Quantitative convergence of Monte Carlo methods for Boltzmann models

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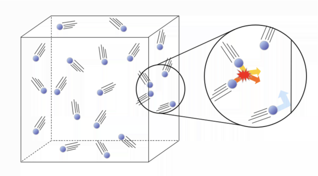
Motivation

Boltzmann equation models evolution of a rarefied gas

Particles undergo **binary collisions**

$$v' = \mathscr{C}(v, v_*, \omega) \qquad \omega \in \mathbb{S}^2$$
$$v_*' = \mathscr{C}(v_*, v, -\omega)$$



[J. James, merithub.com]

$\{x^i(t), v^i(t)\}_{i=1}^N$ approximated via $f = f(t, x, v) \geq 0$

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f, f) \qquad t \geq 0,\ x,\ v \in \mathbb{R}^3$$

- Aerospace applications (atmospheric entry, spacecraft drag,...)
- Micro- and nanoscale gas flows
- Semiconductor carrier transport

Monte Carlo methods is the standard engineering method

DSMC and Outline

Direct Simulation Monte Carlo [Bird '76]: MC approximation of $Q(f,f)$

$$\{x_i(t), v_i(t)\}_{i=1}^{N} \quad \to \quad f(t,x,v) \quad \to \quad \{X_t^i, V_t^i\}_{i=1}^{N_{\text{DSMC}}} \qquad N_{\text{DSMC}} \ll N$$

$$\int \phi(x,v)\mathrm{d}f(t,x,v) \quad \approx \quad \frac{1}{N_{\text{MC}}}\sum_{i=1}^{N_{\text{MC}}} \phi(X_t^i, V_t^i) \qquad f \approx f^N = \frac{1}{N_{\text{MC}}}\sum_i \delta_{(X^i, V^i)}$$

Quantitative error in terms of N

$$\mathbb{E}\|f - f^N\|_? \leq \textbf{?} \qquad \mathrm{Var}(f^N) = \mathbb{E}\|f^N - \mathbb{E}f^N\|_?^2 \leq \textbf{?}$$

Outline:

1) Review of DSMC methods
2) Convergence analysis proposed in [B & Pareschi '26]
3) Extension to Genetic Algorithm in optimization [B '26]

Simplified settings

- **homogeneous** in position space: $f = f(v,t)$
- Collision map $\mathscr{C}$ is Lipschitz and grows linearly

$$\begin{aligned} v' &= \mathscr{C}(v, v_*, \theta) \\ v'_* &= \mathscr{C}(v_*, v, -\theta) \end{aligned} \qquad \theta \sim b \in \mathcal{P}_\infty(\Theta)$$

- **Maxwellian** particles $B = b(\theta) \in \mathcal{P}(\Theta)$

$$\begin{aligned} \langle \phi, Q(f,f) \rangle &= \frac{1}{2} \iiint b(\mathrm{d}\theta) \left(\phi(v') + \phi(v'_*) - \phi(v) - \phi(v_*) \right) \mathrm{d}f \, \mathrm{d}f_* \\ &= \iiint b(\mathrm{d}\theta) \left(\phi(v') - \phi(v) \right) \mathrm{d}f \, \mathrm{d}f_* \quad = \quad \langle \phi, Q^+(f,f) \rangle - \langle \phi, f \rangle \end{aligned}$$

$$\frac{\partial f}{\partial t} = Q^+(f,f) - f$$

Models: Kac's [Kac, 1956], Boltzmann's, Wealth distribution [Pareschi and Toscani, 2013], Opinion formation [Toscani, 2006], optimization [Benfenati et al., 2022]

Bird DSMC [Bird '76]

Sample N particles

$V_0^i \sim f_0. \quad i \in [N]$

$t = 0$

while $t < t_{\max}$ **do**

  $\Delta t \sim \text{Exp}(2/N)$

  select two distinct indices $i \neq j$

  if $k \neq i, j$ **then**

    $V_{t+\Delta t}^k = V_t^k$

  else

    $\theta \sim b(\theta)$

    $V_{t+\Delta t}^i = \mathscr{C}(V_t^i, V_t^j, \theta)$

    $V_{t+\Delta t}^j = \mathscr{C}(V_t^j, V_t^i, -\theta)$

  end

  $t \leftarrow t + \Delta t$

end

- Collision rate per particle is $\mathcal{O}(1)$
- Physical intuition
- MC approach:

$$2\langle \phi, Q^+(f,f)\rangle \approx$$

$$\iiint b(\mathrm{d}\theta)\Big(\phi(v') + \phi(v'_*)\Big)\mathrm{d}f\,\mathrm{d}f_*$$

$$\sum_{(i,j)} \phi\left(\mathscr{C}(V_t^i, V_t^j, \theta)\right) + \phi\left(\mathscr{C}(V_t^j, V_t^i, -\theta)\right)$$

