Quantitative Mean-Field Limits for Singular Interactions: Entropy, Modulated Energies, and Recent Results

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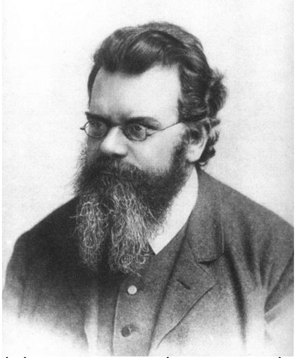
Large N limit

From Microscopic Descriptions to Macroscopic Descriptions: Maxwell (1866), Boltzmann (1872)...

(a) Maxwell (1831-1879)



(b) Boltzmann (1844-1906)

Deriving Boltzmann/Landau and (McKean-)Vlasov-type Kinetic Equations. Bose-Einstein Condensation. Consistency of discrete algorithms...

Mean Field Limit for Newton Dynamics

Consider the classical Newton dynamics for N indistinguishable point particles in the mean filed scaling in the classical regime. Denote (x_i, v_i) the position and the velocity of particle number i. Then

$$\dot{x}_i = v_i, \quad \dot{v}_i = \frac{1}{N}\sum_{j\neq i} K(x_i - x_j), \quad i = 1, 2, \cdots, N,$$

where $x_i, v_i \in \mathbb{R}^3$.
As $N \to \infty$, the conjectured PDE is the famous Vlasov(-Poisson) equation

$$\partial_t f + v \cdot \nabla_x f + K \star_x \rho_f \cdot \nabla_v f = 0,$$

where $\rho_f(t, x) = \int_{\mathbb{R}^3} f(t, x, v)\, \mathrm{d}v$.
Gravitational or Coulomb force: $K(x) = \pm C \frac{x}{|x|^3}$, i.e. the inverse-square law.
Hauray & Jabin ('07 and '15) , Lazarovici & Pickl ('15), Jabin & W. ('16), Bresch, Jabin & Soler ('25), Bresch, Duerinckx & Jabin ('24+)

Mean Field Limit for the First Order Model

Consider the weakly interacting particle system for N indistinguishable point particles. Denote $x_i \in \mathbb{R}^d$ the position of particle number i. The dynamics reads

$$\mathrm{d}x_i = \frac{1}{N}\sum_{j\neq i} K(x_i - x_j)\,\mathrm{d}t + \sqrt{2\sigma}\,\mathrm{d}W_t^i, \quad i = 1, 2, \cdots, N, \quad (IPS)$$

where $x_i \in \mathbb{R}^d$, and W^i are N independent Brownian motions which may model random collisions on particles with rate σ. Let us assume that $\sigma > 0$. The interaction kernels K model 2−body interaction forces between particles. In the point vortex model, one takes $K(x) = \frac{1}{2\pi}\frac{(x_2, -x_1)}{|x|^2}$ in $\mathbb{R}^2$.
The (proved) limit PDE reads (as $N \to \infty$)

$$\partial_t f + \operatorname{div}_x(f K \star_x f) = \sigma \Delta_x f. \qquad (MFD)$$

Goal: Establish and quantify the convergence: Any k−marginal $F_{N,k}(t)$ converges weakly to $f_t^{\otimes k}$, or the empirical measure $\mu_N^t = \frac{1}{N}\sum_{i=1}^N \delta_{x_i^t}$ converges in law to δ_{f_t}.

Literature Overview

- Classical Results: McKean ('67), Braun& Hepp ('77), Dobrusin ('79), Sznitmann ('91) ... $K \in W^{1,\infty}$ (K is Lipschitz!) (Coupling Method).
- Compactness method: 2D Navier-Stokes: Osada ('87), Fournier, Hauray and Mischer ('14). (The Biot-Savart kernel $K(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2}$.)
- Relative Entropy/Modulated (Free) Energy Method: Jabin and W. ('18). Serfaty and Duerinckx ('20). Bresch, Jabin and W. ('23) (general singular kernels). Nguyen-Rosenzweig-Serfaty ('22). Guillin, Le Bris & Monmarché ('24).
- Central Limit Theorem: Fernandez and Méléard ('97): $K \in C_b^{2+d/2}$. W., Zhao and Zhu ('23), Chen, Holzinger and Jüngel ('26), Nikolaev ('25+), Hao, Zhang and Zhao ('26+)...
- Large Deviation Principle: Liu and Wu ('19). Chen and Ge ('25)...

