x_2}{y_1^3} - \frac{x_3}{y_0^3}\right)\Big/ p_3(x,y)\right]$	
$\binom{1}{-1}$:	$\mathfrak{s}_{3b}(x,y) = \left[\left(\frac{x_0}{y_1} - \frac{x_1 y_1^4}{y_0^5} - \frac{x_2}{y_0} - \frac{x_3 y_1^3}{y_0^4}\right)\Big/ p_3(x,y)\right]$	$\begin{array}{ccccc} X_1 & X_2 & X_3 & X_4 & X_5 \\ \hline 1 & 1 & 1 & 0 & 0 \\ -3 & -1 & -1 & 1 & 1 \end{array}$
$\binom{1}{-1}$:	$\mathfrak{s}_{3c}(x,y) = \left[\left(\frac{x_0 y_0^4}{y_1^5} - \frac{x_1}{y_0} + \frac{x_2 y_0^3}{y_1^4} + \frac{x_3}{y_1}\right)\Big/ p_3(x,y)\right]$	
	$p_4(x,y) = p_0(x,y) + x_2\, y_0^4 y_1 + x_3\, y_0^3\, y_1^2$	$F_{3,1,1b}^{(3)} \approx_{\mathbb{R}} F_2^{(3)}$
$\binom{1}{-3}$:	$\mathfrak{s}_{4a}(x,y) = \left[\left(\frac{x_0\, y_0^2}{y_1^5} - \frac{x_1}{y_0^3} + \frac{x_2 y_0}{y_1^4} + \frac{x_3}{y_1^3}\right)\Big/ p_4(x,y)\right]$	$\begin{array}{ccccc} X_1 & X_2 & X_3 & X_4 & X_5 \\ \hline 1 & 1 & 1 & 0 & 0 \\ -3 & -1 & -1 & 1 & 1 \end{array}$
$\binom{1}{-1}$:	$\mathfrak{s}_{4b}(x,y) = \left[\left(\frac{x_0}{y_1} - \frac{x_1 y_1^4}{y_0^5} - \frac{x_2}{y_0} - \frac{x_3 y_1}{y_0^2}\right)\Big/ p_4(x,y)\right]$	
$\binom{1}{-1}$:	$\mathfrak{s}_{4c}(x,y) = \left[\left(\frac{x_0 y_0}{y_1^2} - \frac{x_1 y_1^3}{y_0^4} + \frac{x_2}{y_1} - \frac{x_3}{y_0}\right)\Big/ p_4(x,y)\right]$	equivalent to $p_3(x,y)$
	$p_5(x,y) = p_0(x,y) + x_2 y_0^3 y_1^2 + x_3 y_0^2 y_1^3$	$F_{2,2,1}^{(3)} \approx_{\mathbb{R}} F_{1,1}^{(3)}$
$\binom{1}{-2}$:	$\mathfrak{s}_{5a}(x,y) = \left[\left(\frac{x_0}{y_1^2} - \frac{x_1 y_1^3}{y_0^5} - \frac{x_2}{y_0^2} - \frac{x_3 y_1}{y_0^3}\right)\Big/ p_5(x,y)\right]$	
$\binom{1}{-2}$:	$\mathfrak{s}_{5b}(x,y) = \left[\left(\frac{x_0 y_0^3}{y_1^5} - \frac{x_1}{y_0^2} + \frac{x_2 y_0}{y_1^3} + \frac{x_3}{y_1^2}\right)\Big/ p_5(x,y)\right]$	$\begin{array}{ccccc} X_1 & X_2 & X_3 & X_4 & X_5 \\ \hline 1 & 1 & 1 & 0 & 0 \\ -2 & -2 & -1 & 1 & 1 \end{array}$
$\binom{1}{-1}$:	$\mathfrak{s}_{5c}(x,y) = \left[\left(\frac{x_0 y_0^2}{y_1^3} - \frac{x_1 y_1^2}{y_0^3} + \frac{x_2}{y_1} - \frac{x_3}{y_0}\right)\Big/ p_5(x,y)\right]$	



[BH:1606.07420, 1611.10300 & 2205.12827]
+more



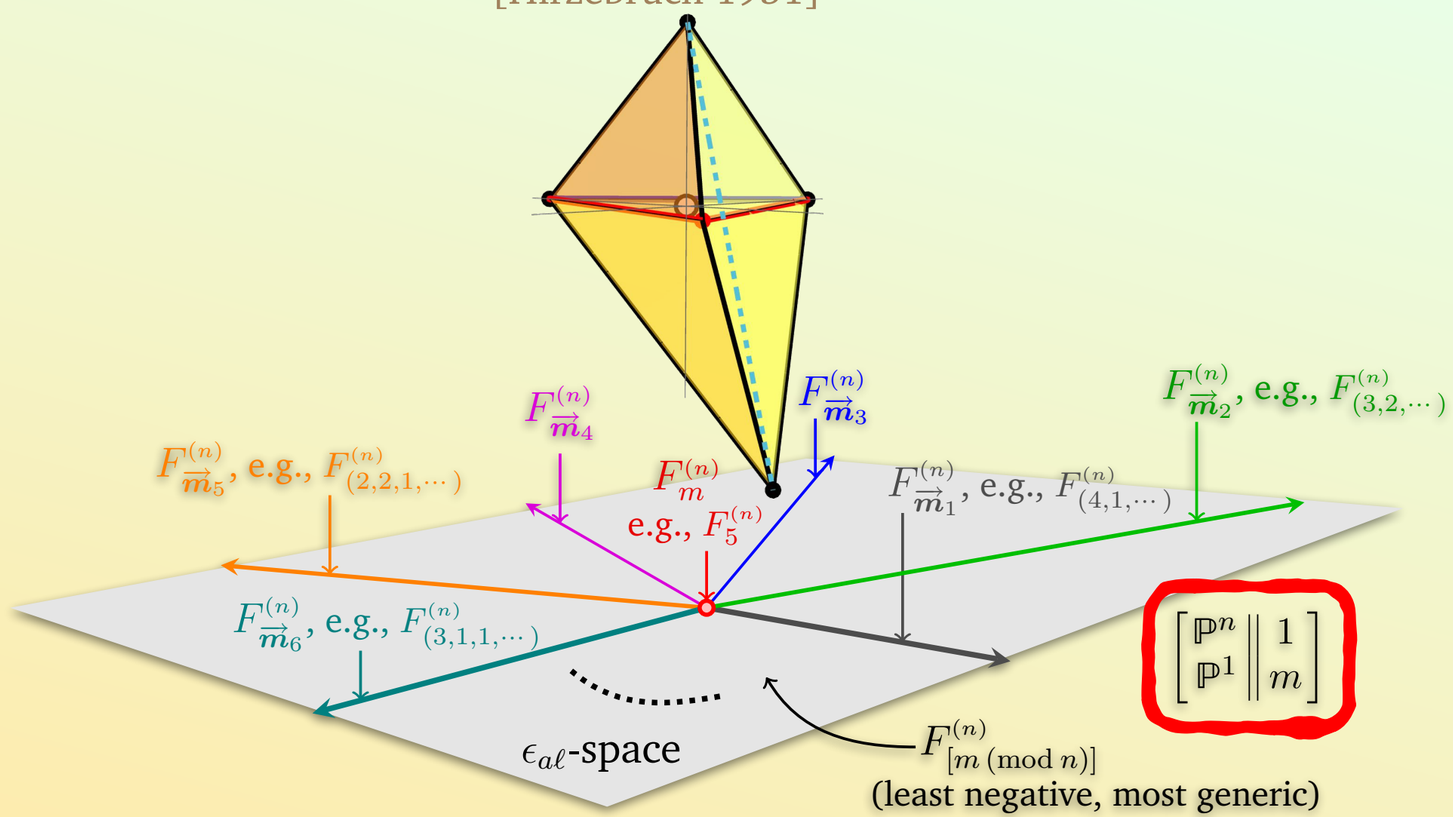
A Family of Scrolls

In a deformation reunion

[BH:1606.07420, 1611.10300 & 2205.12827]
+more

- A deformation *family picture*:

[Hirzebruch 1951]



A Family of Scrolls

In a deformation reunion

[BH:1606.07420, 1611.10300 & 2205.12827]
+more

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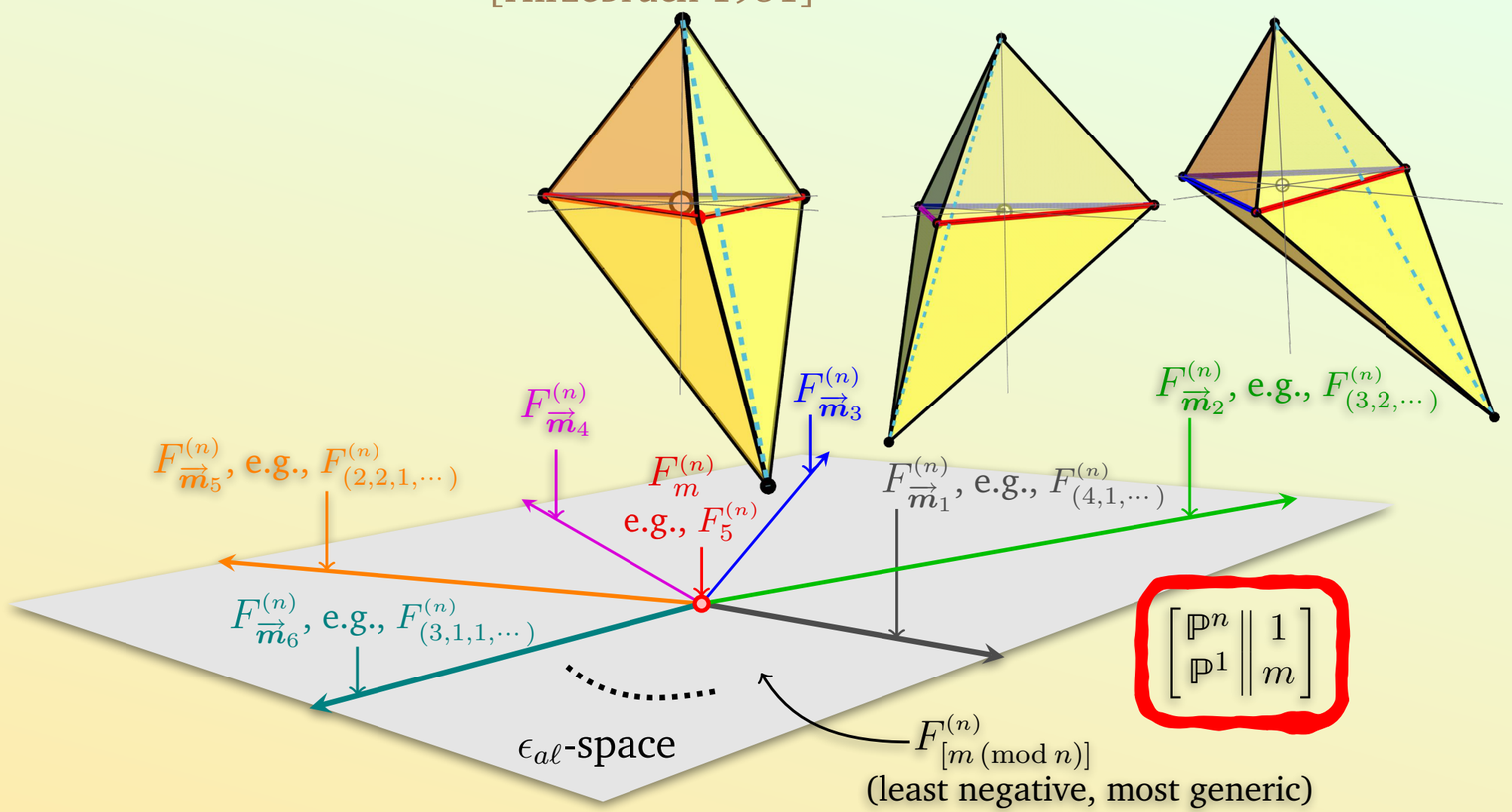


