

The Variation of Calabi-Yau Hodge Structures and Feynman Integrals

Mirror Symmetry, Calabi-Yau Threefolds, and Connections to Physics

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30 September 2026



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I. Introduction to perturbative QFT: ϕ^4 Theory

$$Z[J] = \int \mathcal{D}\phi \exp \left[\frac{i}{\hbar} \int d^D x (\mathcal{L} + J\phi) \right] .$$

In ϕ^4 with Lagrangian $\mathcal{L} = \int d^D x \left[\frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2 - \frac{1}{4!} \lambda \phi^4 \right]$.

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Generally the connected physical correlators are obtained by functional derivatives w.r.t to the sources $J(x)$ ($W[J] = -i \log Z[J]$)

$$\langle \phi(x_1) \dots \phi(x_n) \rangle = Z[J]^{-1} \left(\frac{\delta}{\delta J(x_1)} \right) \dots \left(\frac{\delta}{\delta J(x_n)} \right) W[J] \Big|_{J=0}$$

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In interacting theories $\lambda \neq 0$ this is expanded in Feynman graphs and gives an **asymptotic** series in λ

$$\langle \phi(x_1) \dots \phi(x_n) \rangle = \underbrace{\text{tree}}_{\lambda} + \underbrace{\text{self-energy}}_{\lambda^2} + \underbrace{\text{exchange}}_{\lambda^2} + \underbrace{\text{bubble}}_{\lambda^2} + \dots$$

$$+ \underbrace{\text{triangle}}_{\lambda^3} + \dots + \underbrace{\text{pentagon}}_{\lambda^5} + \dots$$

Limitations and demand for perturbation theory

In QFT the contributions to $\langle \phi(x_1) \dots \phi(x_n) \rangle$ to (loop) order ℓ in λ can be shown to diverge with $CA^\ell \ell!$. Here C and A are of order one. One can conclude that the order of optimal truncation is $1/(A\lambda)$.

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Examples for physically demanded higher loop calculations:

- 1.) The electromagnetic coupling $\alpha = \lambda \sim 1/137$ is small and **the anomalous magnetic moment of the electron**

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Both processes involve the variation of CY mixed Hodge structures!

I. Introduction to perturbative QFT: Dirac process

Collider process: Probability for $e^- e^+$ to annihilate to two photons $P(e^- e^+ \rightarrow \gamma\gamma) \sim |\mathcal{A}(e^- e^+ \rightarrow \gamma\gamma)|^2$, $\alpha \sim \frac{1}{137}$

$$\mathcal{A}(e^- e^+ \rightarrow \gamma\gamma) = \text{[Feynman diagram: } e^- e^+ \text{ annihilation into two photons via a single photon exchange]} + \dots + \kappa \left(\text{[Feynman diagram: } e^- e^+ \text{ annihilation into two photons via a single photon exchange]} + \dots \right) + \kappa^2 \left(\text{[Feynman diagram: } e^- e^+ \text{ annihilation into two photons via two-photon exchange]} + \dots \right) + \dots$$

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$$\mathcal{A}(e^- e^+ \rightarrow \gamma\gamma) = \text{[tree-level diagrams]} + \dots + \alpha \left(\text{[one-loop diagrams]} + \dots \right) + \alpha^2 \left(\text{[two-loop diagrams]} + \dots \right) + \dots$$

The scalar integral relevant e.g. for the one loop (box) integral $I(\epsilon, m^2, p_i \cdot p_j)$ (Propagators $1/D_I$ with $D_I = \frac{1}{k_I^2 - m^2 + i0}$) is:

$$\begin{array}{c} p_1 \\ \swarrow \\ k+p_1 \\ \downarrow \\ p_2 \end{array} \quad \begin{array}{c} k \\ \leftarrow \\ \text{---} \\ \rightarrow \\ k-p_4 \end{array} \quad \begin{array}{c} p_4 \\ \swarrow \\ k-p_4 \\ \downarrow \\ p_3 \end{array}$$

$\sum_{i=1}^4 p_i = 0$ momentum conservation

$$\sim \int \frac{d^D k}{(k^2 - m^2) (k+p_1)^2 ((k+p_1+p_2)^2 - m^2) ((k-p_4)^2 - m^2)}$$

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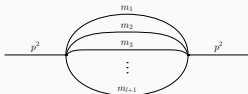
$$\text{[Box diagram]} \sim \int \frac{d^D k}{(k^2 - m^2) ((k+p_1)^2 - m^2) ((k+p_1+p_2)^2 - m^2) ((k-p_4)^2 - m^2)}$$

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Dimensional regularisation: $D = D_0 - 2\epsilon \rightarrow I = \sum_{k=-n}^{\infty} I_k \epsilon^n$ with $I_k(m^2, p_i \cdot p_j)$ functions of (dim' less ratios of) **masses** and Lorentz invariant products of the **external momenta** that we need to know!