The Liouville Equation

Statistical approach: The coupled law of $N-$particle $F_N(t, x_1, \cdots, x_N)$ is governed by the Liouville equation/forward Kolmogorov equation

$$\partial_t F_N + \sum_{i=1}^{N} \mathrm{div}_{x_i}\Big(F_N \frac{1}{N}\sum_{j\neq i} K(x_i - x_j)\Big) = \sigma \sum_{i=1}^{N} \Delta_{x_i} F_N.$$

The coupled law F_N is symmetric/exchangeable, i.e. $F_N \in \mathcal{P}_{sym}(\mathbb{R}^{dN})$, since those particles are indistinguishable.
The observable (statistical information: temperature, pressure for instance) is contained in the marginals $F_{N,k}$ of F_N as

$$F_{N,k}(t, x_1, \cdots, x_k) = \int_{\mathbb{R}^{d(N-k)}} F_N(t, x_1, \cdots, x_N)\,\mathrm{d}x_{k+1} \cdots \mathrm{d}x_N,$$

for fixed $k = 1, 2, \cdots$.
BBGKY hierarchy: The evolution of $F_{N,k}$ involves $F_{N,k+1}$.

Formal Derivation Assuming Molecular Chaos

BBGKY hierarchy: The evolution of $F_{N,k}$ involves $F_{N,k+1}$. For example, integrating the Liouville Eq. w.r.t. $x_2, \cdots, x_N$ and using the symmetry of F_N, one obtains that

$$\partial_t F_{N,1} + \frac{N-1}{N} \int_E \mathrm{div}_x(F_{N,2}\, K(x-y))\, \mathrm{d}y = \sigma \Delta_x F_{N,1}.$$

If we assume that $F_{N,2}(t,x,y) = F_{N,1}(t,x) F_{N,1}(t,y)$ (Molecular Chaos) for any $t \geq 0$, then we recover the limit PDE (MFD) as $N \to \infty$.
But for fixed N, $F_{N,2}(t) \neq F_{N,1}(t)^{\otimes 2}$, even we start from i.i.d. initial data $F_N(0) = F_{N,1}(0)^{\otimes N}$, since correlation does exist since particles do interact!

Relaxation: **Kac's chaos** ('56). Introduced by Kac to derive the spatially homogeneous Boltzmann equation.

Propagation of Chaos

Definition 1 (Kac's chaos)

Let $E = \mathbb{R}^d$ or any Polish space. A sequence $(F_N)_{N\geq 2}$ of symmetric probability measures, *i.e.* $F_N \in \mathcal{P}_{Sym}(E^N)$, is said to be $f-$chaotic for a probability measure f on E, if for any fixed $k = 1, 2, 3, \cdots$, $F_{N,k} \rightharpoonup f^{\otimes k}$, as $N \to \infty$.

"Asymptotic independence" for a finite group.

Definition 2 (Propagation of (Kac's) chaos)

The diagram commutes.

$$\begin{array}{ccc} F_{N,k}(0) & \rightharpoonup & f^{\otimes k}(0) \\ \Downarrow_{\text{IPS}} & & \Downarrow_{\text{MFD}} \\ F_{N,k}(t) & \rightharpoonup & f^{\otimes k}(t) \end{array}$$

See the works by Laure Saint-Raymond's group and also the recent breakthrough by Deng-Hani-Ma ('24+).

Relative Entropy

We use the (scaled) relative entropy to quantify chaos

$$0 \leq \mathcal{H}_N(F_N|f^{\otimes N})(t) = \frac{1}{N}\int_{E^N} F_N \log \frac{F_N}{f^{\otimes N}}\,\mathrm{d}x_1\cdots\mathrm{d}x_N.$$

Thanks to the monotonicity of the (scaled) relative entropy

$$\mathcal{H}_k(F_{N,k}|f^{\otimes k}) := \frac{1}{k}\int_{E^N} F_{N,k} \log \frac{F_{N,k}}{f^{\otimes k}}\,\mathrm{d}x_1\cdots\mathrm{d}x_k \leq \mathcal{H}_N(F_N|f^{\otimes N})$$

and the classical Csiszár-Kullback-Pinsker inequality

$$\|F_{N,k} - f^{\otimes k}\|_{L^1} \leq \sqrt{2k\mathcal{H}_k(F_{N,k}|f^{\otimes k})},$$

one can obtain *propagation of chaos* given a vanishing sequence of $H_N(F_N|f^{\otimes N})$.
See also Ben Arous & Zeitouni ('99).