A Family of Scrolls

In a deformation reunion

[BH:1606.07420, 1611.10300 & 2205.12827]
+more

- A deformation *family picture*:

[Hirzebruch 1951]

$F^{(3)}_{(2,2,1)} \approx_{\mathbb{R}} F^{(3)}_{(1,1,0)}$
[~Veronese]

$F^{(n)}_{\vec{m}_4}$ $F^{(n)}_{\vec{m}_3}$ $F^{(n)}_{\vec{m}_2}$, e.g., $F^{(n)}_{(3,2,\cdots)}$

$F^{(n)}_{\vec{m}_5}$, e.g., $F^{(n)}_{(2,2,1,\cdots)}$

$F^{(n)}_{m}$
e.g., $F^{(n)}_{5}$

$F^{(n)}_{\vec{m}_1}$, e.g., $F^{(n)}_{(4,1,\cdots)}$

$F^{(n)}_{\vec{m}_6}$, e.g., $F^{(n)}_{(3,1,1,\cdots)}$

$\epsilon_{a\ell}$-space

$F^{(n)}_{[m \,(\mathrm{mod}\, n)]}$
(least negative, most generic)

$$\left[\begin{array}{c|c} \mathbb{P}^n & 1 \\ \mathbb{P}^1 & m \end{array}\right]$$

A Family of Scrolls

In a deformation reunion

[BH:1606.07420, 1611.10300 & 2205.12827]
+more

- A deformation *family picture*:
- One complete intersection
- to "rule them all" (toric varieties)
- …and CY hypersurfaces therein
- …and a *web* of connected GLSMs

[Hirzebruch 1951]

$F^{(3)}_{(2,2,1)} \approx_{\mathbb{R}} F^{(3)}_{(1,1,0)}$ [~Veronese]

$F^{(n)}_{\vec{m}_4}$

$F^{(n)}_{\vec{m}_3}$

$F^{(n)}_{\vec{m}_2}$, e.g., $F^{(n)}_{(3,2,\cdots)}$

$F^{(n)}_{\vec{m}_5}$, e.g., $F^{(n)}_{(2,2,1,\cdots)}$

$F^{(n)}_{m}$
e.g., $F^{(n)}_{5}$

$F^{(n)}_{\vec{m}_1}$, e.g., $F^{(n)}_{(4,1,\cdots)}$

$F^{(n)}_{\vec{m}_6}$, e.g., $F^{(n)}_{(3,1,1,\cdots)}$

$\epsilon_{a\ell}$-space

$F^{(n)}_{[m \,(\mathrm{mod}\, n)]}$
(least negative, most generic)

$\left[\begin{array}{c|c} \mathbb{P}^n & 1 \\ \mathbb{P}^1 & m \end{array}\right]$

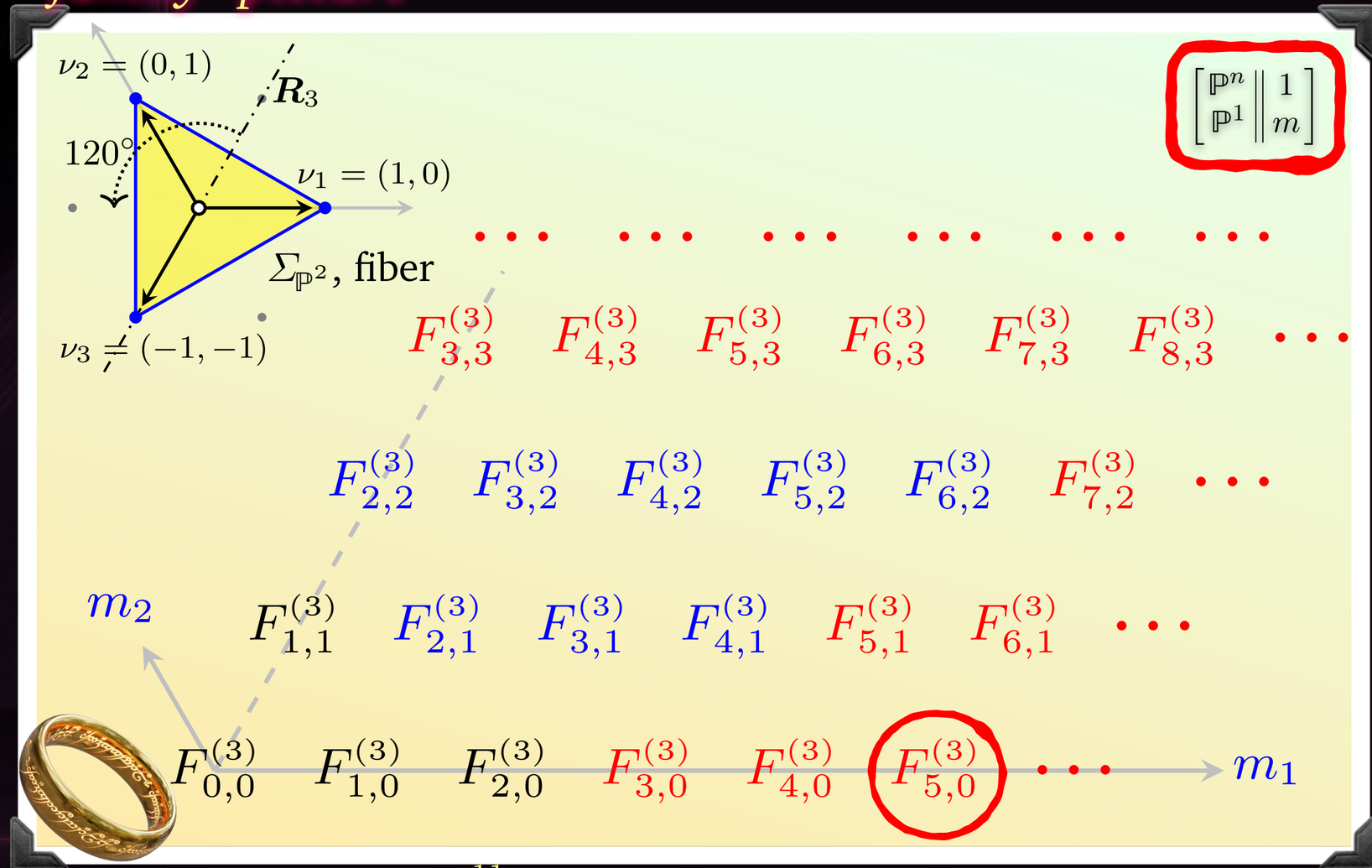
A Family of Scrolls

…with combinatorial counterpoint

[BH:1606.07420,… 2605.06998]

- A dihedral deformation *family picture*:

- 1951, Hirzebruch; here, $m = 5$



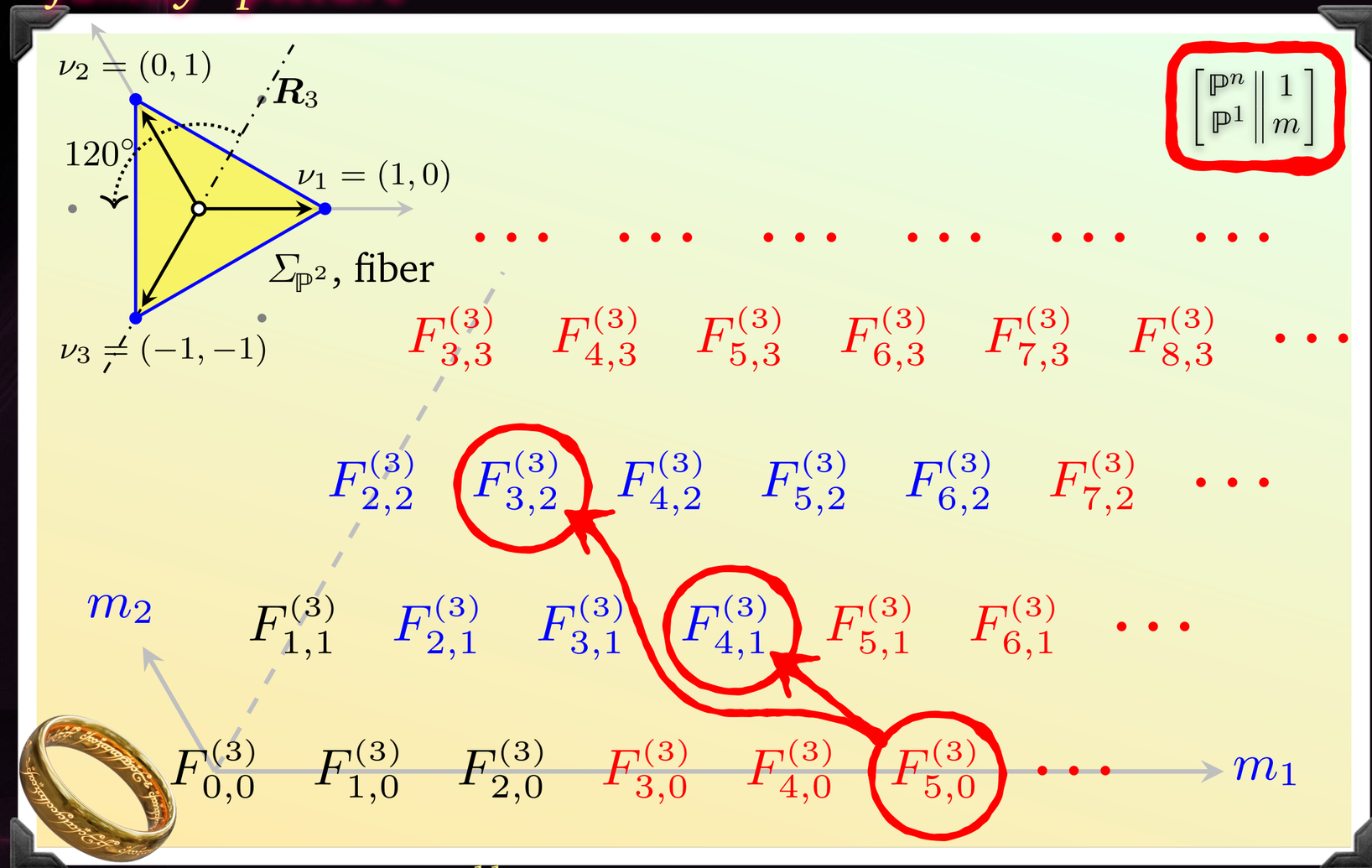
A Family of Scrolls

…with combinatorial counterpoint

[BH:1606.07420,… 2605.06998]

- A dihedral deformation *family picture*:

- 1951, Hirzebruch; here, $m = 5$
- solid arrows: explicit deformations, by trial and directrix identification



A Family of Scrolls

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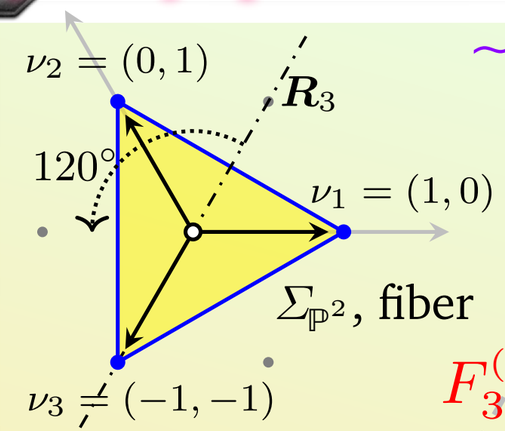
[BH:1606.07420,… 2605.06998]

- A dihedral deformation *family picture*:

$F^{(4)}_{2,1,1,0} \rightsquigarrow F^{(4)}_{0,0,0,0}$



~ Gross [alg-geom/9402014]
& Ruan [*DMJ* **83**.2 (1996) 461]

