

$\mathbb{Z}_3$ -orbifold K3s, their symmetries, and how to track them

MIRROR SYMMETRY, CALABI-YAU THREEFOLDS,
AND CONNECTIONS TO PHYSICS

Institute for Computational and Experimental Research in Mathematics (ICERM)

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joint work with **Kasia Budzik**, **Anne Taormina**, **Mara Ungureanu** and **Ida Zadeh**:

Tracking the symmetries of $\mathbb{Z}_3$ -orbifold K3s within the Mathieu groups,
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Plan:

- 1 $\mathbb{Z}_3$ -orbifold K3s
- 2 Lattice techniques: K3 cohomology
- 3 Lattice techniques: Niemeier lattices
- 4 Towards Mathieu Moonshine

Reminder: The classical Kummer construction

Let $T = \mathbb{C}^2/L$, $L \subset \mathbb{C}^2$ a **lattice** of rank 4.

The $\mathbb{Z}_2$ -**action** on $\mathbb{C}^2$ which is generated by $\mathbb{C}^2 \rightarrow \mathbb{C}^2$, $z \mapsto -z$, **descends** to T , yielding $T/\mathbb{Z}_2$ with **16** singularities of type A_1 .

The minimal resolution $\widetilde{T/\mathbb{Z}_2}$
is a **K3 surface** called **KUMMER SURFACE**.

[Nikulin75+...]

provides a detailed study of **Kummer surfaces**
in terms of their underlying **tori** T .

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We always equip $\widetilde{T/G}$ ($G \subset SU(2)$) with the **(degenerate) Kähler class** that is induced from the **standard Euclidean metric** on $\mathbb{C}^2$.

Kummer-like K3 surfaces

(see also [Bertin88, W00])

Let $T = \mathbb{C}^2/L$, $L \subset \mathbb{C}^2$ a lattice of rank 4; G a finite group with $G \subset \mathrm{SU}(2)$, $1 < |G|$, $GL = L$; then $\widetilde{T/G}$ is a K3 surface.

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For us:

$G = \{1, \gamma, \gamma^2\} \cong \mathbb{Z}_3$, with $\gamma = \mathrm{diag}(\xi, \xi^{-1})$, $\xi = \exp(\frac{2\pi i}{3})$;
 L generated by $\lambda_1 = (1, 0)$, $\lambda_2 = (\xi, 0)$, $\lambda_3 = (0, 1)$, $\lambda_4 = (0, \xi)$.

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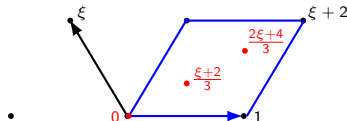
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$[(z_1, z_2)] \in T^{\mathbb{Z}_3}$

$$\iff [(z_1, z_2)] = [(\xi z_1, \xi^{-1} z_2)] \iff \xi z_k - z_k \in \mathbb{Z} + \xi \mathbb{Z}$$



$$\iff z_k \in \mathbb{Z} + \frac{\xi+2}{3} \mathbb{Z},$$

so $T^{\mathbb{Z}_3} \cong \mathbb{F}_3^2$.

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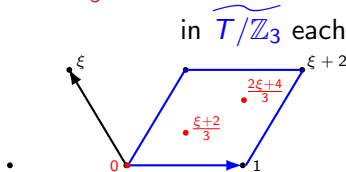
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With $T^{\mathbb{Z}_3} \cong \mathbb{F}_3^2$,

in $\widetilde{T/\mathbb{Z}_3}$ each $t \in T^{\mathbb{Z}_3}$ is replaced by **two transversally intersecting (-2) -curves**.



Symmetries that descend to the $\mathbb{Z}_3$ -orbifold K3

Symmetries (i.e. biholomorphic maps preserving the Kähler class)
of $T = \mathbb{C}^2/L$ that descend to $\widetilde{T}/\mathbb{Z}_3$:

Subgroup $\mathcal{G} \subset \mathrm{SU}(2)$ with $\mathcal{G}L = L$:

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Translations:

for $t \in T$, the translation $[z] \mapsto [z] + t$ descends to $\widetilde{T/\mathbb{Z}_3}$
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Induced symmetry group: $\mathbb{Z}_4 \ltimes \mathbb{Z}_3^2$.

Lattice techniques – why?

X : a K3 surface,

$T = \mathbb{C}^2/L$ as above.

Facts:

- $H^2(X, \mathbb{Z})$ and $H^2(T, \mathbb{Z})$ are even, self-dual lattices of signatures $(3, 19)$ and $(3, 3)$.