Nanbu MC [Nanbu '80]

Forward Euler scheme

$$f_{n+1} = f_n + \Delta t \left(Q^+(f_n, f_n) - f_n\right) \quad = \quad (1-\Delta t) f_n + \Delta t Q^+(f_n, f_n)$$

Sample N particles $V_0^i \sim f_0$, $i = 1, \dots, N$
$n = 1$
repeat
  for $i = 1, \dots, N$ **do**
    with probability $1 - \Delta t$:
      - $V_{n+1}^i = V_n^i$

    with probability Δt :
      - select a random particle j, V_n^j
      - $\theta \sim b(\theta)$
      - $V_{n+1}^i = \mathscr{C}(V_n^i, V_n^j, \theta)$

  end
  $n = n + 1$
until $n = n_{\max}$

- Physical intuition is weaker
- Many collisions performed in parallel
- Quantities conserved in expectation
- Exactly conservative version proposed in [Babovsky '89]

Convergence analysis results

Consider the empirical measure $f^N(t) = \frac{1}{N}\sum_{i=1}^{N} V_t^i$

- **Consistency**, ϕ bounded and Lipschitz

$$V_{t_n}^i \sim f(t_n) \quad \Longrightarrow \quad \mathbb{E}\int \phi \, \mathrm{d}f^N(t_{n+1}) \xrightarrow{N\to 0} \int \phi \, \mathrm{d}f(t_{n+1})$$

Bird [Wagner '92] Nanbu & Nanbu–Babovsky [Babovsky 1989], [Babovsky & Illner 1989] [Rjasanow & Wagner 2005]

- **Convergence with rate** for Bird's method, see Theorem 4.1 in [Pulvirenti, Wagner, Rossi 1994], which is of type

$$\left\|\mathrm{Law}(V_{t_n}^1) - f(t_n)\right\|_{L^1} \lesssim n\left(\frac{1}{N}\right)^{\varepsilon^n}$$

OUR AIM: $\sup_{[0,T]} \mathbb{E}\mathbb{W}(f_n^N, f(t_n)) \to 0$ with a rate for Nanbu MC

Best error we can hope for

$$\mathbb{W}_p(\mu,\nu) := \left(\min_{\gamma\in\Gamma(\mu,\nu)} \int |v-w|^p \,\gamma(\mathrm{d}v,\mathrm{d}w)\right)^{1/p} \qquad \mu,\nu\in\mathcal{P}_p(\mathbb{R}^{\mathsf{d}})$$

Consider $f\in\mathcal{P}(\mathbb{R}^{\mathsf{d}})$ and $p>0$ and its empirical approximation

$$\mu^N = \frac{1}{N}\sum_{i=1}^N \delta_{V^i} \quad \text{with } V^i\sim f \quad \text{i.i.d.}\,.$$

Theorem. [Fournier & Guillin, 2015] $M_q(f)$ be the q-th moment of $f\in\mathcal{P}(\mathbb{R}^{\mathsf{d}})$.

$$\mathbb{E}\left[\mathbb{W}_p(f,\mu^N)\right] \le C M_q^{p/q}(f)\varepsilon_p(N)$$

$$\varepsilon_p(N) := \begin{cases} N^{-1/2} + N^{-(q-p)/q} & \text{if } p>d/2 \text{ and } q\neq 2p,\\ N^{-1/2}\log(1+N) + N^{-(q-p)/q} & \text{if } p=d/2 \text{ and } q\neq 2p,\\ N^{-p/d} + N^{-(q-p)/q} & \text{if } p\in(0,d/2) \text{ and } q\neq d/(d-p)\end{cases}$$

Main Theorem [B & Pareschi, 2026]

Assumptions $\mathcal{C}$ Lipschitz + linear growth + bounded kernel b

Let f_n^N be constructed with the Nanbu algorithm. For $n\Delta t \in [0,T]$ it holds

$$\mathbb{E}\mathbb{W}_1(f_n, f_n^N) \leq C\left(1 + C_1 T e^{TC_2}\right) M_q^{1/q}(f_0)\varepsilon_1(N)$$

For $p = 1$ it holds

$$\mathbb{E}\mathbb{W}_1(f_n, f_n^N) \lesssim (1 + Te^T) \times \begin{cases} N^{-1/2} & \text{if } d = 1 \text{ and } q > 2, \\ N^{-1/2}\log(1+N) & \text{if } d = 2 \text{ and } q > 2, \\ N^{-1/d} & \text{if } d > 2 \text{ and } q > d/(d-1) \end{cases}$$