$\begin{bmatrix} \mathbb{P}^n & 1 \\ \mathbb{P}^1 & m \end{bmatrix}$

~ Wall & Veronese: $\vec{m} \rightsquigarrow \vec{m} \bmod n$

$F^{(3)}_{3,3}$ $F^{(3)}_{4,3}$ $F^{(3)}_{5,3}$ $F^{(3)}_{6,3}$ $F^{(3)}_{7,3}$ $F^{(3)}_{8,3}$ …

$F^{(3)}_{2,2}$ $F^{(3)}_{3,2}$ $F^{(3)}_{4,2}$ $F^{(3)}_{5,2}$ $F^{(3)}_{6,2}$ $F^{(3)}_{7,2}$ …

m_2

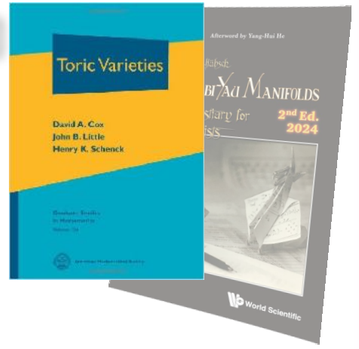
$F^{(3)}_{1,1}$ $F^{(3)}_{2,1}$ $F^{(3)}_{3,1}$ $F^{(3)}_{4,1}$ $F^{(3)}_{5,1}$ $F^{(3)}_{6,1}$ …

$F^{(n)}_{3,1,1} \rightsquigarrow F^{(n)}_{2,0,0}$

$F^{(3)}_{0,0}$ $F^{(3)}_{1,0}$ $F^{(3)}_{2,0}$ $F^{(3)}_{3,0}$ $F^{(3)}_{4,0}$ $F^{(3)}_{5,0}$ … m_1

- 1951, Hirzebruch; here, $m = 5$
- solid arrows: explicit deformations, by trial and directrix identification
- dashed arrows: non-algebraic map (Wall "mod m" diffeo; (…perhaps…) ~Veronese-twisted, smooth-deformed)

[BH:1606.07420,… 2605.06998]

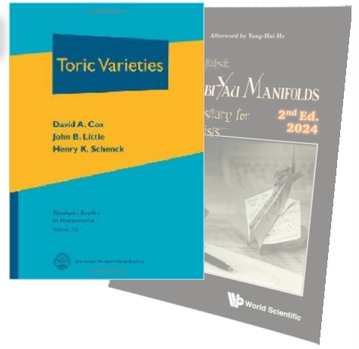


and Their Offspring

…back to the main motif

- CY: $F_m^{(n)}[c_1]$, i.e., $F_m^{(n)} \in \left[\begin{matrix}\mathbb{P}^n \\ \mathbb{P}^1\end{matrix}\middle\| \begin{matrix}1 \\ m\end{matrix} \middle| \begin{matrix}n \\ 2-m\end{matrix}\right]$, i.e., need $\deg = \binom{n}{2-m}$
- $f_0(X) = \Pi X := (X_1 = \mathfrak{s}(x,y))\cdots(X_n = x_n)(X_{n+1} = y_0)(X_{n+2} = y_1)$ = "fundamental monom."
  - is not transverse & $\{\Pi X = 0\}$ is singular ("$(n+1)$-horon")

[BH:1606.07420,… 2605.06998]



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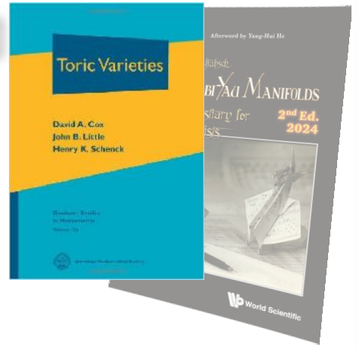
  where $\deg[\mathfrak{m}_i(X)] = \deg[X_i]$ & $\partial_i \mathfrak{m}_i(X) = 0$ (*~Demazure roots*)

[BH:1606.07420,… 2605.06998]



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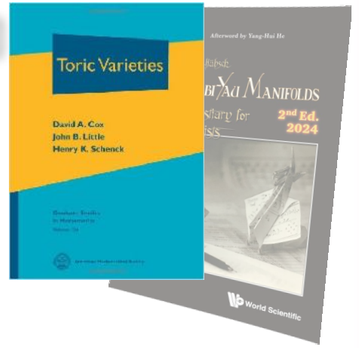
  - in QM, $|n\rangle^{(1)} = \Sigma_{m\neq n} c_m |m\rangle^{(0)}$; also, $X_i = 0 \not\Longrightarrow \delta_i \Pi X = 0$
  - $(\delta_i \Pi X) \cap (\delta_j \Pi X)$ are X_i- *and* X_j-independent and delimit $[\delta_i \Pi X]$ & $[\delta_j \Pi X]$ so $\langle X_i \rangle, \langle X_j \rangle = 0 \not\Rightarrow \langle [\delta_i \Pi X] \cap [\delta_j \Pi X] \rangle = 0$; etc.

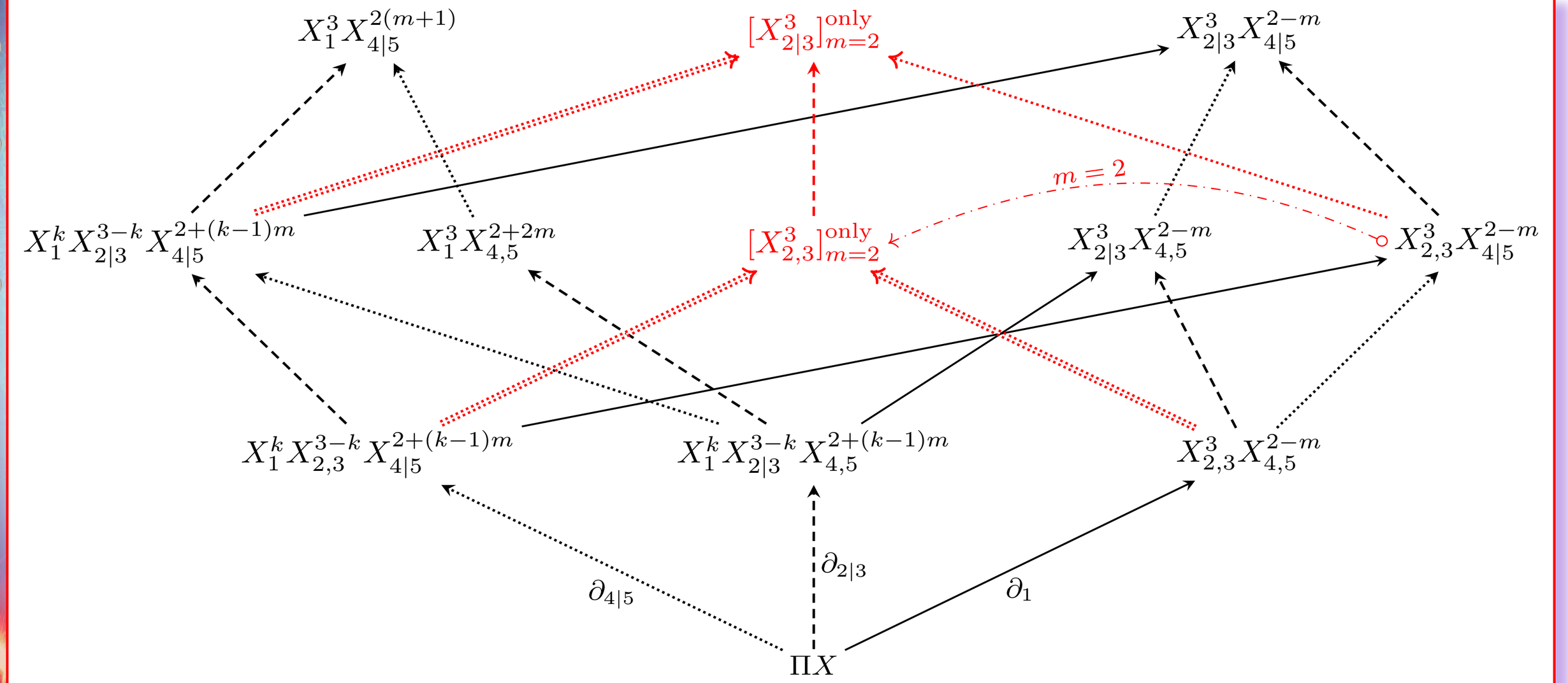
they form a poset

[BH:1606.07420,… 2605.06998]

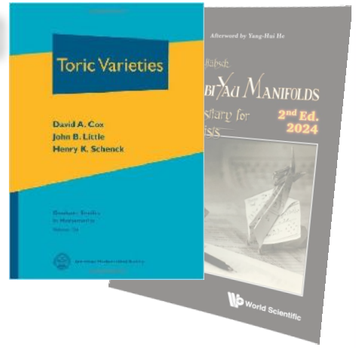


and Their Offspring

…back to the main motif



[BH:1606.07420,… 2605.06998]



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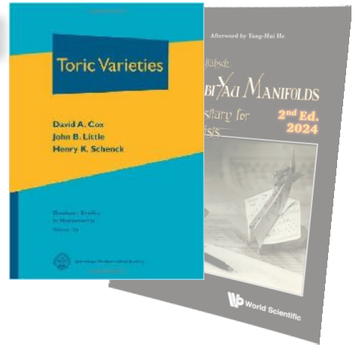
  they form a poset
- Combinatorially, $\Pi X \sqcup \bigsqcup_i [\delta_i \Pi X] \overset{\nabla}{\simeq} \{\Pi X = 0 \,|\, \text{S.R.}, \ldots\}$
- Also, $\partial_i \perp \delta_i \Pi X$ & generate a dual poset

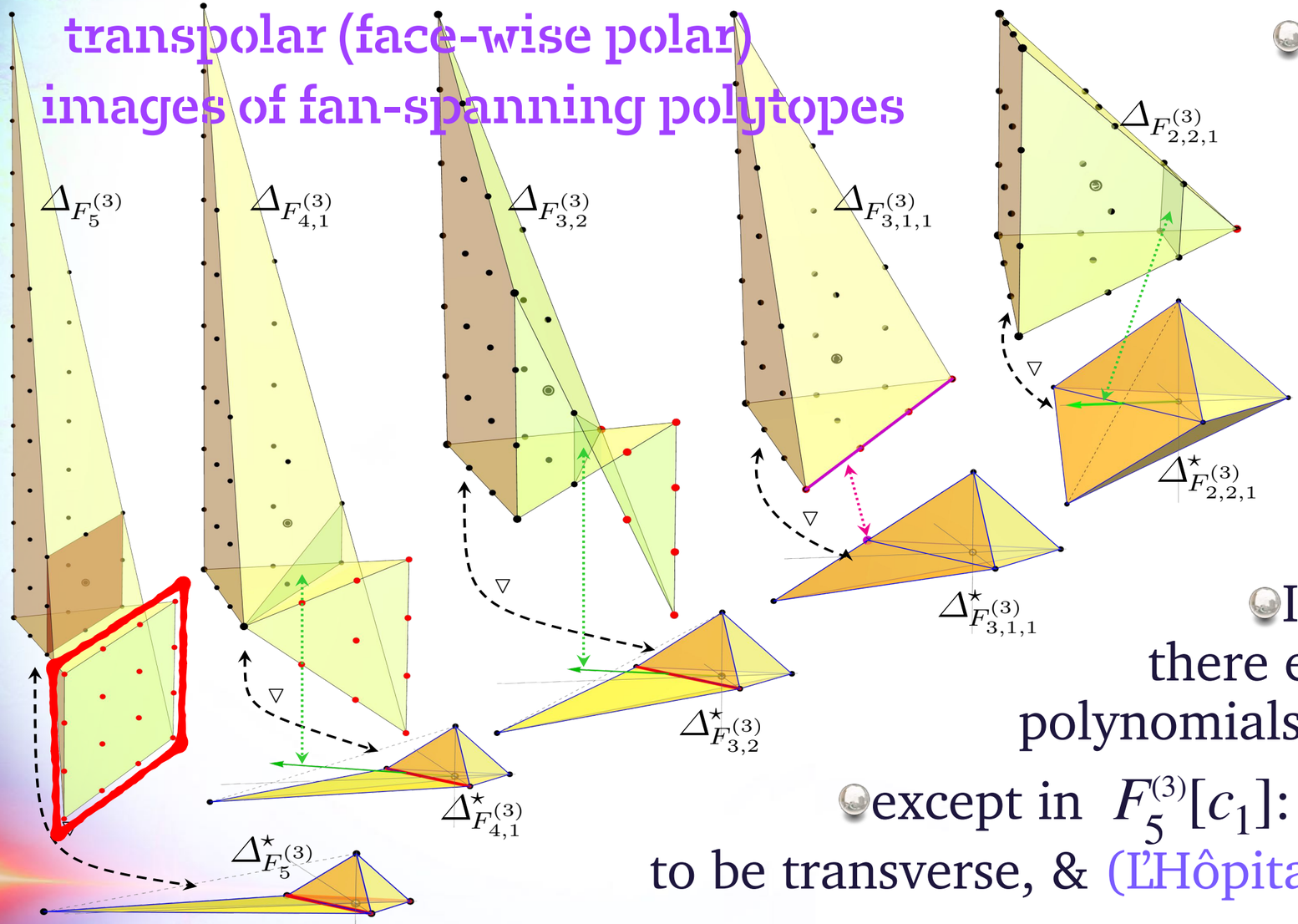
[BH:1606.07420,… 2605.06998]