Example: Banana Integrals in 2d:

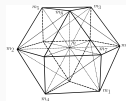
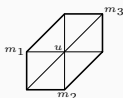
The maximal cut Banana integrals with $t = p^2/\mu^2$, $\xi_i = m_i/\mu$ or $z_i = \xi_i^2/t$.



in [Feynman-Symanzik parametrisation](#) in 2d are directly identified with CY hypersurface periods Π_Γ

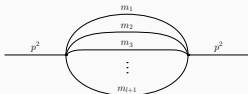
$$I(z_i) = \int_\Gamma \oint_{S^1} \frac{\mu_\ell}{\left(t - \left(\sum_{i=1}^{\ell+1} \xi_i^2 x_i\right) \left(\sum_{i=1}^{\ell+1} x_i^{-1}\right)\right) \prod_{i=1}^{\ell+1} x_i}$$

with reflexive Newton Polyhedra for e.g. for elliptic curve and $K3$



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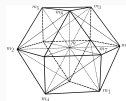
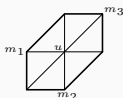
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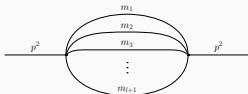
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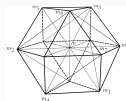
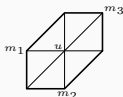
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What is the key advantage of the geometric reformulation?

The periods $\Pi_\Gamma(\underline{z})$ fulfil Picard-Fuchs differential equations. For equal masses, $z_1 = \cdots = z_{\ell+1} = z$ and $\theta = z \frac{d}{dz}$, these reduce to a single n -th order ordinary differential equation:

$$\mathcal{L}^{(\ell)} \Pi_\Gamma(z) = 0, \quad k = 1, \dots, n = \dim(H_n(M_n, \mathbb{Z})) = h_n.$$

The h_n solutions define **an extended class of higher transcendental (period) functions**, generalising algebraic and elliptic functions, which govern higher-order perturbation theory.

$\ell=1$	$\mathcal{L}^{(1)} = 1 - 2z - (1 - 4z)\theta,$	algebraic functions
2	$\mathcal{L}^{(2)} = 1 - 3z - (2 - 10z)\theta + (1 - z)(1 - 9z)\theta^2,$	elliptic functions
3	$\mathcal{L}^{(3)} = 1 - 4z - (3 - 18z)\theta + (3 - 30z)\theta^2 - (1 - 4z)(1 - 16z)\theta^3,$	K3-periods
4	$\mathcal{L}^{(4)} = 1 - 5z - (4 - 28z)\theta + (6 - 63z + 26z^2 - 225z^3)\theta^2 -$ AESZ34 $(4 - 70z + 450z^3)\theta^3 + (1 - z)(1 - 9z)(1 - 25z)\theta^4,$	CY-3fold periods

Better representation as complete intersection

In K. Boehnisch, F. Fischbach, C.Nega, AK 2020 a representation without redundant deformations as complete intersection of two degree $(1, \dots, 1)$ polynomials $P_1 = P_2 = 0 \subset \mathbb{P}_{l+1} := \bigotimes_{i=1}^{l+1} \mathbb{P}_{(i)}^1$ was identified.

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According to S. Hosono, AK, S. Theisen and S.-T. Yau 1994 the periods of this geometry fulfil a simple GKZ system with $2(\ell + 2)$ redundant deformation parameters and linear relations

$$\begin{aligned} l^{(1)} &= (-1, -1; 1, 1, 0, 0, \dots, 0, 0, 0, 0) \\ &\vdots \\ l^{(\ell+1)} &= (-1, -1; 0, 0, 0, 0, \dots, 0, 0, 1, 1) \end{aligned}$$

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and Frobenius solution

$$\omega_0(\underline{z}; \underline{\epsilon}) = \sum_{n_1, \dots, n_{l+1} \geq 0} c(\underline{n} + \underline{\epsilon}) \underline{z}^{\underline{n} + \underline{\epsilon}}, \quad c(\underline{n}) = \frac{\prod_{j=1}^2 \left(-\sum_{k=1}^{l+1} l_{0j}^{(k)} n_k \right)!}{\prod_{i=1}^{2l+2} \left(\sum_{k=1}^{l+1} l_i^{(k)} n_k \right)!}.$$

in the $\ell + 1$ invariant MUM coordinates $z_k = m_i^2 / p^2 = \prod_{i=1}^{2(\ell+4)} a_i^{l_i^{(k)}}$.

Advantages for the geometric formulation

At the point of maximal unipotent monodromy mirror symmetry predicts via the $\hat{\Gamma}$ -class formalism an integral monodromy basis of the periods.

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An extended $\hat{\Gamma}$ -class for the Fano variety $P_2 = 0 \subset \bigotimes_{i=1}^{l+1} \mathbb{P}_{(i)}^1$

$$\hat{\Gamma}_F(TF) = \frac{\hat{A}(TF)}{\hat{\Gamma}^2(TF)} = \frac{\Gamma(1 - c_1)}{\Gamma(1 + c_1)} \cos(\pi c_1) .$$

fixes also an integral basis of the solutions to the inhomogeneous Picard fuchs differential ideal including the full Banana integral. This basis choice was confirmed by [Hiroshi Iritani 2021](#).

Inhomogeneities from subgraphs

The inhomogeneities come from all possible subgraphs



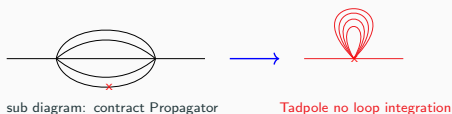
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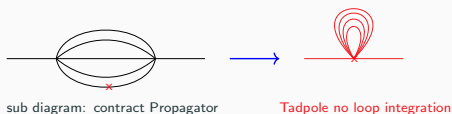
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$$\mathcal{L}^{(\ell)} I_{\sigma_\ell}(z) = -(\ell + 1)!z$$

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From a geometric point of view, $I_{\sigma_\ell}(a)$ are chain integrals of Griffiths normal functions.