Proof sketch 1/3: The particle systems

Write the Nanbu particles as **Markov Chains**

$$V_0^i \sim f_0\,, \qquad \tau_n^i \sim \mathrm{Bern}(\Delta t)\,, \qquad \alpha_n^i \sim \mathrm{Unif}[0,N)\,, \qquad \theta_n^i \sim b \qquad i = 1, \dots, N$$

$$\mathbf{j} : [0,N) \to \{1,\dots,N\} \qquad \mathbf{j}(\alpha) := \lfloor \alpha \rfloor + 1$$

$$V_{n+1}^i = (1-\tau_n^i)V_n^i + \tau_n^i \mathscr{C}\left(V_n^i, V_n^{\mathbf{j}(\alpha_n^i)}, \theta_n^i\right) \qquad \text{(Nanbu)}$$

$W_0^i = V_0^i$ and $W_n^*(\alpha) \sim f_n$ if $\alpha \sim \mathrm{Unif}[0,N)$

$$W_{n+1}^i = (1-\tau_n^i)W_n^i + \tau_n^i \mathscr{C}\left(W_n^i, W_n^*(\alpha_n^i), \theta_n^i\right) \qquad \text{(IID)}$$

$W_n^i \sim f_n\,, \qquad i = 1, \dots, N$ and i.i.d. **!!!!**

Proof sketch 2/3: The coupling (visualization)

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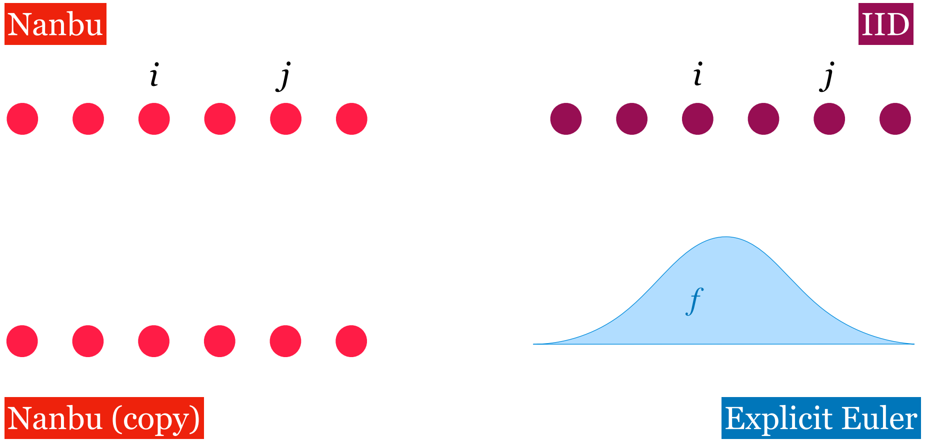

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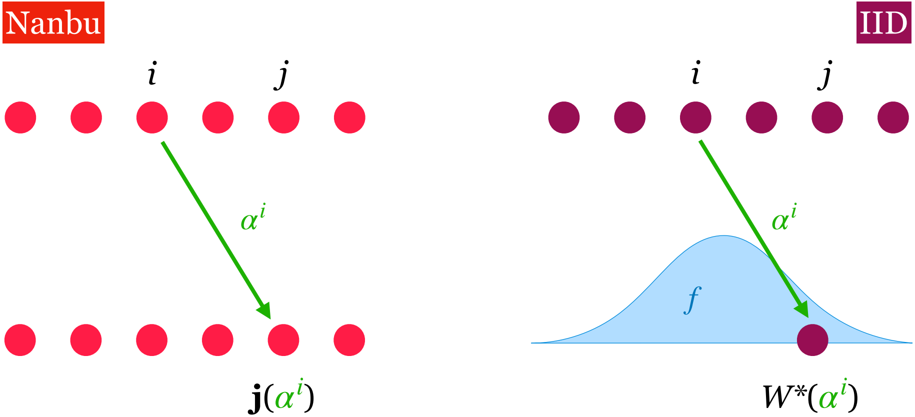

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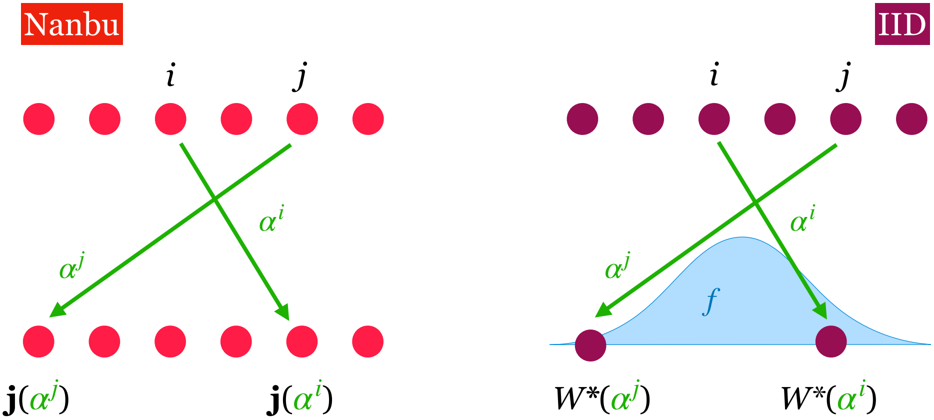