and Their Offspring

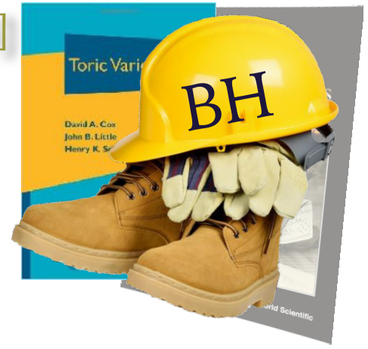
(bird's eye view)

transpolar (face-wise polar) images of fan-spanning polytopes



- Monomial collections with $U(1)^2$-charges of ΠX form posets, with "deformation fronts" = facets
  - The shape changes throughout the ϵ_{ik}-space indicate that — *convexity is overrated*
- In all deformed models, there exist transverse *regular* polynomials (*nonnegative exponents*)…
- except in $F_5^{(3)}[c_1]$: requires rational monomials to be transverse, & (L'Hôpital/intrinsic limit) smooth

[BH:1606.07420,… 2605.06998]



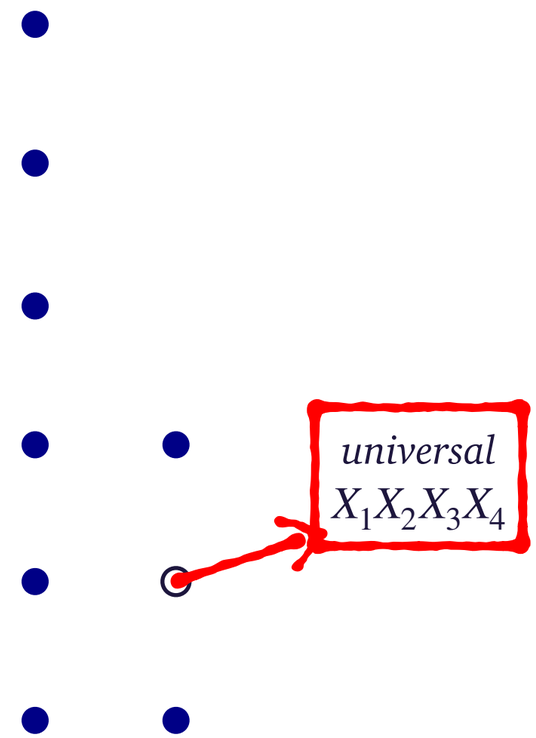
…in the Mirror

& Non-Convex Mirrors

—2D *Proof-of-Concept*—

- $X_1^{\,2} X_2^{\,0} (X_3 \oplus X_4)^{2+1m} \;\oplus\; X_1^{\,1} X_2^{\,1} (X_3 \oplus X_4)^{2+0m}$

$m=3$



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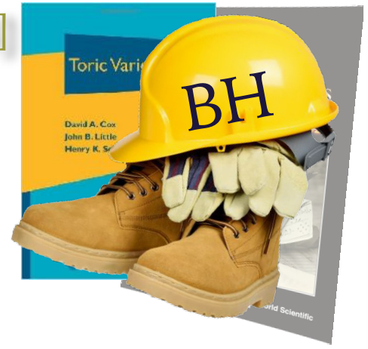

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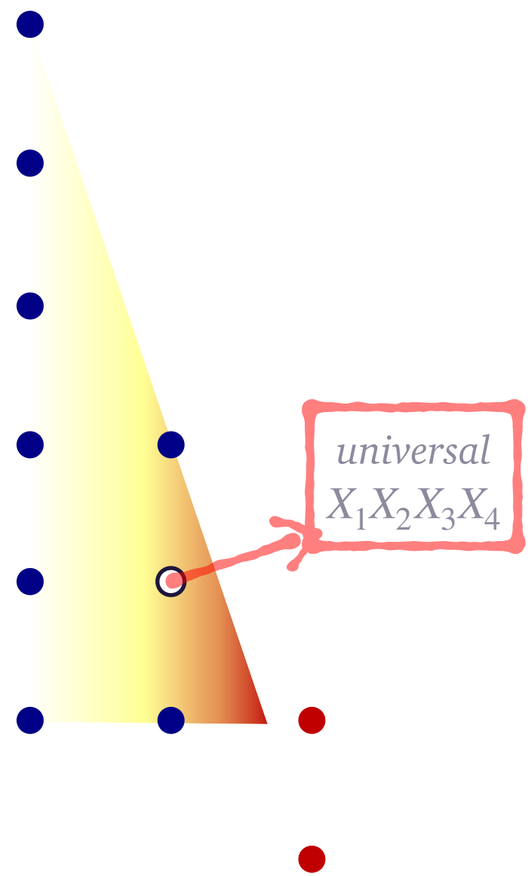

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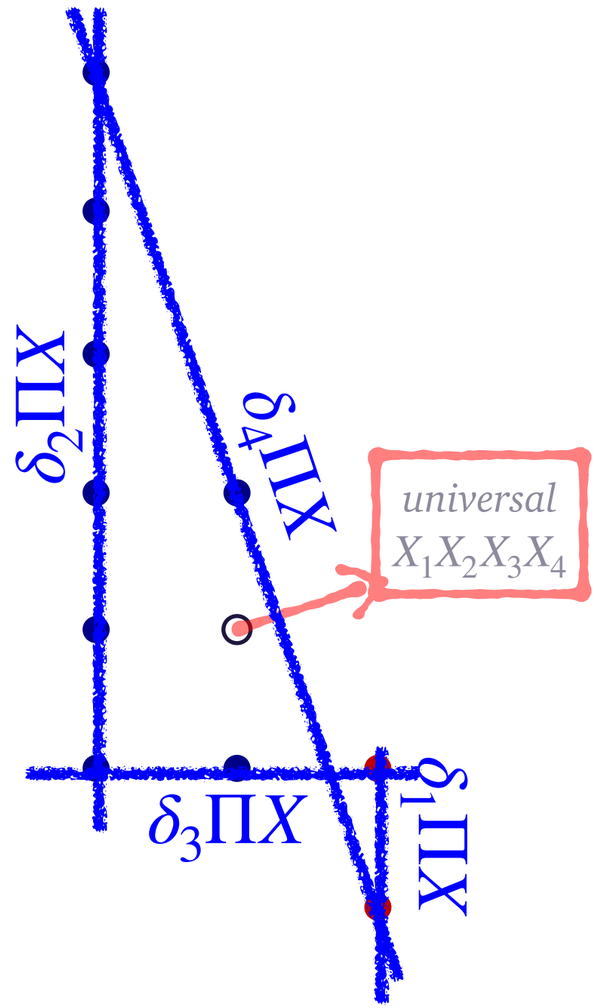

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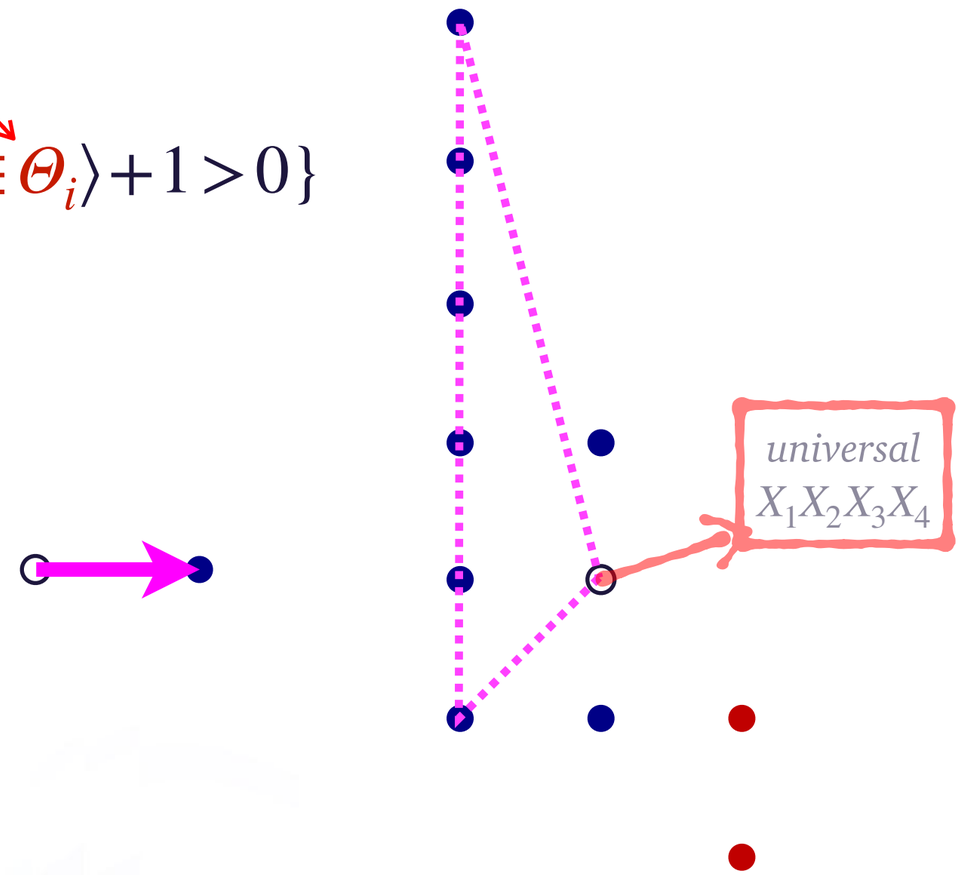

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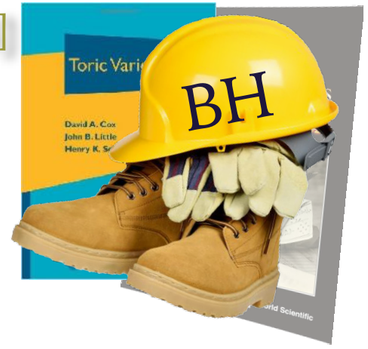

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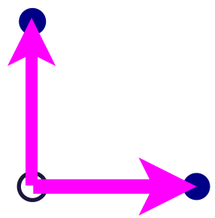

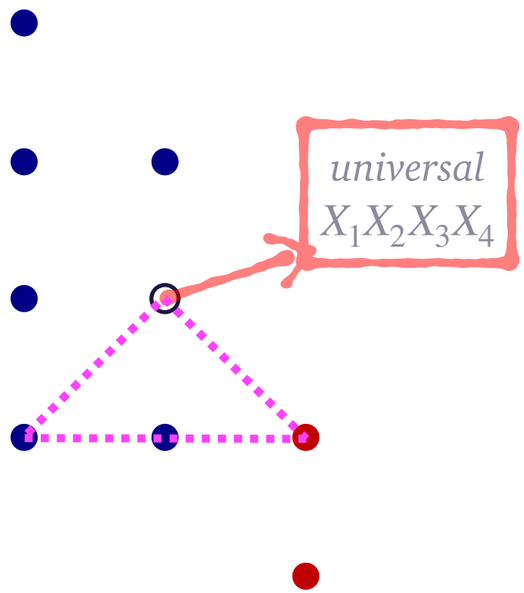

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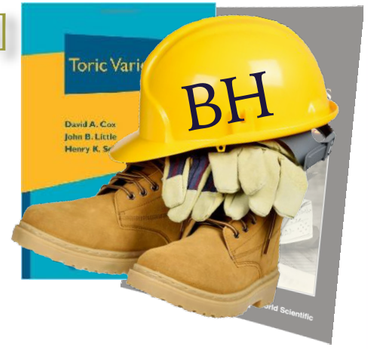