Analytic structure of the banana integrals

The $\hat{\Gamma}_F(TF)$ class and the inhomogeneous differential equation for I_{σ_ℓ} control the analytic structure of the full equal-mass banana graph extremely well through **fast convergence and analytic continuation**.

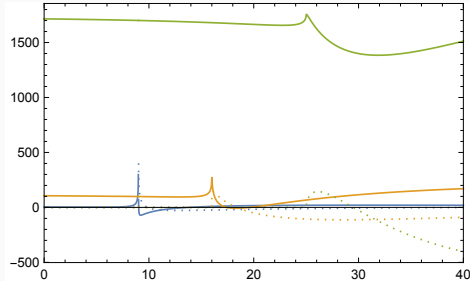


Figure 1: Loop orders $\ell = 2, 3, 4$: blue, orange, green. Solid curves: real part; dashed curves: imaginary part. $t = 1/z = p^2/m^2$; $t = (\ell + 1)^2$: nearest conifold; $t = \infty$: maximal unipotent monodromy point. 10 digits in μs .

Picard-Fuchs equations, flatness and monodromy

The fact that $\vec{\Pi}(\underline{z})$ fulfils the Picard-Fuchs equations $\mathcal{L}^{(\ell)}\Pi_{\Gamma_k} = 0$ is equivalent to its being a **flat section** of the Hodge bundle:

$$D_i \vec{\Pi} = (\partial_{z_i} - A_i(\underline{z}))\vec{\Pi} = 0, \quad i = 1, \dots, h^{n-1,1}(M),$$

where D_i is the **Gauss-Manin connection** and $A_i \in \mathbb{Q}[\underline{z}]_{h_n \times h_n}$.

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The **appropriate section—the Feynman integral**—is determined by the integral global monodromy structure

$$\mathfrak{M}_i \in \begin{cases} \mathrm{Sp}(h_n, \mathbb{Z}), & n \text{ odd}, \\ \mathrm{O}(h_n, \mathbb{Z}), & n \text{ even}, \end{cases}$$

and its singular behaviour in the **ultraviolet, on-shell and infrared limits**.

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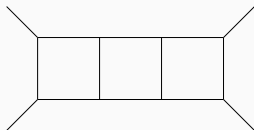
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and its singular behaviour in the **ultraviolet, on-shell and infrared limits**. These correspond to **degenerations of the Hodge structure** of the CY: maximally unipotent points, conifolds, etc.

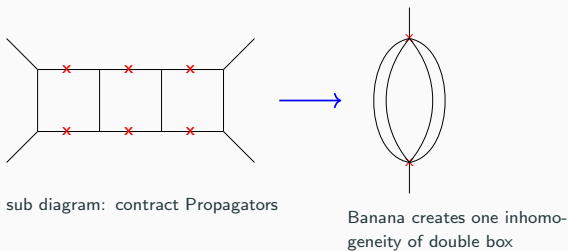
Why are Banana graphs relevant?

Consider the triple box as a Standard Model graph



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The iterated integrals of banana integrals (K3-periods) appear generically as inhomogeneties of in 3-loop standard model graphs. E.g. in the calculation of the anomalous magnetic moment of the electron a_e Forner, Nega, Tancredi (24).

Summary: Dictionary Feynman integrals and geometry

Perturbative QFT	Geometry M	Differential eq.	Arithmetic Geometry
maximal cut Feynman integral	Period integral $\underline{\Pi}$ (ϵ -deformed)	Homogeneous Gauss Manin ($d - A(z))\underline{\Pi} = 0$	Motive defined by l -adic coh $H_{et}^k(\overline{X}, \mathbb{Q}_l)$
	⊙ Monodromy group $\in \Gamma(\mathbb{Z})$; irre- ducible ?		⊙ Galois group $\text{Gal}(\overline{K}/K)$ ir- reducible ?
actual Feynman integral	Chain integral (ϵ -deformed)	Inhomogeneous Gauss Manin connection ($d - A(z))\underline{\Pi} = B(z)$	Extended motive

This picture emerged due to work of I. Gel'fand, D. Broadhurst, D. Kreimer, S. Bloch, P.

Vanhove, M.Kerr, C. Duran, S. Weinzierl, F. Brown, O. Schnetz, J. Bourjaily, A. McLeod, M. Hippel, M. Wilhelm,

J. Broedel, L. Trancredi, S. Müller-Stach, ... + at least 248 cits. in K. Bönisch, C. Duhr, F. Fischbach, C. Nega, AK

Period geometry on CY n-folds

The main constraints which govern the period geometry of CY-folds are **the Riemann bilinear** relations

$$e^{-K} = i^{n^2} \int_{M_n} \Omega \wedge \bar{\Omega} > 0 \quad (1)$$

defining the real positive exponential of the **Kähler potential** $K(z)$ for the **Weil–Petersson metric** $G_{i\bar{j}} = \partial_{z_i} \bar{\partial}_{\bar{z}_j} K(z)$ on $\mathcal{M}_{cs}(M_n)$.