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Even, self-dual lattice Γ of signature (p, q) :

$\Gamma = \{n_1 f_1 + \cdots + n_{p+q} f_{p+q} \mid n_1, \dots, n_{p+q} \in \mathbb{Z}\},$
 with $(f_1, \dots, f_{p+q})$ a basis of $(\mathbb{R}^{p,q}, \langle \cdot, \cdot \rangle)$ of signature $(p, q),$
 $A = (\langle f_j, f_k \rangle)_{j,k}$ with $\langle f_j, f_k \rangle \in \mathbb{Z}, \langle f_j, f_j \rangle \in 2\mathbb{Z} \forall j, k, |\det A| = 1.$

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- The Kummer-like construction of $X = \widetilde{T/\mathbb{Z}_3}$ induces an injection $\pi_* H^2(T, \mathbb{Z})^{\mathbb{Z}_3} \hookrightarrow H^2(X, \mathbb{Z}).$

Generalizing Nikulin's results to $\mathbb{Z}_3$ -orbifold K3s: input

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$$\gamma: (\mu_1, \mu_2, \mu_3, \mu_4) \longmapsto (-\mu_2, \mu_1 - \mu_2, \mu_4 - \mu_3, -\mu_3).$$

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- For every $t \in \mathbb{F}_3^2$: $E_t^{(1)}, E_t^{(2)} \in (\pi_* H^2(T, \mathbb{Z})^{\mathbb{Z}_3})^\perp$;
if $s, t \in \mathbb{F}_3^2$, then $\left(\langle E_s^{(k)}, E_t^{(\ell)} \rangle \right)_{k, \ell} = \delta_{s, t} \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$;
so $A_2^9 \subset (\pi_* H^2(T, \mathbb{Z})^{\mathbb{Z}_3})^\perp$.

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Task:

determine $K := (\pi_* H^2(T, \mathbb{Z})^{\mathbb{Z}_3} \otimes \mathbb{Q}) \cap H^2(X, \mathbb{Z})$,
 $P := K^\perp \cap H^2(X, \mathbb{Z})$, and $K \oplus P \hookrightarrow H^2(X, \mathbb{Z})$.

Generalizing Nikulin's results to $\mathbb{Z}_3$ -orbifold K3s: results

Results [Budzik/Taormina/Ungureanu/W/Zadeh25]

(see also [Shioda/Inose77, Bertin88, W00]):

- $H^2(T, \mathbb{Z})^{\mathbb{Z}_3}$ is generated by

$$\{\mu_{12}, \mu_{34}, \mu_{13} - \mu_{24}, \mu_{14} + \mu_{23} - \mu_{24}\},$$
 so $\pi_* H^2(T, \mathbb{Z})^{\mathbb{Z}_3} \oplus A_2^9$ has rank $4 + 18 = 22 = \text{rk } H^2(X, \mathbb{Z})$;
 $K := (\pi_* H^2(T, \mathbb{Z})^{\mathbb{Z}_3} \otimes \mathbb{Q}) \cap H^2(X, \mathbb{Z})$ is generated by

$$\{\pi_* \mu_{12}, \pi_* \mu_{34}, \pi_* (\mu_{13} - \mu_{24}),$$

$$\frac{1}{3} \pi_* (\mu_{14} + \mu_{23} - 2\mu_{13} + \mu_{24})\};$$
- $P := K^\perp \cap H^2(X, \mathbb{Z})$ is generated by the $E_t^{(k)}$ with $t \in \mathbb{F}_3^2$, $k \in \{1, 2\}$, and the $\frac{1}{3} \left(\sum_{s \in L} E_s - \sum_{t \in L'} E_t \right)$, where $L, L' \subset \mathbb{F}_3^2$ are parallel affine lines, and for $t \in \mathbb{F}_3^2$, $E_t = E_t^{(1)} + 2E_t^{(2)}$.

Gluing techniques

For Γ an even, self-dual lattice,
and a sublattice $R \subset \Gamma$ of same rank, yielding $R \subset \Gamma \subset R^*$:

GLUE CODE: $\Gamma/R \subset R^*/R$;

GLUE VECTORS: a choice of representatives of the
glue codewords that generate Γ from R .

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Special case [Nikulin80]:

Let $R = R_1 \oplus R_2$ with $R_1 = R_2^\perp \cap \Gamma$ and $R_2 = R_1^\perp \cap \Gamma$.

$\exists g: R_1^*/R_1 \xrightarrow{\cong} R_2^*/R_2$, s. th. $\Gamma/R = \{(\kappa, g(\kappa)) \mid \kappa \in R_1^*/R_1\}$, so
 $\Gamma = \{(r_1, r_2) \in R_1^* \oplus R_2^* \mid g([r_1]) = [r_2]\}$.

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 $\Gamma = \{([r_1], [r_2]) \in R_1^* \oplus R_2^* \mid g([r_1]) = [r_2]\}$.

Result [Budzik/Taormina/Ungureanu/W/Zadeh25]:

$H^2(X, \mathbb{Z})$ is obtained from $K \oplus P$ by gluing
according to a purely geometric prescription.