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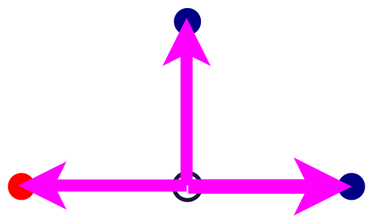

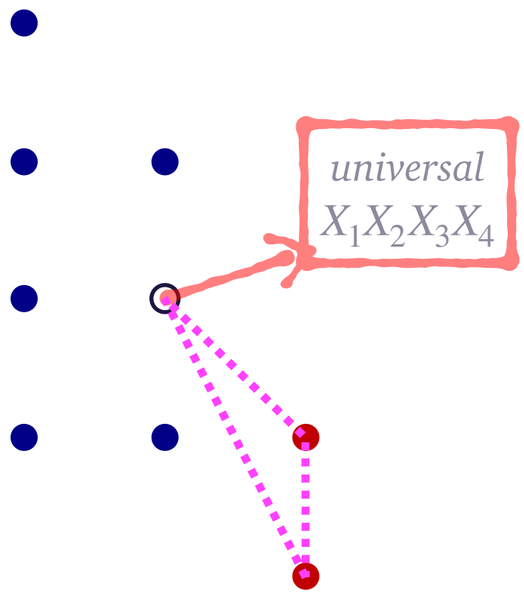

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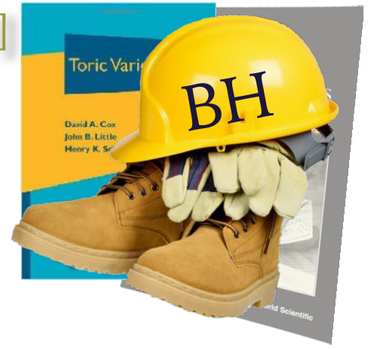

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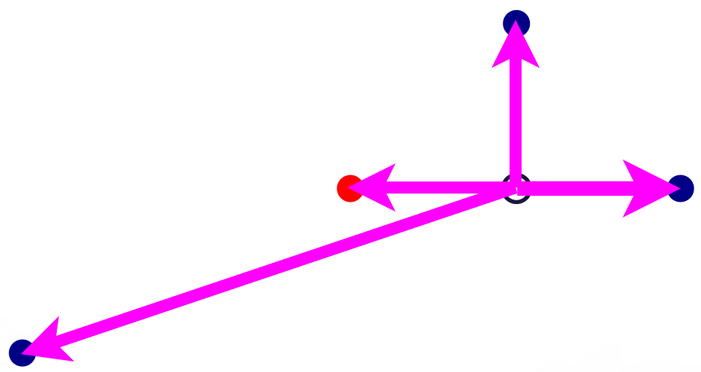

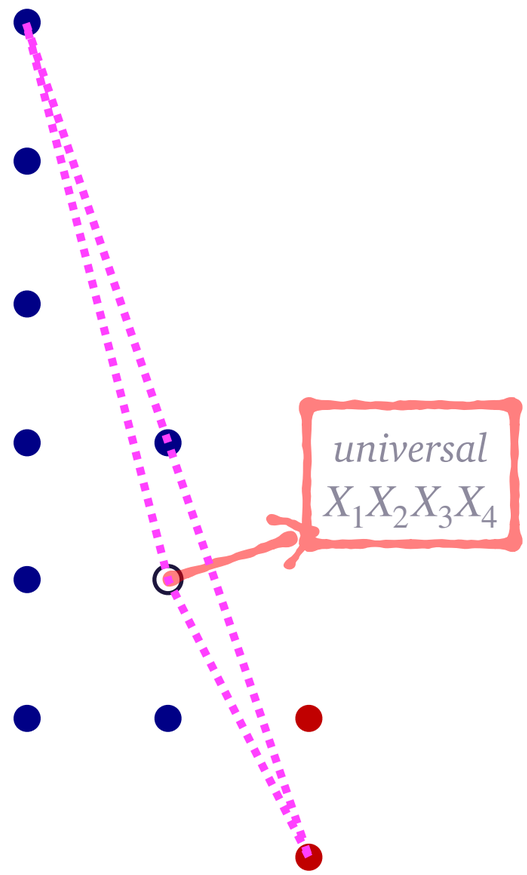

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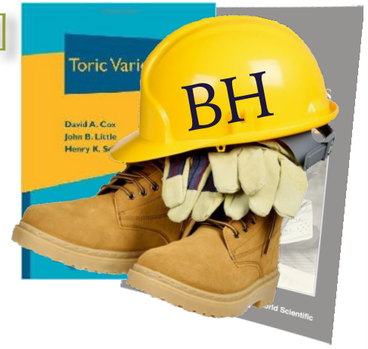

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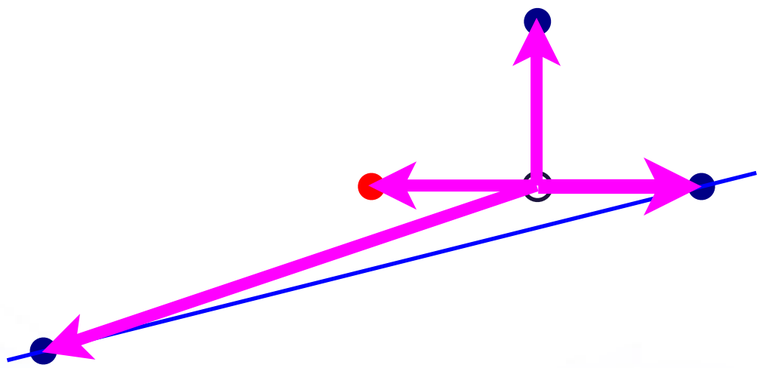

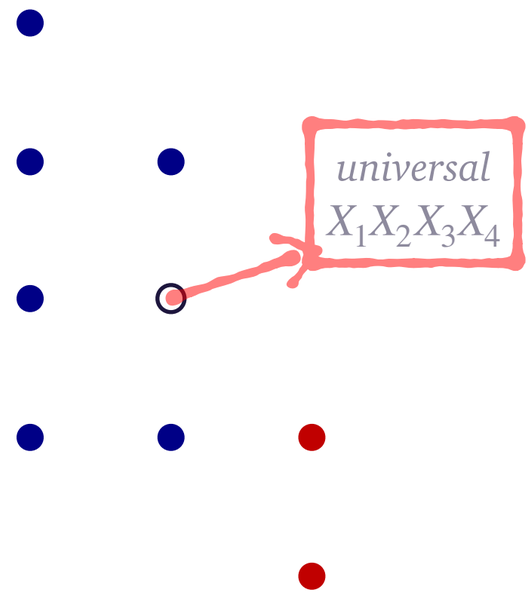

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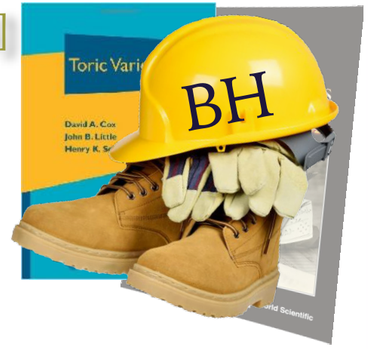

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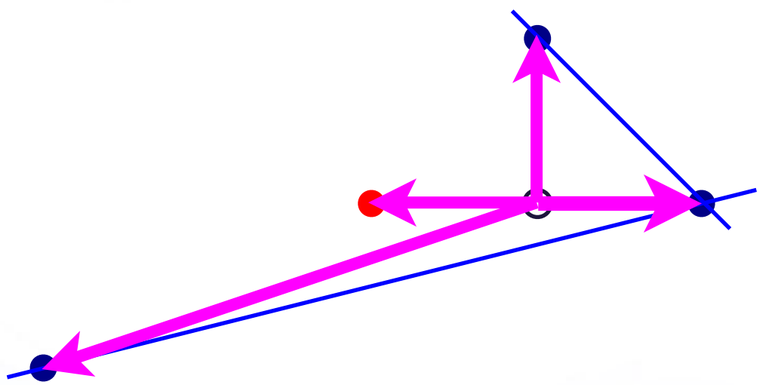
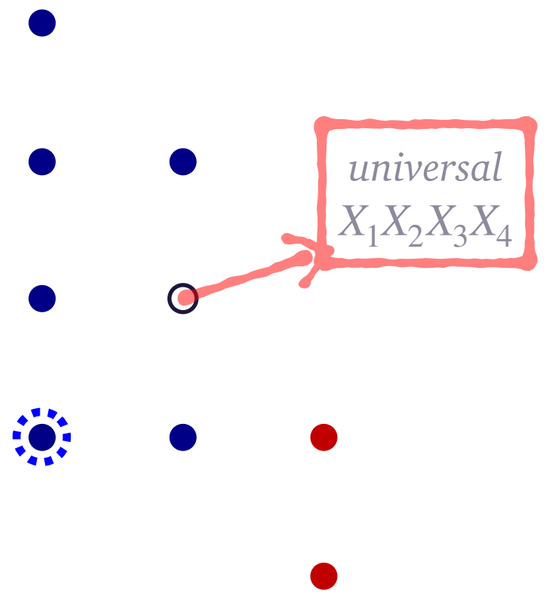
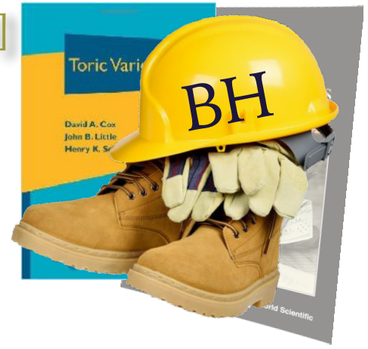




[BH:1606.07420,… 2605.06998]



…in the Mirror

& Non-Convex Mirrors

—2D *Proof-of-Concept*—

$$X_1^{\,2}X_2^{\,0}(X_3\oplus X_4)^{2+1m} \;\oplus\; X_1^{\,1}X_2^{\,1}(X_3\oplus X_4)^{2+0m} \;\oplus\; X_1^{\,0}X_2^{\,2}(X_3\oplus X_4)^{2-1m} \qquad m=3$$

- *Trans*polar (≈ dual):
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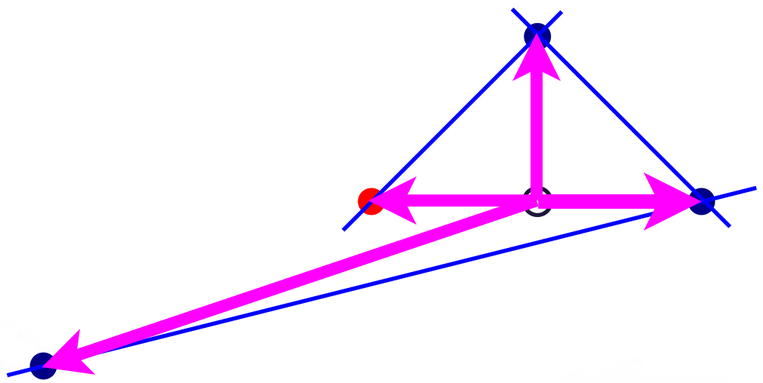
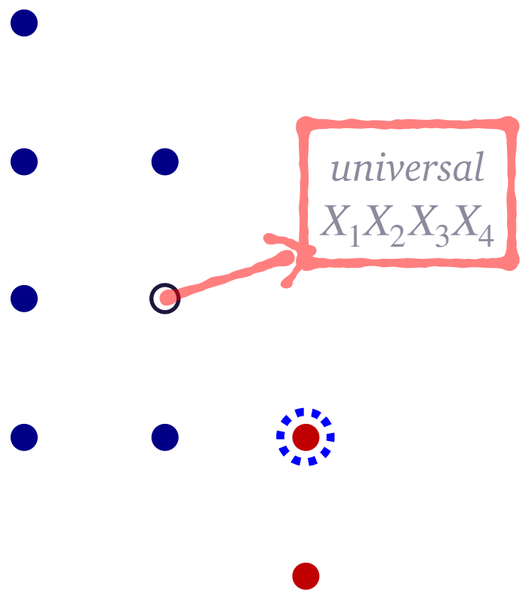
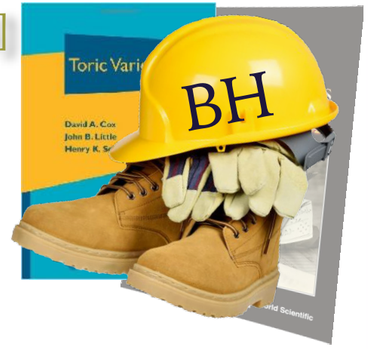