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defining the real positive exponential of the **Kähler potential** $K(z)$ for the **Weil–Petersson metric** $G_{i\bar{j}} = \partial_{z_i} \bar{\partial}_{\bar{z}_j} K(z)$ on $\mathcal{M}_{cs}(M_n)$. As well as the relations on holomorphic bilinears and their derivatives that follow from **Griffiths transversality**¹

$$\int_{M_n} \Omega \wedge \underline{\partial}_{l_k}^k \Omega = \begin{cases} 0 & \text{if } k < n \\ C_{l_n}(\underline{z}) \in \mathbb{Q}(\underline{z}) & \text{if } k = n . \end{cases} \quad (2)$$

Here $\underline{\partial}_{l_k}^k \Omega = \partial_{z_{l_1}} \dots \partial_{z_{l_k}} \Omega \in F^{n-k} := \bigoplus_{p=0}^k H^{n-p,p}$ are sections of holomorphic sub bundles $F^n \subset F^{n-k} \subset \dots \subset F^1 \subset F^0$ determining the **Hodge filtration** of the Hodge bundle F^0 . ¹R. Bryant and P. Griffiths, Prog.

Griffiths transversality and iterated integrals

Set $\ell = n + 1$. Let ϖ_k , $k = 0, \dots, \ell - 1$, be a basis of $\ker \mathcal{L}^{(\ell)}$, with the $\ell \times \ell$ Wronskian

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Here $\mathbf{b}$ denotes the source in the associated first-order system.

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These generalise the **polylogarithms** that appear in one-loop Feynman integrals.

We have seen that important maximal cut Feynman integrals are CY periods and that the full Feynman graphs are determined by inhomogeneous Gauss-Manin connections, whose inhomogeneities are determined by subgraphs.

€ deformed Gauss Manin connection and sub-sectors

We have seen that important maximal cut Feynman integrals are CY periods and that the full Feynman graphs are determined by **inhomogeneous Gauss-Manin connections**, whose inhomogeneities are determined by **subgraphs**.

The key way to systematize this and obtain results in any dimension is to introduce so called **master integrals** for ℓ -loop **Feynman integrals** in general dimensions $D \in \mathbb{R}_+$ Johannes Henn (13)

$$I_{\underline{\nu}}(\underline{z}, D) := \int \prod_{r=1}^{\ell} \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^p \frac{1}{D_j^{\nu_j}} \quad (5)$$

$D_j = q_j^2 - m_j^2 + i \cdot 0$ for $j = 1, \dots, p$ are the propagators, q_j is the j^{th} momentum through D_j , $m_j^2 \in \mathbb{R}_+$ are masses. Subject to momentum conservation the q_j are linear in the external momenta and the internal loop momenta k_r . We define $\epsilon := \frac{D_{\text{cr}} - D}{2}$.

Master Integrals and integration by parts relations

Note that the propagator exponents in the **master integrals** are arbitrary non-negative integers and together with $D_{cr} \in \mathbb{Z}$ span a **non-negative cone in a lattice** $(\underline{\nu}, D_{cr}) \in \mathbb{Z}_{\geq 0}^{p+1}$.

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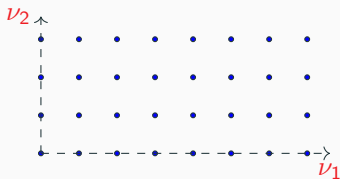
The master integrands define a **finite** cohomology, which sometimes extends the cohomology $H^{\ell}(M_{\ell}, \mathbb{Z})$ of a CY M_{ℓ} .

Master Integrals and integration by parts relations

In particular there is a **finite region in $\mathbb{Z}_{\geq 0}^{p+1}$** **Smirnov and Petukhov LMP 2011** that contains all relevant master integrals. I.e. in those master integrals one can express all derivatives w.r.t. the z_k as a linear combination with **rational coefficients** by the IBP relations.

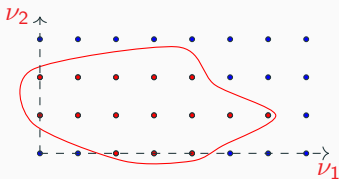
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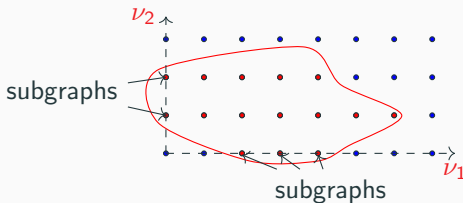
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A complete set of IBP relations corresponds to the complete Picard Fuchs ideal or Gauss-Manin connection for the period integrals.

IBP relation summary:

Hence the IBP relations characterise a suitable finite set of master integrals

$$I(\underline{x}, D) := \int \prod_{r=1}^l \frac{d^D k_r}{i\pi^{\frac{D}{2}}} \prod_{j=1}^p \frac{1}{D_j^{\nu_j}},$$

in a finite region in $\mathbb{Z}_{\geq 0}^{p+1}$, by a generalised first order Gauss Manin connection

$$dI(\underline{x}, \epsilon) = \mathbf{A}(\underline{x}, \epsilon) I(\underline{x}, \epsilon)$$

$$\epsilon = (D_{cr} - D)/2.$$

Basis change on master integrals ϵ -factorized form

$$\underline{I}(\underline{x}, \epsilon) \rightarrow \underline{I}^{better}(\underline{z}(x); \epsilon) = R_0(\underline{z}(x); \epsilon) \underline{I}(\underline{z}(x); \epsilon)$$

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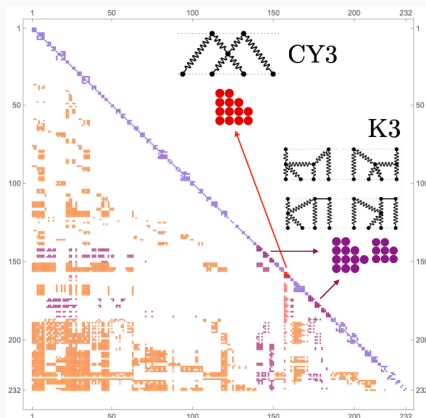


Figure 2: Non-zero entries of the 232×232 extended Gauss-Manin connection matrix $A^{best}(x)$. The not-magnified diagonal sectors give rise to polylogarithms.