Classification of symmetries of $\mathbb{Z}_3$ -orbifold K3s

Result [Budzik/Taormina/Ungureanu/W/Zadeh25]:

$\alpha \in \text{Aut}(H^2(X, \mathbb{Z}))$ is induced by a symmetry of X if and only if α is induced by a symmetry of T , that is, if and only if α is a composition of β^k , $k \in \mathbb{Z}$, and a translation on T by some $t \in T^{\mathbb{Z}_3}$, $T^{\mathbb{Z}_3} \cong \mathbb{F}_3^2$:

the symmetry group of $\mathbb{Z}_3$ -orbifold K3s is $\mathbb{Z}_4 \ltimes \mathbb{Z}_3^2$.

Note: α is uniquely determined by its action on the $E_t^{(k)}$.

Tracking K3 symmetries on Niemeier lattices

[Kondo88]:

$\mathcal{G}$: a finite, non-trivial symmetry group of a K3 surface Y ,

$$M_{\mathcal{G}} := (H^2(Y, \mathbb{Z})^{\mathcal{G}})^{\perp}.$$

Then there exists an even, self-dual lattice N of signature $(0, 24)$
with $M_{\mathcal{G}} \hookrightarrow N$ such that $(M_{\mathcal{G}} \otimes \mathbb{Q}) \cap N = M_{\mathcal{G}}$.

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A modification of this:

Result [Budzik/Taormina/Ungureanu/W/Zadeh25]:

Up to lattice automorphisms, there exists
 a unique even, self-dual lattice N of signature $(0, 24)$
 with $P \hookrightarrow N$ such that $(P \otimes \mathbb{Q}) \cap N = P; \quad N \supset A_2^{12}.$

Tracking K3 symmetries in the Mathieu group M_{12}

Let N : the even, self-dual lattice of signature $(0, 24)$ with $A_2^{12} \subset N$,
and $\widetilde{W} \triangleleft \text{Aut}(N)$ generated by the Weyl group of A_2^{12} and $-\mathbb{1}$.

Results [Budzik/Taormina/Ungureanu/W/Zadeh25]:

- The embedding $P \hookrightarrow N$ with $(P \otimes \mathbb{Q}) \cap N = P$ is unique up to lattices automorphisms.
- We obtain an explicit description of the symmetries of X in terms of automorphisms of N ;
the symmetry group of X injects into $\text{Aut}(N)/\widetilde{W} \cong M_{12}$.
- Up to lattices automorphisms, there exists
a unique even, self-dual lattice $\widetilde{N}$ of signature $(0, 24)$
with $P \hookrightarrow \widetilde{N}$ such that $(P \otimes \mathbb{Q}) \cap \widetilde{N} \supsetneq P$; $\widetilde{N} \supset E_6^4$.

Towards Mathieu Moonshine

Fact:

The K3 elliptic genus $\mathcal{E}$ is a supercharacter of conformal field theories called K3 THEORIES, hence a character of the $N = 4$ super-Virasoro algebra at central charge $c = 6$.

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Observation [Eguchi/Ooguri/Tachikawa10]:

The **decomposition** of $\mathcal{E}$ into $N = 4$ **supercharacters**

$$\chi_{m.m.}, \quad \chi_0, \quad \chi_1, \quad \chi_2, \quad \chi_3, \quad \dots$$

of **irreducible** representations reads

$$\mathcal{E} = 20\chi_{m.m.} - 2\chi_0 + 2(A_1\chi_1 + A_2\chi_2 + A_3\chi_3 + \dots),$$

where

n	1	2	3	4	5	...
A_n	45	231	770	2277	5796	...

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This is the **finger print** of the **Mathieu group** M_{24} !

Symmetry Surfing

Results

- By [Mukai88], if $\mathcal{G}$ is a symmetry group of a K3 surface, then $\mathcal{G} \hookrightarrow M_{24}$, and $|\mathcal{G}| \leq 960 \ll 244.823.040 = |M_{24}|$.

Symmetry Surfing

Results

- By [Mukai88], if $\mathcal{G}$ is a symmetry group of a K3 surface, then $\mathcal{G} \hookrightarrow M_{24}$, and $|\mathcal{G}| \leq 960 \ll 244.823.040 = |M_{24}|$.
- By [Gaberdiel/Hohenegger/Volpato11], the Mathieu group M_{24} is not the symmetry group of any K3 theory.
- By [Gannon12], a representation $\mathbb{H}$ exists with supercharacter $\mathcal{E}$ and yielding all predicted M_{24} -twisted characters.

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By [Song17,W17], as an $N = 4$ representation, $\mathbb{H}$ is isomorphic to the cohomology of the chiral de Rham complex of K3.

Combining K3 symmetries in M_{24}

**Conjecture & evidence [Taormina/W11,13,15,19;
Gaberdiel/Keller/Paul16; Keller/Zadeh19]**

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Result [Budzik/Taormina/Ungureanu/W/Zadeh25]:

We find an embedding $M_{12} \hookrightarrow M_{24}$, such that **combining the symmetries** of all **Kummer surfaces** with those of the $\mathbb{Z}_3$ -orbifold K3s, we **generate** the **Mathieu group** M_{24} .

The End

THANK YOU
FOR YOUR ATTENTION!