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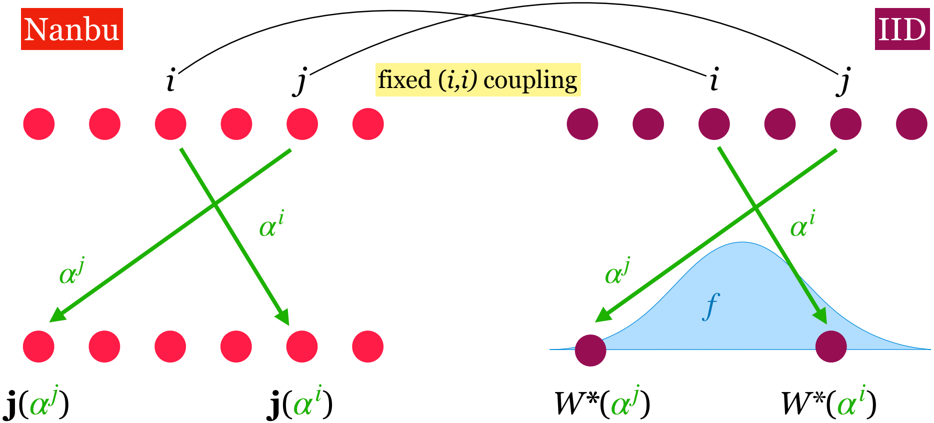
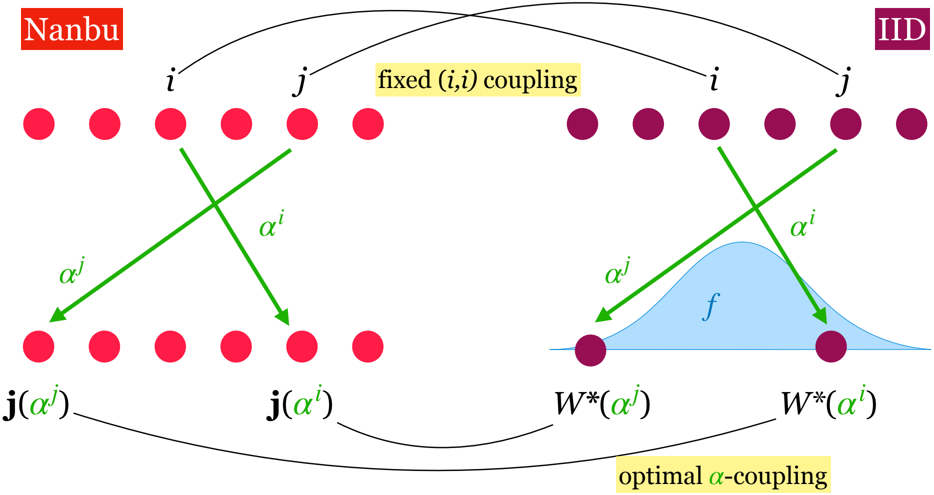


Proof sketch 2/3: The coupling (visualization)



Proof sketch 2/3: The coupling

$$\begin{cases} V_{n+1}^i = (1-\tau_n^i)V_n^i + \tau_n^i \mathscr{C}\left(V_n^i, V_n^{\mathsf{j}(\alpha_n^i)}, \theta_n^i\right) \\ W_{n+1}^i = (1-\tau_n^i)W_n^i + \tau_n^i \mathscr{C}\left(W_n^i, W_n^*(\alpha_n^i), \theta_n^i\right) \end{cases} \qquad i = 1, \ldots, N$$

Lemma. [BP26] Fix n and consider $f \in \mathcal{P}_p(\mathbb{R}^{\mathsf{d}})$, $\boldsymbol{v} = (v^i, \ldots, v^N) \in (\mathbb{R}^{\mathsf{d}})^N$. There exists a measurable mapping

$$\begin{aligned} W^* : (\mathbb{R}^{\mathsf{d}})^N \times [0, N) &\to \mathbb{R}^{\mathsf{d}} \\ (\boldsymbol{v}, \alpha) &\mapsto W^*(\boldsymbol{v}, \alpha) \end{aligned}$$

such that, if $\alpha \sim \mathrm{Unif}([0, N))$, then $(W^*(\boldsymbol{v}, \alpha), v^{\mathsf{j}(\alpha)})$ is an optimal coupling:

$$\mathbb{E}|W^*(\boldsymbol{v}, \alpha) - v^{\mathsf{j}(\alpha)}|^p = \mathbb{W}_p^p(f, f^N)\,.$$

