

Beyond Calabi-Yau Threefolds

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Workshop on Mirror Symmetry, Calabi-Yau Threefolds,
and Connections to Physics

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The Calabi-Yau theorem gives a parameter space for Ricci-flat metrics on a (differentiable) Calabi-Yau manifold X : namely, The bundle of Kähler cones over the moduli space of complex structures on X . In physics, this parameter space is extended by taking the product with $H^2(X, \mathbb{R})/H^2(X, \mathbb{Z})$, thereby “complexifying” the Kähler cones. Mirror symmetry should exchange the roles of complex structure and complexified Kähler class in the physical theory. Since those two parts of the parameter space do not have the same form, if mirror symmetry is to hold globally and not just locally, there must be deformations of the physical theories which go beyond Calabi-Yau manifolds, and there must be some sub-structure of the complex structure moduli space which reflects the extension of the mirror complexified Kähler space. We will survey what is known about this question.

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Mirror symmetry is, at its foundation, a statement about certain quantum field theories in physics. We do not currently know a mathematically rigorous foundation for such theories, but we do know how to translate many features of these theories into mathematical language, and can thus formulate precise mathematical statements which are consequences of mirror symmetry.

Physical background con.

In its original formulation, mirror symmetry was a statement about 2-dimensional superconformal quantum field theories, and one important class of such theories is the 2-dimensional sigma models. These require a compact Ricci-flat Riemannian manifold X^n to be specified, and the amount of supersymmetry possessed by the model is encoded in the Riemannian holonomy of the manifold. The amount of supersymmetry is twice the minimal amount exactly when the holonomy is $SU(n/2)$ ($n \geq 6$), it is four times the minimal amount exactly when the holonomy is $Sp(n/4)$, and it is eight times the minimal amount exactly when the holonomy is trivial (i.e., X is flat). The physical theory is constructed out of maps from a surface Σ to the target manifold X , and there is a natural enlargement of this class of theories to include not only a metric on X but also a real harmonic 2-form (up to an integral 2-form), known as the B -field.

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Physical background con.

In this talk, we shall only treat the twice-minimal case, so that we are excluding hyperkähler manifolds such as K3 surfaces (which are qualitatively quite different). One consequence of this is that the complex structure on X is unique, and thanks to Calabi and Yau, the Ricci-flat metric is Kähler and the complex structure plus Kähler class uniquely determine the metric. If we include the B -field by complexifying the Kähler class, then the physical parameter space splits naturally as the product of two complex manifolds.

The basic symmetry (super)algebra of such a theory acts on things such as the physical parameter space and respects the product structure there. Mirror symmetry began from the observation that the relevant superconformal algebra (for twice-minimal supersymmetry) has an outer automorphism which reverses the two factors in the product structure on the parameter space.

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The question then arose: could there be pairs of target manifolds (X, Y) together with an isomorphism of the physical theories which reversed the roles of the two factors? In other words, the complex structures on X would correspond to the complexified Kähler classes on Y and vice versa.

There have been a number of extensions of the original physical context which have led to extensions of the mathematical framework as well. For example, one may formulate the physical theories on surfaces with boundary $(\Sigma, \partial\Sigma)$, with the maps constrained to take $\partial\Sigma$ to particular calibrated submanifolds $Z_X \subset X$; there are then mirror constraints which say that the Z_X s must somehow map to the Z_Y s.

Or, one may compactify one of the “type II” superstring theories in dimension 10 to get a theory in dimension $10 - n$. In this case, the “type IIA” superstring theory must be exchanged with the “type IIB” superstring theory.

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Early examples

The first concrete examples were constructed using representation theory of the superconformal algebra. It was shown that (1) there are “discrete series”-type representations of this algebra, (2) some tensor products of these discrete series reps could also be described as sigma-models on finite quotients of Fermat-type hypersurfaces in projective spaces, and (3) this could produce mirror pairs of theories (Greene–Plesser).

The Greene–Plesser construction finds a mirror pair (X, Y) at certain particular points of the parameter spaces. It’s not hard to show in general that a particular mirror pair (X_0, Y_0) can be extended to a mirror pair (X_s, Y_t) for all s and t near the starting points, where $s \mapsto t$ is a “mirror map” on open neighborhoods of the original points.

Early examples, con.

Notice that if the Kähler cone has an integral basis, then coordinatewise exponentiation of the complexified Kähler cone maps it 1-1 to a complex polydisk, while the set of complex structures on X is a quasi-projective variety over $\mathbb{C}$ (since X itself is a compact/complete algebraic variety over $\mathbb{C}$ and the local algebraic moduli of X is well-understood).

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The quintic 3-fold

One of the Greene–Plesser examples was the quintic 3-fold in $\mathbb{CP}^4$, paired with a finite quotient of a 1-parameter family of quintics.

This was studied in great detail by Candelas–de la Ossa–Green–Parkes, who showed how to encode the physics depending on the complex structure of the 1-parameter family in terms of the (multi-valued) periods of the family of holomorphic 3-forms. In the mirror story, one has a (multi-valued) function on the polydisk corresponding to the complexified Kähler cone. The multivaluedness of periods is a well-known story in mathematics (variation of Hodge structure) and comparing the multivaluednesses led to the identification of the complexified Kähler polydisk with a polydisk in $\overline{\mathcal{M}}_Y$ near the boundary of complex moduli. Candelas et al. also identified the coefficients in an expansion of the function on the Kähler side with the numbers of rational curves of fixed degree on the quintic, thereby giving the famous mirror symmetry prediction of the latter

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Approaching the boundary of the Kähler polydisk corresponds to approaching the boundary of the Kähler cone itself, and the limit of the Calabi-Yau metric in that situation should be ill-behaved. But there are physical arguments that at least in some cases, there should be well-defined physical limits. So in the quintic case, one would hope that crossing the Kähler boundary would yield another polydisc that we could glue in next to the Kähler one, giving a tiling of $\overline{\mathcal{M}_Y}$ matching extensions of the Kähler parameters of X yielding new theories.

There are a few cases where its known how to do this in detail.

I) The birational geometry of Calabi-Yau threefolds is well understood. Kawamata has defined the “movable cone” inside $H^2(X, \mathbb{R})$, which is essentially the union of all the Kähler cones of birational models of X (up to taking closures). Although this construction provides a kind of tiling structure, its not enough to explain the full story because there are other ways a Kähler class can degenerate.

II) Among degenerations of a different kind, there are cases where a divisor E on the threefold X will contract to a singular curve. If the fiber Γ of that contraction map satisfies $E \cdot \Gamma = -2$, one can reflect the movable cone in the boundary wall $\Gamma^\perp$ to obtain a further cone or cones continuing the tiling.

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I will close by mentioning another tool which can help address this issue. There is a class of 2-dimensional quantum field theories with twice-minimal supersymmetry called gauged linear sigma models (GLSM) which are not conformally invariant, but which are expected to flow to superconformal theories under renormalization. The parameter spaces for these theories naturally divide into cone-like pieces, and in fact cases I) and II) from the previous section can be described in GLSM terms. It is hoped that further structures of the desired kind can be found using GLSMs.