The use of the ϵ factorized form

$$\nabla_{\epsilon} I(z, \epsilon) = (d_z - \epsilon \mathbf{A}(z)) I(z, \epsilon) = 0$$

Suppose we have boundary conditions $I(z_0, 0) = I_0$ then we can write the solution vector $I(z, \epsilon)$ as path ordered exponential

$$I(z, \epsilon) = \mathbb{P} \exp\left[\epsilon \int_{z_0}^z \mathbf{A}(z) dz\right] I_0$$

in terms of iterated integrals in the form

$$I(z, \epsilon) = \left[\mathbf{1} + \epsilon \int_{z_0}^z \mathbf{A}(z') dz' + \epsilon^2 \int_{z_0}^z \int_{z_0}^{z'} \mathbf{A}(z') dz' \mathbf{A}(z'') dz'' + \dots \right] I_0$$

For example for the 1 loop diagrams this formalism gives rise to the *function class* of multiple poly logarithms. For higher loop amplitudes this class generalizes to iterated Calabi-Yau periods.

Applications to General Relativity

"Emergence of Calabi Yau manifolds in high precision black hole scattering"

Driesse, Jakobsen, AK, Mogull,
Nega, Plefka, Sauer



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"Calabi-Yau periods for black hole scattering in classical general relativity"

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Exact approaches to General Relativity

Motivation: Experimental
gravitational wave detection:
LIGO, Virgo, KAGRA,



Virgo detector, Cascina, Italy.

Interpretation requires theoretical understanding of source wave formation. (BH mergers)

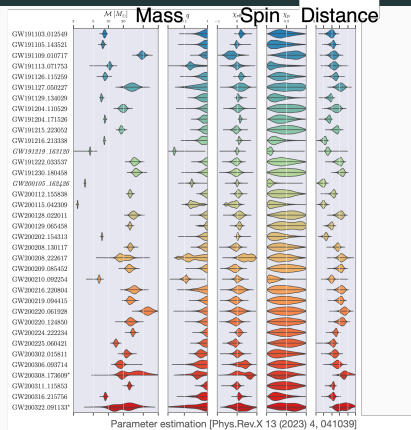
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Aim: Analytical description in an effective Field theory approach.

Effective field theory

Quantum Gravity as an effective Theory: Donoghue (1993)

Classical Gravity from Amplitudes: Cheung, Rothstein, Solon (2018);
Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove (2018); Bern,
Cheung, Roiban, Shen, Solon, Zeng (2019)

Frameworks for classical gravity from amplitudes: KMOC: Kosower,
Maybee, O'Connell (2018)

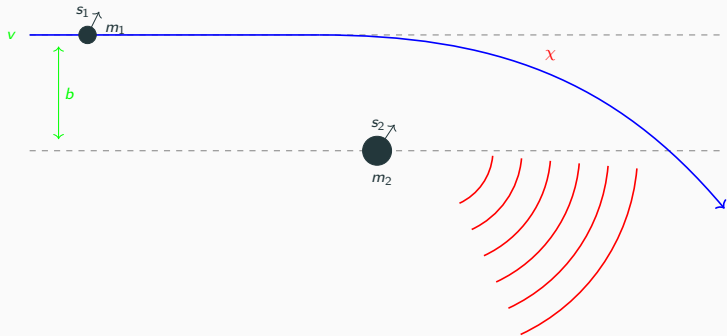
Heavy Black Hole Effective Theory: Damgaard, Haddad, Helset (2019)

Eikonal Operator: Di Vecchia, Heissenberg, Russo, Veneziano (2021)

Heavy Mass EFT: Brandhuber, Brown, Chen, De Angelis, Gowdy, Travaglini
(2023)

Treat scattering of two black holes (BH) as starter to the description of
BH mergers in a perturbative expansion effective QFT long range
description to gain some analytic control.

Worldline QFT approach to General Relativity



Worldline Quantum Field Theory approach to General Relativity

The action for the scattering process [Kälin, Porto '06](#), [Dlapa, Kälin, Liu, Porto '12](#), [Jakobsen, Mogull, Plefka, Steinhoff '21, '23, '24](#)

$$S = - \sum_{i=1}^2 m_i \int d\tau \left[\frac{1}{2} g_{\mu\nu} \dot{x}_i^\mu \dot{x}_i^\nu \right] + \frac{1}{16\pi G} \int d^4x \sqrt{-g} R(g) + S_{\text{g.f.}}^{\text{Donder}}$$

is expanded in Post Minkowskian (PM) approximation in the Worldline Quantum Field Theory (WQFT) approach around the non-interacting background configurations

$$x_i^\mu = b_i^\mu + v_i^\mu \tau + z_i^\mu(\tau), \quad g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{32\pi G} h_{\mu\nu}(x).$$

The dynamical fields z_i^μ and $h_{\mu\nu}$ are integrated out in the path integral. $\langle x_i^\mu \rangle$ fulfills the classical e.o.m.

Worldline QFT approach to General Relativity

The goal is to calculate from the initial data: the impact parameter $b^\mu = b_1^\mu - b_2^\mu$ and the incoming velocities v_1, v_2 the physical quantity of interest, which is the radiation induces change in the momentum say $\Delta p_1^\mu = m_1 \int d\tau \ddot{x}(\tau)$ of the first particle.