Proof follows [Fournier & Mischler, 2016] [Cortez & Fontbona 2016]

Proof sketch 3/3: Grönwall's estimate

$$\mathbb{E}|V^i_{n+1} - W^i_{n+1}| \leq (1 - \Delta t + \Delta t L_c)\mathbb{E}|V^i_n - W^i_n| + \Delta t L_c \mathbb{E}|V_n^{\mathsf{j}(\alpha^i_n)} - W^*_n(\boldsymbol{V}_n, \alpha^i_n)|$$
$$\leq (1 - \Delta t + \Delta t L_c) \underbrace{\mathbb{E}|V^i_n - W^i_n|}_{\checkmark \text{Grönwall}} + \Delta t L_c \underbrace{\mathbb{E}\mathbb{W}_1\left(f^N_n, f_n\right)}_{??} .$$

$$\mathbb{W}_1\left(f^N_n, f_n\right) \leq \mathbb{W}_1\left(f^N_n, \overline{f}^N_n\right) + \mathbb{W}_1\left(\overline{f}^N_n, f_n\right) \leq \frac{1}{N}\sum_{i=1}^{N} \underbrace{|V^i_n - W^i_n|}_{\checkmark \text{Grönwall}} + \underbrace{\mathbb{W}_1\left(\overline{f}^N_n, f_n\right)}_{\checkmark \varepsilon(N)}$$

+ Moment's estimates $\Rightarrow$ $\mathbb{E}\mathbb{W}_1\left(f_n, f^N_n\right) \lesssim \left(1 + Te^T\right) M_q^{1/q}(f_0)\varepsilon_1(N)$

Full error analysis

$$\partial_t f = Q(f,f) \qquad \langle \phi, Q(f,f)\rangle = \iiint b(\theta)\left(\phi(v') - \phi(v)\right) \mathrm{d}\theta f(\mathrm{d}v) f \mathrm{d}v_*)$$

ASM: $v' = \mathscr{C}(v, v_*, \theta)$ Lipschitz + linear growth + b bounded

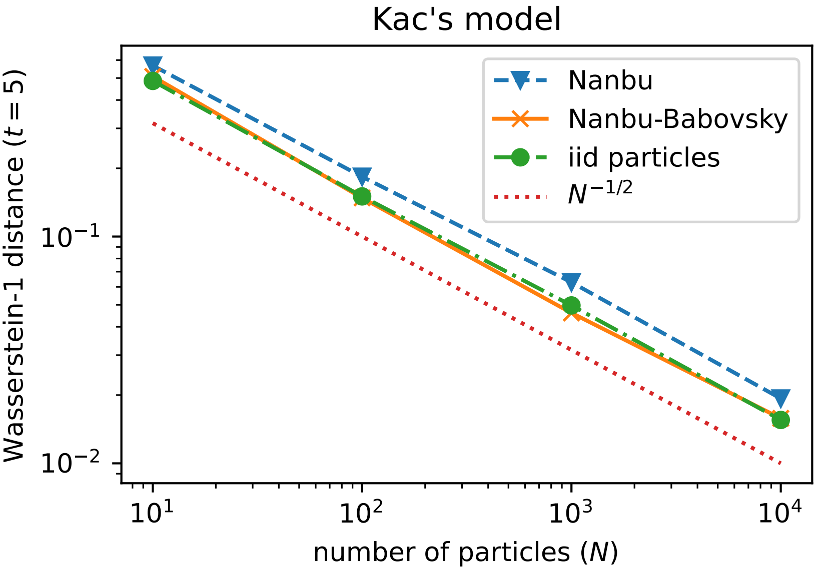
Theorem 2. [BP26] $\exists!$ **measure solution** $f \in \mathrm{Lip}([0,T], \mathcal{P}_1(\mathbb{R}^{\mathsf{d}}))$ (take $\phi \in \mathcal{C}_0^\infty$). Let $f^{\Delta t}$ be the interpolation of the **forward Euler** method. It holds

$$\sup_{t\in[0,T]} \mathbb{W}_1\left(f^{\Delta t}(t), f(t)\right) \lesssim_T \Delta t$$

Corollary. [BP26] f_n^N = Nanbu algorithm

$$\sup_{t_n = n\Delta t \in [0,T]} \mathbb{E}\mathbb{W}_1\left(f_n^N, f(t_n)\right) \lesssim_T \Delta t + \varepsilon_1(N)$$