[BH:1606.07420,… 2605.06998]



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& Non-Convex Mirrors

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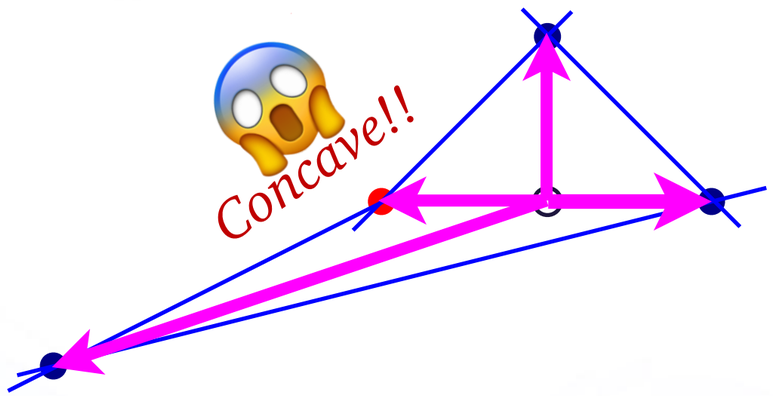

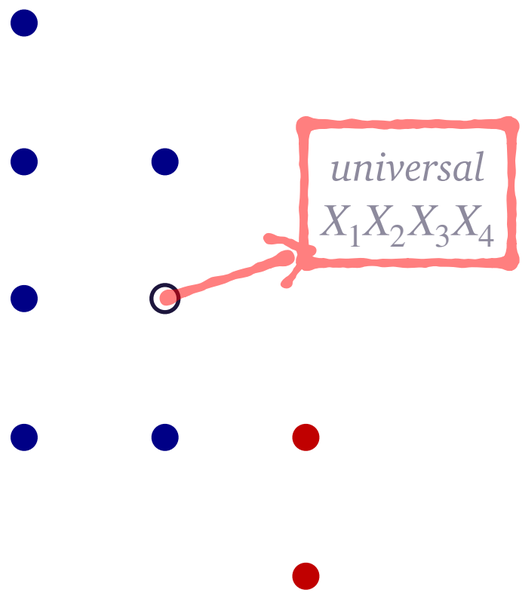

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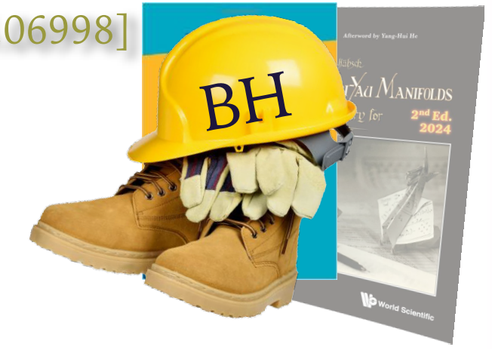

…in the Mirror

& Non-Convex Mirrors

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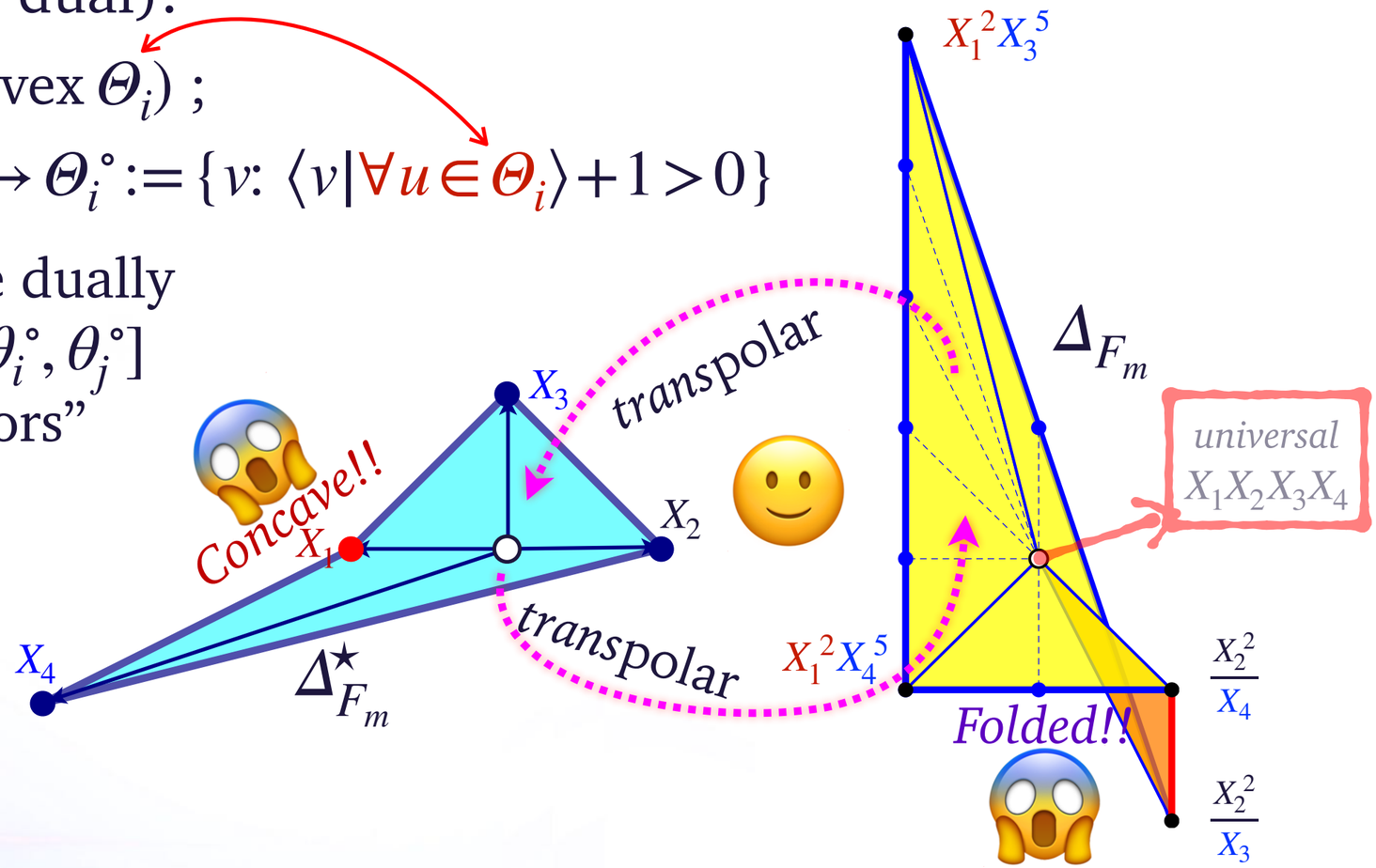

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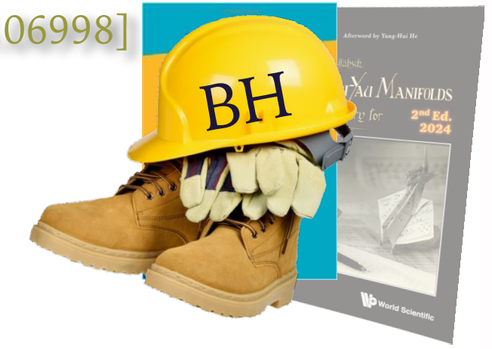

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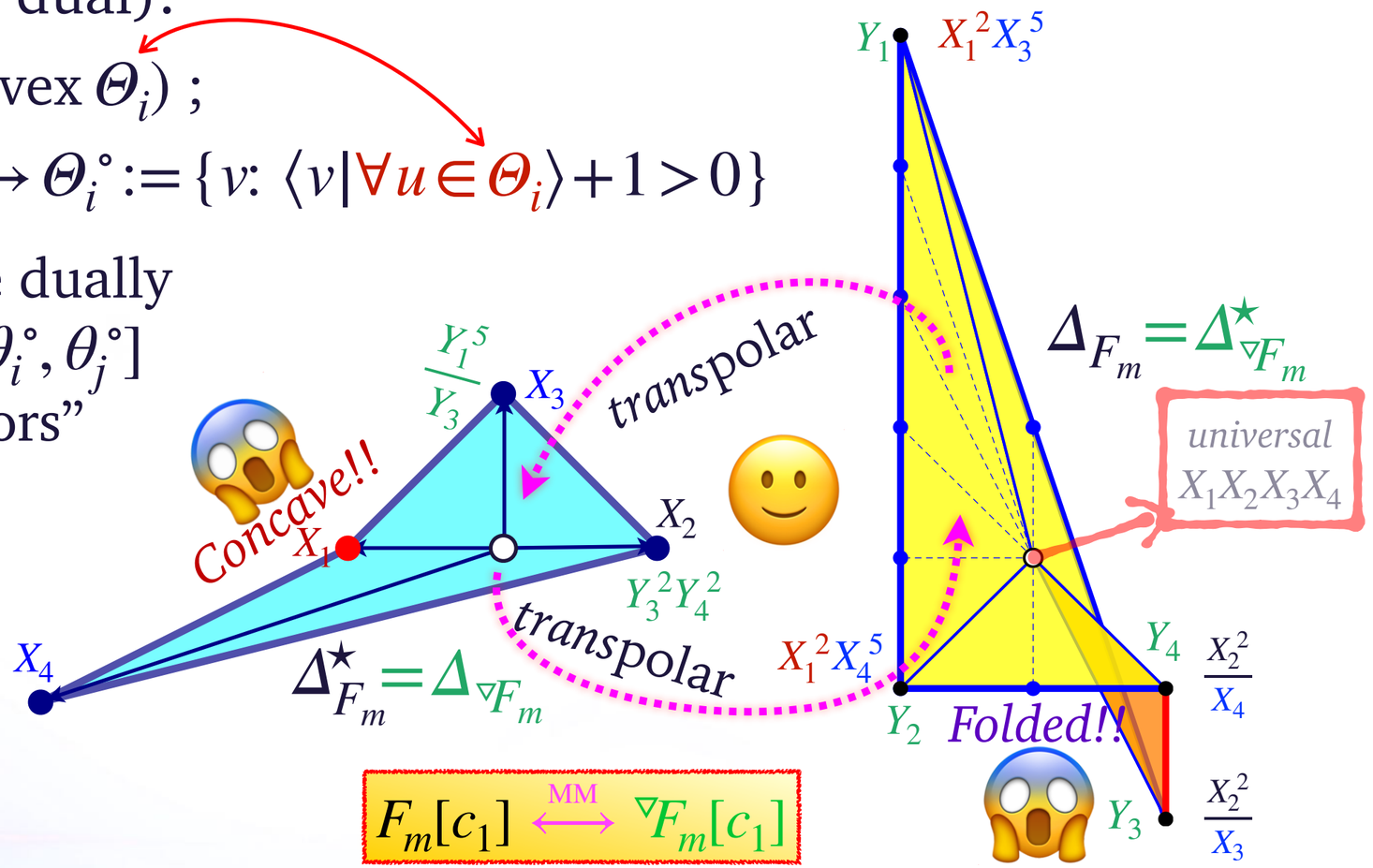
- Swap roles $X_i \leftrightarrow Y_I$



$$\mathfrak{E}_{ij} = \langle u_i, v_j \rangle + 1 \qquad u_i \in \Delta^\star_X,\ v_j \in \Delta_X$$

mirror=transposition

$$F_m[c_1] \overset{\text{MM}}{\longleftrightarrow} {}^\nabla F_m[c_1]$$

[BH:1606.07420,… 2605.06998]