In the PM approximation the latter can be expanded in the gravitational coupling G

$$\Delta p_1^\mu = \sum_{n=1}^{\infty} G^n \Delta p^{(n)\mu}(x) .$$

At each order the contributions $\Delta p^{(n)\mu}(x)$ are calculated in the WQFT approach in the Swinger-Keldysh in-in formalism in terms of a Feynman graph expansion with retarded propagators. Here $x = \gamma - \sqrt{\gamma^2 - 1}$ with γ the Lorentz factor of the relative velocities is the only parameter.

Worldline QFT approach to General Relativity

In the 4PM approximation the Feynman integral in the 1SF sector, contributing to the $\frac{m_1}{m_2}$ order in the corresponding expansion



involve symmetric products of elliptic function, which are periods of the $K3$

$$Y^2 = X(X-1)(X-x)Z(Z-1)(Z-1/x).$$

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In the 5PM approximation we find in [8] that in the 5PM approximation the following graphs in the 1SF sector



Worldline QFT approach to General Relativity

The corresponding smooth CY three-fold one-parameter complex family $x = (2\psi)^{-8}$, can be defined as resolution of four symmetric quadrics

$$x_j^2 + y_j^2 - 2\psi x_{j+1}y_{j+1} = 0, \quad j \in \mathbb{Z}/4\mathbb{Z}$$

in the homogeneous coordinates $x_i, y_j, j = 0, \dots, 3$ of $\mathbb{P}^7$.

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in the homogeneous coordinates $x_i, y_j, j = 0, \dots, 3$ of $\mathbb{P}^7$. Its periods fulfill the hypergeometric ${}_4F_3\left(\begin{smallmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1, 1, 1 \end{smallmatrix} \middle| z\right)$ Picard-Fuchs differential operator

$$\mathcal{L}_1^{(4)} = \theta^4 - 2^8 z \left(\theta + \frac{1}{2}\right)^4$$

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The periods of the above $K3$ and CY threefold fulfill the differential equation and determine all special functions that are necessary to solve for $\Delta p^{(5)\mu}(x)$ and the radiative energy in the 1SF sector.

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The corresponding smooth CY three-fold one-parameter complex family $x = (2\psi)^{-8}$, can be defined as resolution of four symmetric quadrics

$$x_j^2 + y_j^2 - 2\psi x_{j+1} y_{j+1} = 0, \quad j \in \mathbb{Z}/4\mathbb{Z}$$

in the homogeneous coordinates $x_i, y_j, j = 0, \dots, 3$ of $\mathbb{P}^7$. Its periods fulfill the hypergeometric ${}_4F_3 \left(\begin{smallmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1, 1, 1 \end{smallmatrix} \middle| z \right)$ Picard-Fuchs differential operator

$$\mathcal{L}_1^{(4)} = \theta^4 - 2^8 z \left(\theta + \frac{1}{2} \right)^4$$

The periods of the above $K3$ and CY threefold fulfill the differential equation and determine all special functions that are necessary to solve for $\Delta p^{(5)\mu}(x)$ and the radiative energy in the 1SF sector. Note that all solutions are either symmetric or Hadamard products of the second order Legendre differential equation.

In the 5PM 2SF sector there is a graph



that leads [8] to period integrals on the Beukers K3

$$1 - (1 - XY)Z - zXYZ(1 - X)(1 - Y)(1 - Z) = 0$$

with Picard-Fuchs differential operator

$$L_A^{(3)} = \theta^3 + z^2(\theta + 1)^3 - z(2\theta + 1)(17\theta^2 + 17\theta + 5) .$$

Historical this was used to explain geometrically Apéry's proof of the irrationality of $\zeta(3)$ by Beukers (79,85) .

Another CY 3-fold

The graph



was identified using its Baikov integral representation as period integrals of the CY 3-fold

$$P = t^2(WX + Y^2)^2(W + Z)^2(X + Z)^2 \\ + 2^6(1 + t)(WXY)^2Z(W + X + Z)$$

by [Frellesvig, Morales, Wilhelm PRL '24](#). It has Picard-Fuchs operator $t = x^1 - 1$, $z = 2^{-8}x^4$

$$\mathcal{L}_2^{(4)} = \theta^4 - 2^{30}z^3\left(\theta + \frac{1}{2}\right)^4 - 2^4z(192\theta^4 + 128\theta^3 + 112\theta^2 \\ + 48\theta + 7) + 2^{14}z^2(192\theta^4 + 256\theta^3 + 208\theta^2 + 64\theta + 7) .$$

ϵ -factorisation for the CY blocks in 5PM 1SF

To derive the ϵ -factorization we split the periods matrix into its semi-simple and unipotent part. The unipotent part satisfies

$$(d - A^u(x))W^u(x) = 0, \quad A^u(x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & Y_1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $A^u(x)$ is nilpotent, i.e. $(A^u(x))^4 = 0$.

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$$G_1(x) = - \int_0^x \frac{24576x' (1 + 256x'^4)}{(1 - 256x'^4)^2} \frac{\varpi_0(x')^2}{\alpha_1(x')} dx',$$
$$G_3(x) = \int_0^x \frac{x'}{1 - 256x'^4} \frac{G_1(x')\alpha_1(x')^2}{\varpi_0(x')^2} dx'.$$

Here $\alpha_1 = (\theta\varpi_1/\varpi_0)^{-1}$ and $t = \varpi_1/\varpi_0$ is the affine coordinate related to the mirror map.