Empirical error decay



Remarks & extensions

- ⁜ Extension to **conservative** MC (Nanbu–Babovksy, Bird) following [Cortez & Fontbona, 2016, 2018]. Additional correlation due to

$$\mathbf{j}(\alpha_n^i) = j \quad \Rightarrow \quad \mathbf{j}(\alpha_n^j) = i$$

- ⁜ Extension to **non-Maxwellian** case $b = b(\theta, v, v_*)$ [Pulvirenti et al. '94]

$$|b(\theta, v, v_*) - b(\theta, w, w_*)| \lesssim |v - w| + |v_* - w_*|$$

- ⁜ Extension to **space inhomogeneous** models
- ⁜ $\mathrm{Var}\left(\mathbb{W}_1\left(f_n^N, f(t_n)\right)\right) \lesssim$ **?**

✓ **Extension to jump processes in optimization and sampling**

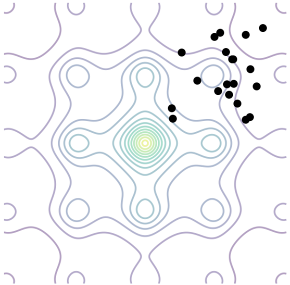
Genetic Algorithms [Holland '75]

$\mathbf{F}(\cdot) \to$ **fitness function**

$$x^\star \in \underset{x \in \mathbb{R}^d}{\operatorname{argmax}} \, \mathbf{F}(x)$$

(x, x_*) picked with probability $\propto \mathbf{F}(x), \mathbf{F}(x_*)$

$$\begin{aligned} x' &= (1-\gamma) \odot x + \gamma \odot x_* \; + \; \sigma \xi \\ x'_* &= \underbrace{(1-\gamma) \odot x_* + \gamma \odot x}_{\text{crossover}} \; + \; \underbrace{\sigma \xi_*}_{\text{mutation}} \end{aligned}$$



Training & inference in ML: **Population-Based Training** [DeepMind '17]; **NeverGrad** [Meta '18]; **Evolutionary Model Merge** [Sakana AI '25]; **Model Soups** [Wortsman et al. '22]; **Mind Evolution** [DeepMind '25] **AlphaEvolve/OpenEvolve** [DeepMind '25]

Boltzmann formulation and literature review

$$\frac{\partial}{\partial t} f(t,x) = Q^+(f,f)(t,x) - f(t,x)$$

$$\int \phi(x) Q^+(f,f)(\mathrm{d}x) = \frac{1}{2} \iiiint \frac{\mathbf{F}(x)\mathbf{F}(x_*)}{\left(\int \mathbf{F}(y) f(\mathrm{d}y)\right)^2} \, \left(\phi(x') + \phi(x'_*)\right) \, f(\mathrm{d}x) \, f(\mathrm{d}x_*) \, \mathrm{d}\gamma \, \mathrm{d}\xi$$

GA literature:

- ⋆ **"Branching-diffusion processes"** [Del Moral–Miclo '01, Del Moral '04]
- ⋆ Related to **Sequential Monte Carlo methods** [Del Moral–Doucet–Jasra '06]
- ⋆ **only selection-mutation** mechanisms considered [Del Moral – Guionnet '99]

$$\sup_{\|\phi\|_\infty < 1/2} \mathbb{E} |\langle \phi, f_n^N \rangle - \langle \phi, f_n \rangle| \lesssim \frac{1}{\sqrt{N}}$$

OUR AIM: Consider crossover, and obtain $\mathbb{EW}\left(f_n^N, f_n\right) \lesssim \varepsilon(N,d)$

A Nanbu-type Genetic Algorithm

Sample N solutions $X_0^i \sim f_0$, $i = 1, \dots, N$
$n = 0$
repeat
  for $i = 1, \dots, N$ **do**
    with probability $1 - \tau$:
      - $\tilde{X}_n^i = X_n^i$

    with probability τ :
      - sample two parents X, X_* according to

$$\mathbb{P}(X = X_n^j) = \mathbb{P}(X_* = X_n^j) = \frac{\mathbf{F}(X_n^j)}{\sum_{\ell=1}^N \mathbf{F}(X_n^\ell)}$$

      - sample $\gamma \sim \mathrm{Unif}[0,1]^d$, $\xi \sim \mathcal{N}(0, I_d)$
      - $\tilde{X}_n^i = (1-\gamma) \odot X + \gamma \odot X_* + \sigma\xi$

  end
  $X_{n+1}^i = \tilde{X}_n^i$ for all $i = 1, \dots, N$
  $n = n + 1$
until $n = n_{\max}$

Chaos propagation result

Assumptions: Fitness $\mathbf{F} : \mathbb{R}^{\mathsf{d}} \to \mathbb{R}_{>0}$ is globally Lipschitz & bounded

$$\mathbb{W}^{\wedge}(f,g) := \inf_{X\sim f, Y\sim g} \mathbb{E}\left[|X-Y| \wedge 1\right]$$