…in the Mirror

& Non-Convex Mirrors

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- Swap roles $X_i \leftrightarrow Y_I$

$$f(X) = X_1^{\,2}X_3^{\,5} + X_1^{\,2}X_4^{\,5} + X_2^{\,2}X_4^{-1} + X_2^{\,2}X_3^{-1}$$
$$f^{\mathsf{T}}(Y) = Y_1^{\,2}Y_2^{\,2} + Y_3^{\,2}Y_4^{\,2} + Y_1^{\,5}Y_4^{-1} + Y_2^{\,5}Y_3^{-1}$$

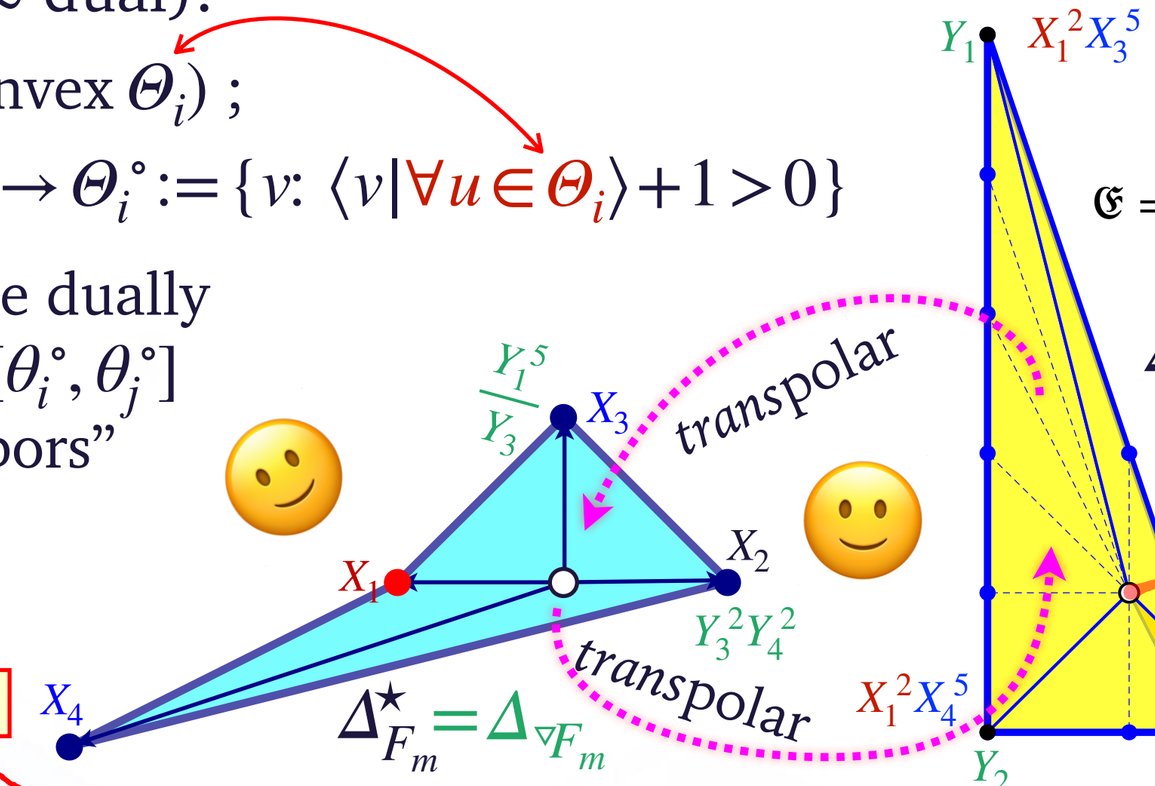
$$\mathfrak{G} = \begin{bmatrix} 2 & 0 & 5 & 0 \\ 2 & 0 & 0 & 5 \\ 0 & 2 & 0 & -1 \\ 0 & 2 & -1 & 0 \end{bmatrix}$$

$$\mathfrak{G}_{ij} = \langle u_i, v_j \rangle + 1 \qquad u_i \in \Delta_X^\star,\ v_j \in \Delta_X$$
mirror=transposition

$\Delta_{F_m} = \Delta^\star_{\nabla F_m}$

rk[$\mathfrak{G}$] = 3, so need to chisel both $\Delta^\star_{F_3}$ and also $\Delta_{F_3} =: \Delta^\star_{\nabla F_3}$ to 0-enclosing simplices

Who's ……
$\nabla F_m^{(n)}$?! ……
…



universal $X_1X_2X_3X_4$

'92: Khovanskii +Pukhlikov
'93: Karshon +Tolman
'99: Hattori +Masuda +lots of

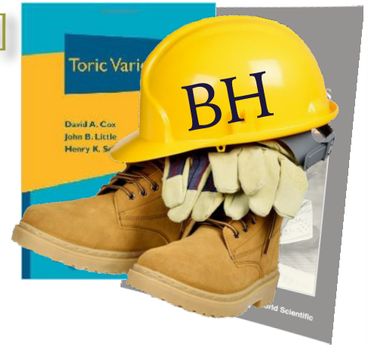
(pre)complex algebraic geometry

~~con~~VEX polytopes

$F_m[c_1] \overset{\text{MM}}{\longleftrightarrow} \nabla F_m[c_1]$

(pre)symplectic geometry

[BH:1606.07420,… 2605.06998]



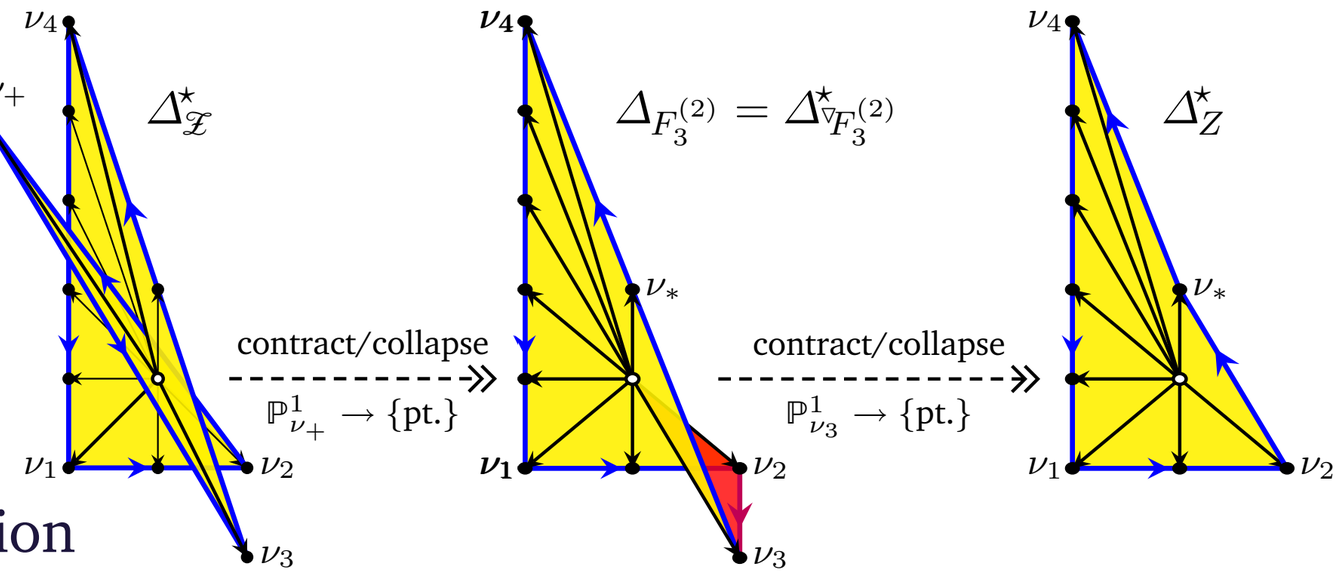
…in the Mirror

& Non-Convex Mirrors

—2D *Proof-of-Concept*—

$m=3$

- What of self-crossing multifans & multitopes that span them?
- Good question!
  - "sandwiched" between toric varieties (far right)
  - and (unitary) torus manifolds (far left) with $(S^1)^n$-equivariant almost complex structure
- $\mathbb{C}^* \approx S^1 \times \mathbb{R}_+$ = "circular" & "radial" action
- Topological Torus Manifolds? [Ishida, Fukukawa & Masuda, 2013]
  - Top cone orientation ↔ rel. orientation btw. M & cpx$[T_M \otimes \underline{\mathbb{R}}^{2k}]$…
  - They have cohomology rings generated by Cox-like coords & "$\mathscr{K}_M^*$"
- Guess: *obverse top-cones* obstruct global almost complex structure (…*& break supersymmetry…?*)



top cones encode characteristic points; obverse top cones encode points where complex structure "breaks"

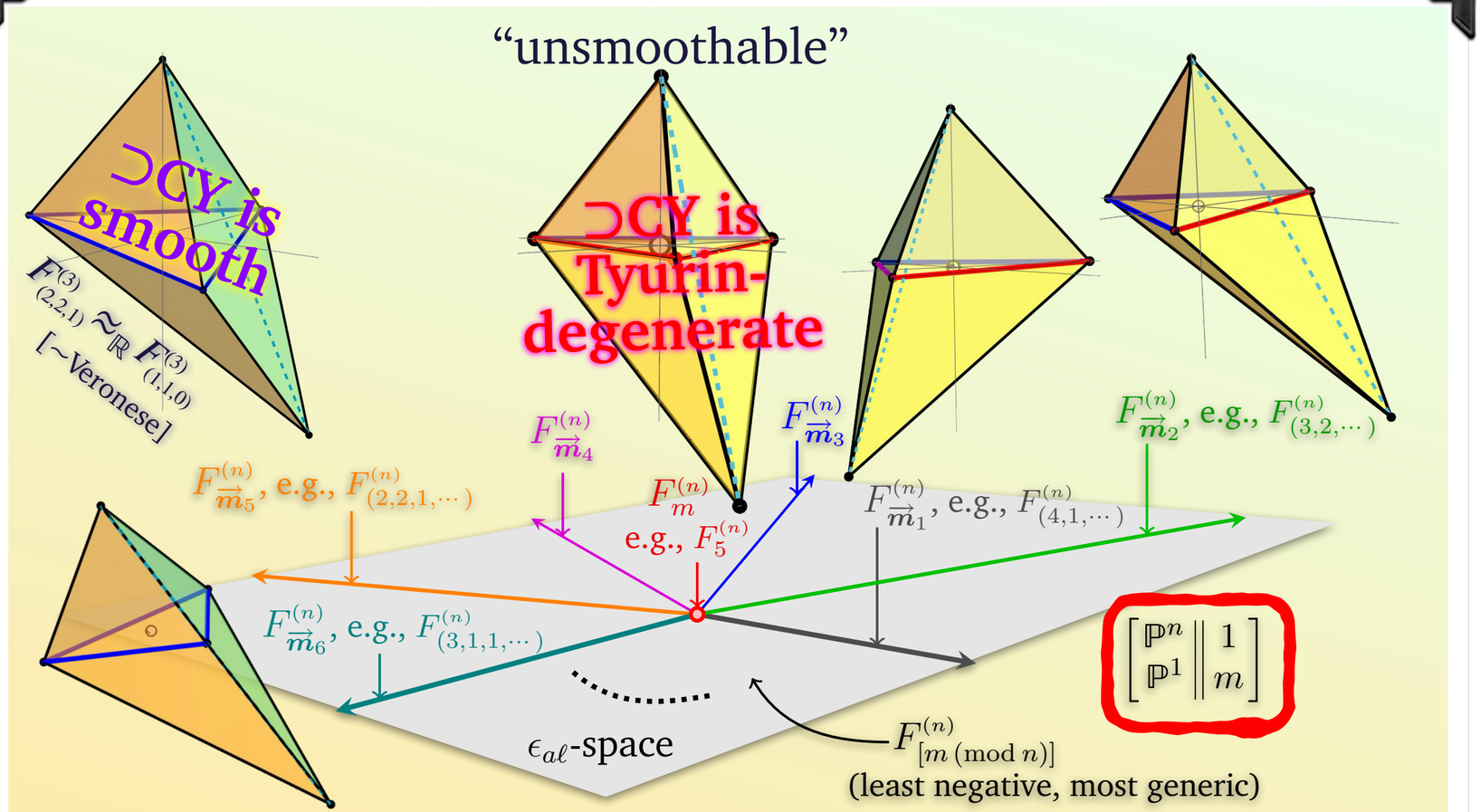
[BH:1606.07420,… 2605.06998]