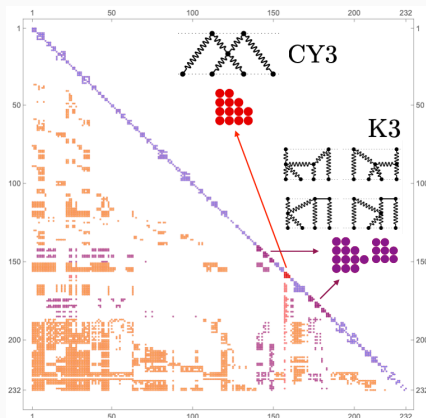


Figure 3: Non-zero entries of the 232×232 extended Gauss-Manin connection matrix $A^{best}(x)$ of odd parity. The not-magnified diagonal sectors give rise to polylogarithms.

Results in the 5PM 1SF Sector

After imposing the physical boundary conditions, determining the l_0 we can extract all regular physical contribution to the **deflections angle** and the **radiation loss** and eventually the *gravitational waveform*.

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The latter are *analytically* given in terms of **iterated integrals of Calabi-Yau 3-fold periods, K3-periods and polylogarithms** available **online**.

Scattering angle versus impact parameter up to 5PM 1SF order

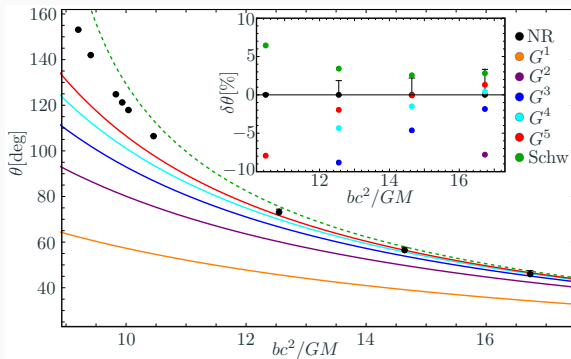


Figure 4: Scattering angle versus impact parameter

Radiation loss versus velocity

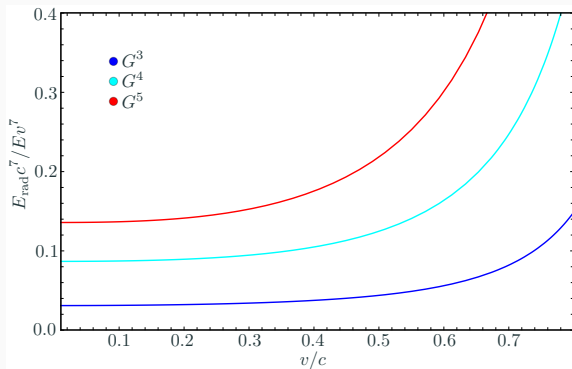
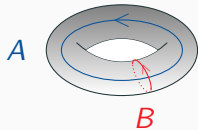
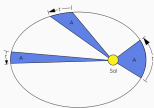
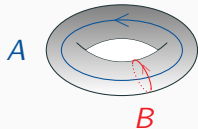
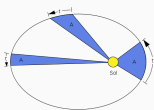


Figure 5: Radiation loss versus velocity

Conclusion and Outlook

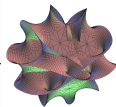
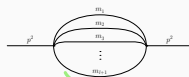


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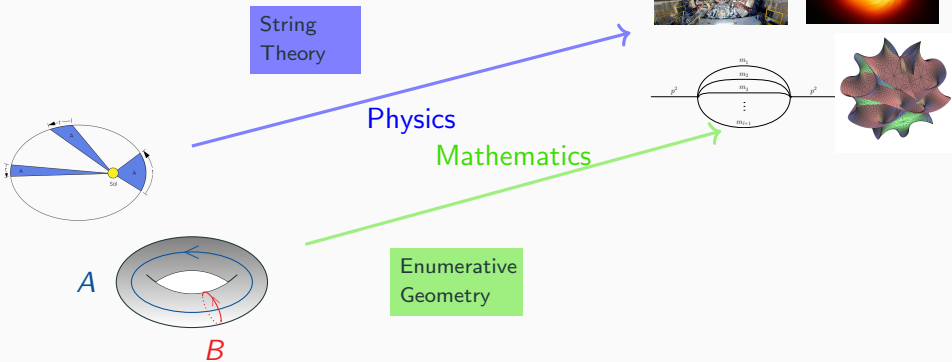