Theorem 1. [B. '26]

Let f_n^N be constructed with the GA algorithm.
$f(t) \in \mathcal{P}_q(\mathbb{R}^{\mathsf{d}}), t \in [0,T]$ measure solution to PDE. For $t_n := n\Delta t \in [0,T]$ it holds

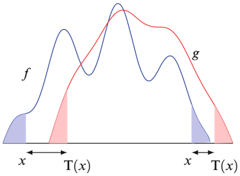
$$\mathbb{E}\mathbb{W}^{\wedge}\left(f_n^N, f(t_n)\right) \lesssim_T \varepsilon(N) + \Delta t$$

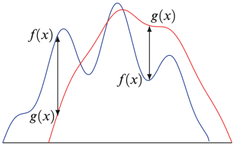
By Kantorovich–Rubinstein duality $\mathbb{W}^{\wedge}(f,g) = \|f-g\|_{\mathrm{BL}}$
$\|\cdot\|_{\mathrm{BL}} =$ **Bounded Lipschitz norm** (AKA Dudley or flat metric)

Rate of i.i.d. particles for Wasserstein-1 distance ! [Fournier–Guillin '15]

Wasserstein distance(s)

$$\mathbb{W}^d(f,g) = \inf_{X\sim f, Y\sim g} \mathbb{E}\left[d(X,Y)\right] \qquad d = \text{cost/distance}$$





[Santambrogio '16]

$d(x,y) = |x-y| \quad \to \quad \mathbb{W}^{|\cdot|}$ "horizontal" distance

✓ Good for (Lipschitz) **transport maps**

$$|x'-y'| \leq |\gamma||x-y| + |1-\gamma||x_* - y_*|$$

$d(x,y) = \delta(x-y) \quad \to \quad \mathbb{W}^{\delta}$ "vertical" TV distance

✓ Good for **densities** $\mathbb{W}^{\delta}(f,g) = \int |f(x)-g(x)|\mathrm{d}x$

$$\left|\frac{\mathbf{F}(x)f(x)}{\langle \mathbf{F}, f\rangle}\right| \leq \frac{\sup \mathbf{F}}{\inf \mathbf{F}}|f(x)|$$

$$\Rightarrow \quad D^{\wedge}(x,y) := \min\{|x-y|, 1\} = |x-y| \wedge 1$$

Stability of selection

$$\mathbb{W}^{\wedge}(f,g) = \sup\{\langle \phi, f-g\rangle \mid \mathrm{Lip}(\phi) \leq 1\,, \|\phi\|_\infty \leq 1/2\} = \|f-g\|_{\mathrm{BL}}$$

Lemma 2. [B. '26] Selection is stable w.r.t. BL norm

$$\mathbb{W}^{\wedge}\left(\frac{\mathbf{F}f}{\langle \mathbf{F}, f\rangle}, \frac{\mathbf{F}g}{\langle \mathbf{F}, g\rangle}\right) \leq C_{\mathbf{F}} \mathbb{W}^{\wedge}(f,g) \quad \text{for any} \quad f,g \in \mathcal{P}(\mathbb{R}^{\mathsf{d}})\,.$$

GA system: $\boldsymbol{X}_n = (X^1, \ldots, X_n^N)$ with associated emprical measure f_n^N

IID system: $\overline{\boldsymbol{X}}_n = (\overline{X}_n^1, \ldots, \overline{X}_n^N)$, $\overline{X}_n^i \sim f_n$, i.i.d., with empirical measure $\overline{f}_n^N$

Euler discretization: $f_{n+1} = (1-\Delta t) f_n + \Delta t Q^+(f_n, f_n)$

Wasserstein concentration inequalities [Fournier–Guillin '15]:

$$\mathbb{E}\mathbb{W}^{|\cdot|}(\overline{f}_n^N, f_n) \lesssim \varepsilon(N) \qquad \Rightarrow \qquad \mathbb{E}\mathbb{W}^{\wedge}(\overline{f}_n^N, f_n) \lesssim \varepsilon(N)$$