…in the Mirror

Summary

—Proof-of-Concept—

- CY($n-1$)-folds in Hirzebruch n-folds, $X_m^{(n-1)} \in F_m^{(n)}[c_1]$



A deformation family picture

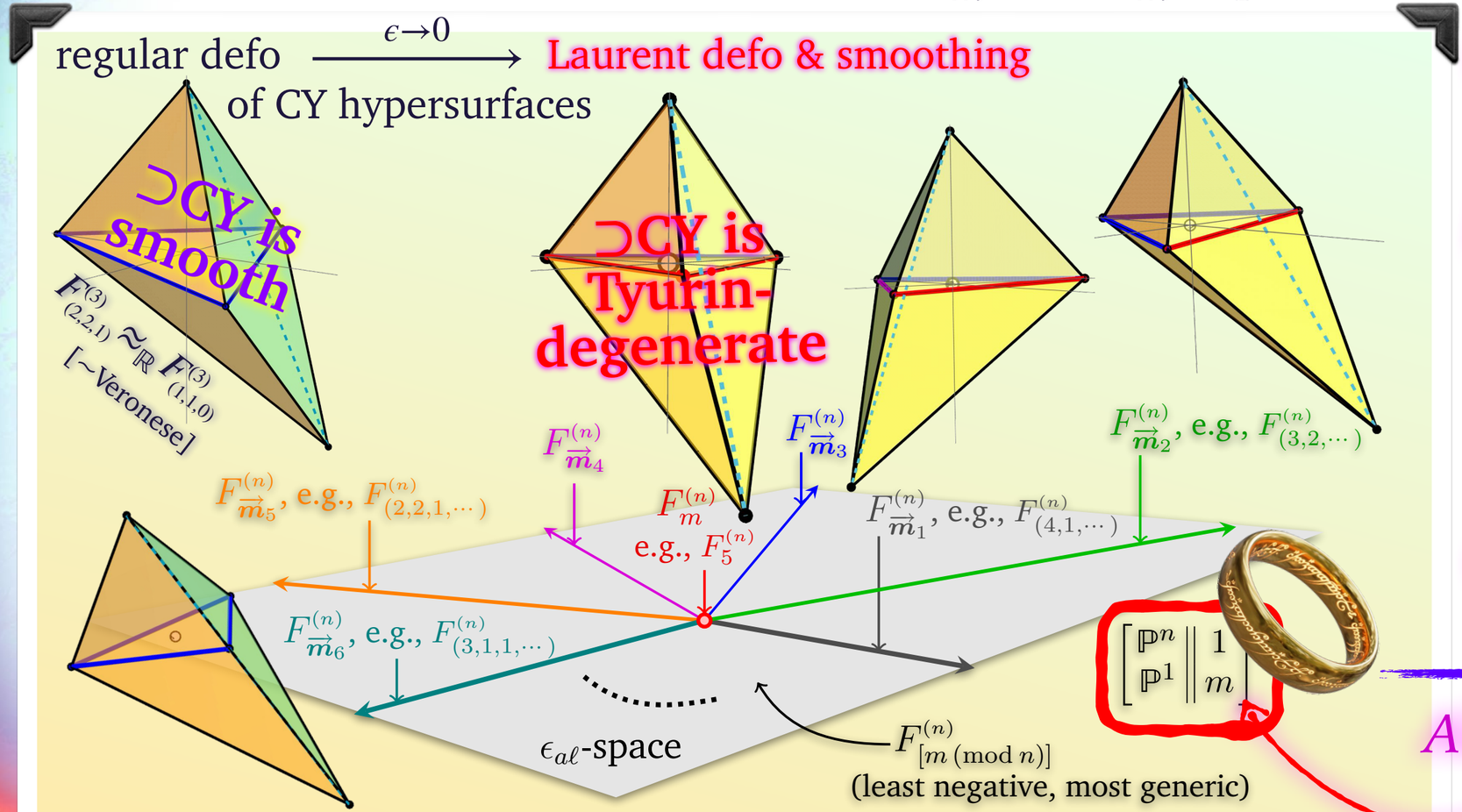
[BH:1606.07420,… 2605.06998]

…in the Mirror

Summary

—Proof-of-Concept—

- CY($n-1$)-folds in Hirzebruch n-folds, $X_m^{(n-1)} \in F_m^{(n)}[c_1]$



A deformation family picture

Each distinct $F^{(n)}_{\vec{m}}$ harbors a combinatorial web of transposition mirror models

In generic $F^{(n)}_{\vec{m}}$, the ΠX has plenty of regular smoothing deformations (with only nonnegative exponents)

No regular smoothing deformations remain in the central model: only rational monomials remain

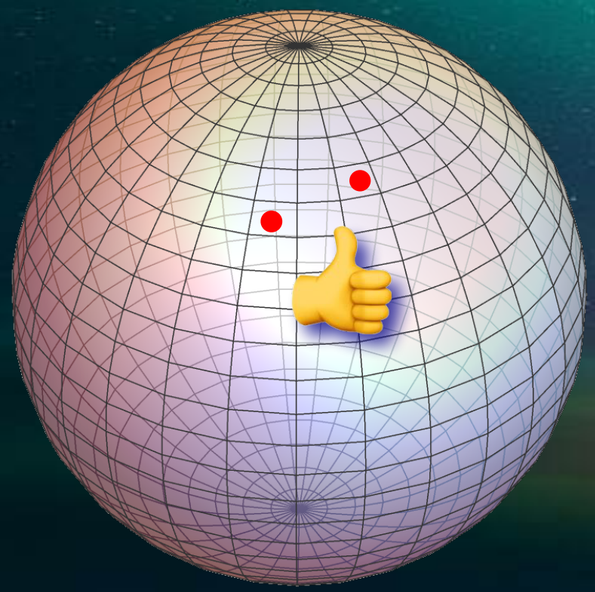
All these distinct toric varieties are found within the same explicit deformation family

How Hard Can it Be?

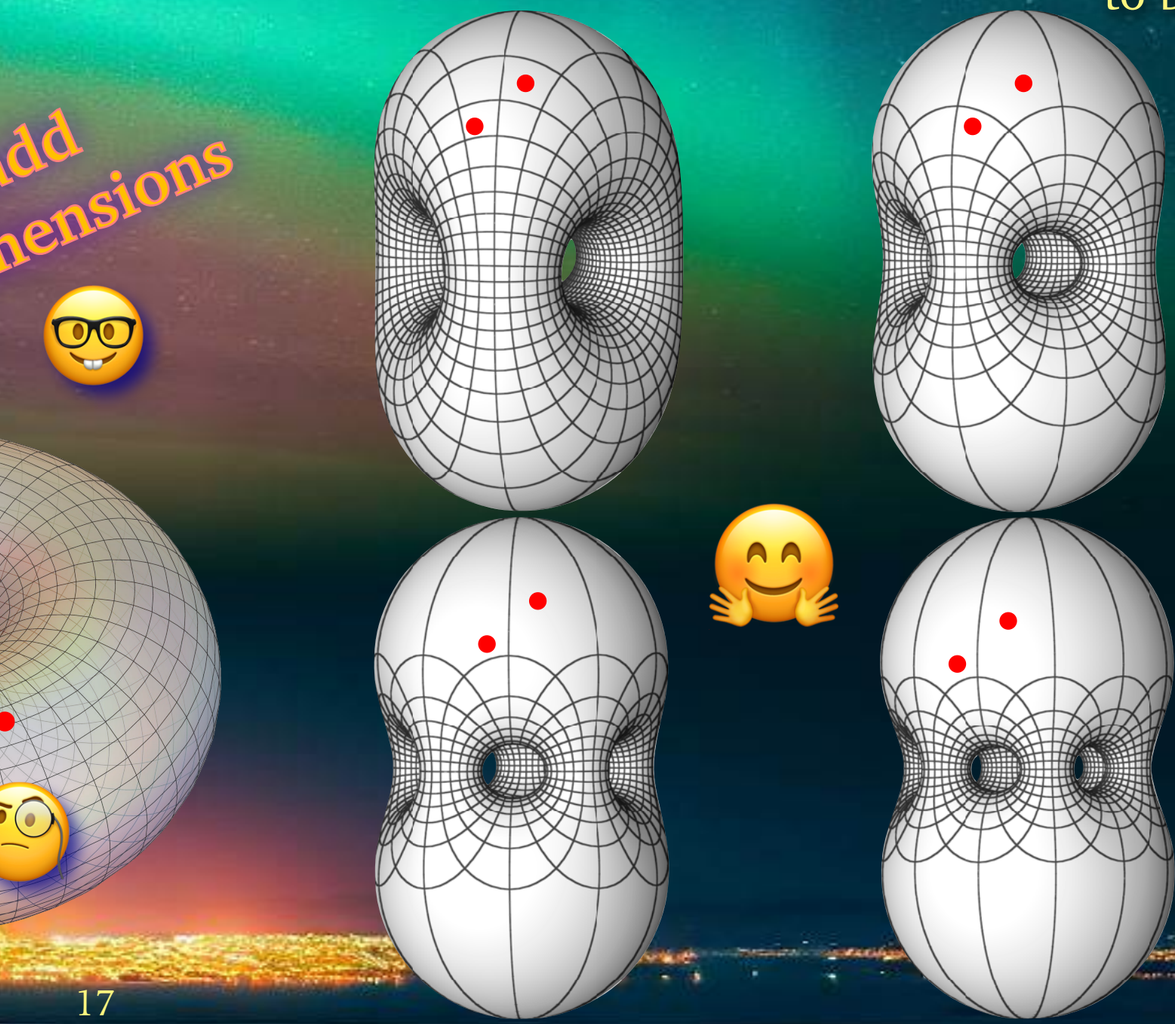
Constructing CY ⊂ Some "Nice" Ambient Space

- Reduce to 0 dimensions: $\mathbb{P}^4[5] \to \mathbb{P}^3[4] \to \mathbb{P}^2[3] \to \mathbb{P}^1[2]$

1611.10300
2205.12827
2403.07139
2501.11684
2502.08002
2605.06998
2605.24724
to be cont'd

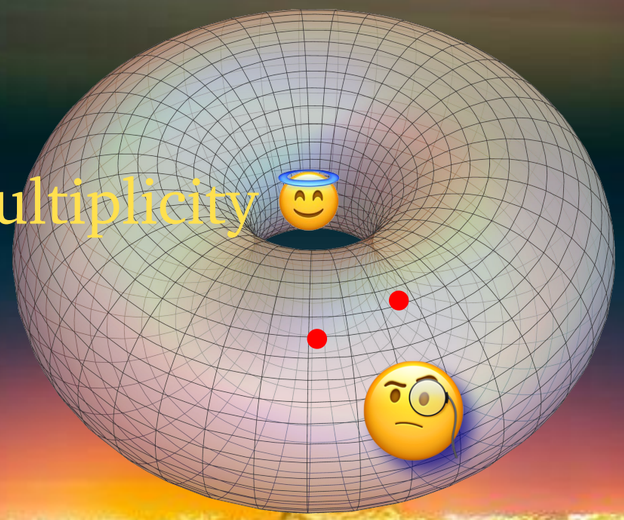


Just add some dimensions



Counting with multiplicity

$\mathscr{X} \in A[c_1]$



Eldorado Ho!

Thank You!

1611.10300
2205.12827
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2501.11684
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Eldorado Ho!