Physics

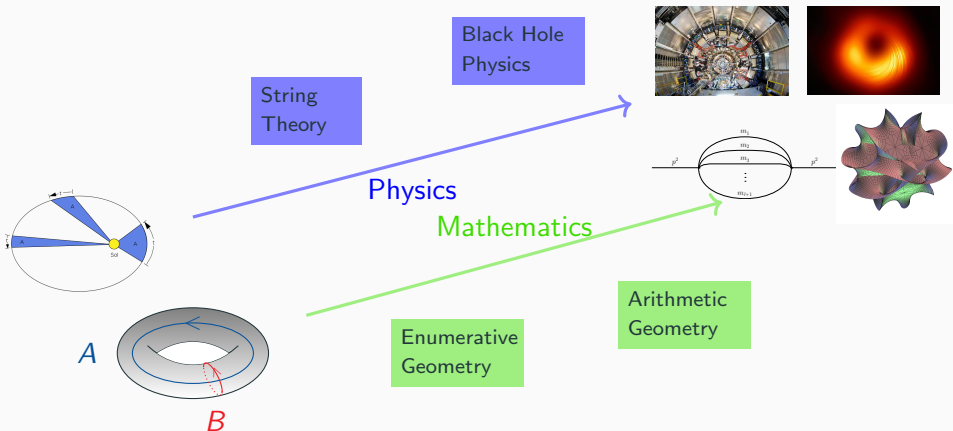
Mathematics



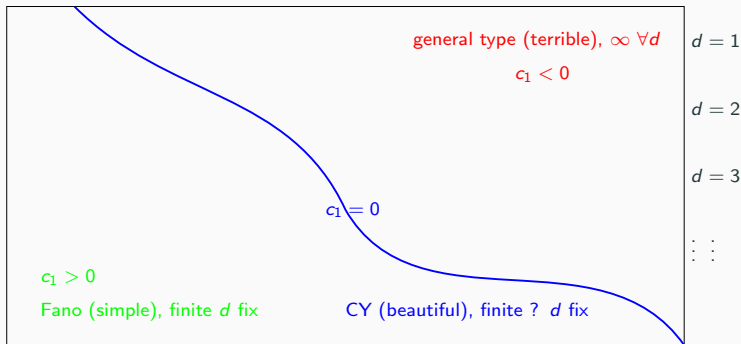
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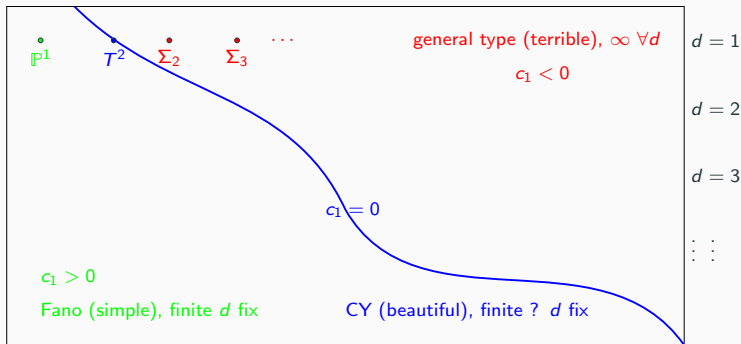


Philosophical Remarks: Kodaira map of algebraic varieties

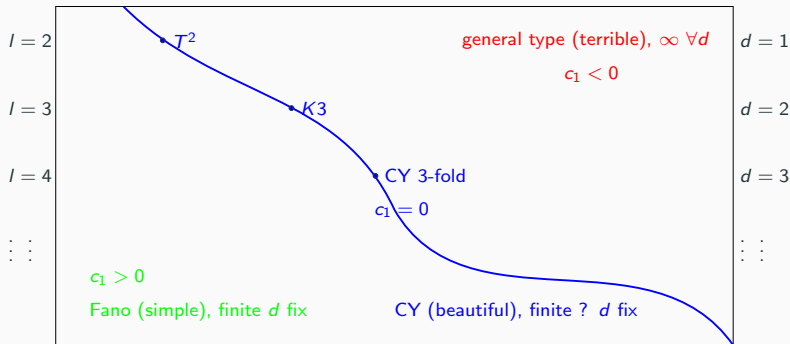


Philosophical Remarks: Kodaira map of algebraic varieties

$$\begin{array}{ccccccc} \ell = 0 & \ell = 1 & \ell = 2 & \ell = 3 & \cdots \\ g = 0 & g = 1 & g = 2 & g = 3 & \cdots \end{array}$$



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Monodromy and iteration over dimensions

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One parameter K3 operators $\mathcal{L}^3(K3)$ of type $(1, 1, 1)$ are always symmetric squares the ones of elliptic curves E

$$\mathcal{L}^3(K3) = \text{Sym}^2(\mathcal{L}^2(E))$$

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Slightly more complicated one has that quite generally

$$\mathcal{L}^4(M_3) = \text{Hadamard}^{(2)} \mathcal{L}^2(E) .$$

E.g. in the 5PM 1SF calculation all CY operators are obtainable from $\mathcal{L}^2(y^2 = x(x-1)(x-z))$ by the above constrs! In which sense do we need Higher dimensional CY period geometries?

A hint from Topological string theory

CY 3-folds are special for topological field theory as $F_g \neq 0 \ \forall g$ due to $\dim_{vir} \overline{M}_g(hol. \ maps, M_3) = 0$ for all g .

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$$\begin{aligned} S^{zz} &= -\frac{1}{C} \left(\frac{w_{0,2}}{w_{0,1}} - q_{zz}^z \right), & \tilde{S}^z &= -\frac{1}{C} \left(\frac{w_{1,2}}{w_{0,1}} - q_{zz} \right), \\ \tilde{S} &= -\frac{1}{2C} \left(\frac{w_{1,3}}{w_{0,1}} - \partial_z q_{zz} - q_{zz} q_{zz}^z \right) + \frac{\partial_z C}{2C^2} \left(\frac{w_{1,2}}{w_{0,1}} - q_{zz} \right) - \frac{q_z^z}{2}. \end{aligned} \tag{6}$$

with $w_{i,j}$ sub determinants of the Wronskian and $C, q_* \in \mathbb{Q}[\underline{z}]$ Gu,

Kashani-Poor, Marino, AK, e-Print: 2305.19916.

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Maybe the topological string is a toy explanation for the conundrum. Why do higher dim CY n-folds instead of higher genus Riemann surfaces appear.