The two correlated particle systems

$$\tau_n^i \sim \mathsf{Bern}(\Delta t)\,, \quad \alpha_n^{1,i}, \alpha_n^{2,i} \sim \mathrm{Unif}[0,N)\,, \quad \gamma_n^i \sim \mathrm{Unif}[0,1]^d\,, \quad \xi_n^i \sim \mathcal{N}(0,I)$$

$$\mathbf{j}(\mathbf{w},\cdot) : [0,N) \to \{1,\dots,N\} \quad \text{s. t.} \quad \mathbb{P}(\mathbf{j}(\mathbf{w}_n,\alpha) = j) \propto \mathbf{F}(X_n^j) \qquad \text{if} \quad \alpha \sim \mathrm{Unif}[0,N)$$

$$X_{n+1}^i = (1-\tau_n^i)X_n^i + \tau_n^i \mathscr{C}\left(X_n^{\mathbf{j}(\mathbf{w}_n,\alpha_n^{1,i})}, X_n^{\mathbf{j}(\mathbf{w}_n,\alpha_n^{2,i})}, \gamma_n^i, \xi_n^i\right) \qquad \text{(GA)}$$

$$\overline{X}_0^i = X_0^i \quad \text{and} \quad \overline{X}_n^*(\alpha) \sim \frac{\mathbf{F} f_n}{\langle \mathbf{F}, f_n\rangle} \quad \text{if} \quad \alpha \sim \mathrm{Unif}[0,N)$$

$$\overline{X}_{n+1}^i = (1-\tau_n^i)\overline{X}_n^i + \tau_n^i \mathscr{C}\left(\overline{X}_n^*(\alpha_n^{1,i}), \overline{X}_n^*(\alpha_n^{2,i}), \gamma_n^i, \xi_n^i\right) \qquad \text{(IID)}$$

Grönwall argument

Lemma 3. [B. '26] It's possible to **optimally** construct $\overline{X}^*_n$ such that

$$\mathbb{E}D^\wedge\left(X_n^{\mathbf{j}(\mathbf{w}_n,\alpha)}, \overline{X}^*_n(\alpha)\right) = \mathbb{E}\mathbb{W}^\wedge\left(\frac{\mathbf{F}f_n^N}{\langle \mathbf{F}, f_n^N\rangle}, \frac{\mathbf{F}f_n}{\langle \mathbf{F}, f_n\rangle}\right) \qquad \text{if} \quad \alpha \sim \mathrm{Unif}[0,N)$$

$$\begin{aligned}
\mathbb{E}D^\wedge(X^i_{n+1}, \overline{X}^i_{n+1}) &\le (1-\Delta t)\mathbb{E}D^\wedge(X^i_n, \overline{X}^i_n) \\
&\quad + \Delta t\left(\mathbb{E}D^\wedge\left(X_n^{\mathbf{j}(\mathbf{w}_n,\alpha_n^{i,1})}, \overline{X}^*_n(\alpha_n^{i,1})\right) + \mathbb{E}D^\wedge\left(X_n^{\mathbf{j}(\mathbf{w}_n,\alpha_n^{i,2})}, \overline{X}^*_n(\alpha_n^{i,2})\right)\right) \\
&\overset{\textbf{Lemma 3}}{\le} (1-\Delta t)\mathbb{E}D^\wedge(X^i_n, \overline{X}^i_n) + 2\Delta t\mathbb{E}\mathbb{W}^\wedge\left(\frac{\mathbf{F}f_n^N}{\langle \mathbf{F}, f_n^N\rangle}, \frac{\mathbf{F}f_n}{\langle \mathbf{F}, f_n\rangle}\right) \\
&\overset{\textbf{Lemma 2}}{\le} (1-\Delta t)\mathbb{E}D^\wedge(X^i_n, \overline{X}^i_n) + 2\Delta t C_{\mathbf{F}}\mathbb{E}\mathbb{W}^\wedge(f_n^N, f_n) \\
&\le (1-\Delta t)\mathbb{E}D^\wedge(X^i_n, \overline{X}^i_n) + 2\Delta t C_{\mathbf{F}}\mathbb{E}\mathbb{W}^\wedge(f_n^N, \overline{f}_n^N) + 2\Delta t C_{\mathbf{F}}\underbrace{\mathbb{E}\mathbb{W}^\wedge(\overline{f}_n^N, f_n)}_{\varepsilon(N)}
\end{aligned}$$

Remarks & Outlook

✓ Monte Carlo methods for Boltzmann model converge with **iid rate**.

✓ Extension to genetic algorithms with **"truncated" Wasserstein distance**

? Ongoing work (with M. Majka, Heriot-Watt)

- un-weighted particle methods for **Wasserstein–Fisher–Rao** flows [Lu-Lu-Nolen '19]

$$\partial_t \rho_t = \nabla \cdot (\nabla V(x)\rho_t) + \Delta \rho_t - \underbrace{\left(\log(\rho_t) + V(x) - \langle \rho, \log(\rho_t) + V(x)\rangle\right) \rho_t}_{\text{birth-death dynamics}}$$

- **filtering algorithms** (Sequential Monte Carlo) [Del Moral–Doucet–Jasra '06]

Thank you.

"***Given the physical basis of the DSMC method**, the existence, uniqueness and **convergence issues** that are important in the traditional mathematical analysis of equations, **are largely irrelevant**.*" [Bird '94]

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