Quantitative Entropic Propagation of Chaos for 2D Vortex Model on $\mathbb{T}^2$

Theorem (Jabin & W. ('18))

Assume that $K \in \dot{W}^{-1,\infty}(\mathbb{T}^d)$ with $\operatorname{div} K \in \dot{W}^{-1,\infty}$. Assume further that $f \in L^\infty([0,\ T],\ W^{2,p}(\mathbb{T}^d))$ for any $p < \infty$ solves (MFD) with $\inf f > 0$ and $\int_{\mathbb{T}^d} f = 1$. Then

$$\mathcal{H}_N(F_N^t \,|\, f_t^{\otimes N}) \leq e^{\bar{M}(\|K\| + \|K\|^2)\, t} \left(\mathcal{H}_N(F_N^0 \,|\, f_0^{\otimes N}) + \frac{1}{N} \right),$$

where we denote $\|K\| = \|K\|_{\dot{W}^{-1,\infty}} + \|\operatorname{div} K\|_{\dot{W}^{-1,\infty}}$ and $\bar{M}$ is a universal constant.

This result applies to the Biot-Savart law, i.e. $K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}$, since $K = \operatorname{div} V$ with

$$V = \frac{1}{2\pi} \begin{bmatrix} -\arctan \frac{x_1}{x_2} & 0 \\ 0 & \arctan \frac{x_2}{x_1} \end{bmatrix}.$$

Uniform-in-time propagation of chaos by **Guillin, Le Bris & Monmarché** ('24).

2D Vortex Model on $\mathbb{R}^2$

Theorem (Feng & W. ('26, '26))

Assume that F_N is an entropy solution to the Liouville equation and that $f \in L^\infty([0,T], L^1 \cap L^\infty(\mathbb{R}^2))$ solves (MFD) with $f \geq 0$ and $\int_{\mathbb{R}^2} f(t,x)\,\mathrm{d}x = 1$. Assume that there exist constants C_1, C_2, C_3 such that the initial data $f_0 \in W^{2,\infty}(\mathbb{R}^2)$ satisfies the growth condition

$$|\nabla \log f_0(x)|^2 \leq C_1(1+|x|^2), \quad |\nabla^2 \log f_0(x)| \leq C_2(1+|x|^2),$$

and the Gaussian upper bound

$$f_0(x) \leq C_3 \exp(-C_3^{-1}|x|^2).$$

Then we have

$$H_N(F_N^t | f_t^{\otimes N}) \leq C(1+t)^C \left(H_N(F_N^0 | f_0^{\otimes N}) + \frac{1}{N} \right),$$

where C is some universal constant that only depends on those initial bounds.

Evolution of the Relative Entropy

Let us focus on the torus case first. We compute the time evolution of the relative entropy

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_N(F_N|f^{\otimes N})(t) \leq -\frac{\sigma}{N}\int_{\Pi^{dN}} |\nabla \log \frac{F_N}{f^{\otimes N}}|^2 \,\mathrm{d}F_N + \int_{\Pi^{dN}} \Big(\frac{1}{N^2}\sum_{i,j=1}^{N} \phi(x_i, x_j)\Big)\,\mathrm{d}F_N,$$

where upon symmetrization, i.e. taking $\frac{1}{2}(\phi(x,y)+\phi(y,x))$ as the new $\phi(x,y)$, one writes

$$\begin{aligned}\phi(x,y) = &-\frac{1}{2}K(x-y)\cdot(\nabla \log f(x) - \nabla \log f(y)) - \operatorname{div}K(x-y)\\ &+\frac{1}{2}\nabla \log f(x)\cdot K * f(x) + \frac{1}{2}\nabla \log f(y)\cdot K * f(y).\end{aligned}$$

At least for the Biot-Savart law, we have that $\phi \in L^\infty$, as long as f is regular enough.

Uniform in N Large Deviation Type Estimate

Recall a Jensen-type (Gibbs) inequality, i.e. for any parameter $\eta > 0$,

$$\int F_N \Phi_N \leq \frac{1}{\eta}\frac{1}{N}\int F_N \log\frac{F_N}{\bar{f}^{\otimes N}} + \frac{1}{\eta}\frac{1}{N}\log\int \bar{f}^{\otimes N}\exp\left(\eta N \Phi_N\right).$$

We only need to show that the 2nd term is $o(1)$ as $N \to \infty$ which is guaranteed by

Theorem (Jabin & W. ('18))

There exists a constant c_0, such that when $\|\phi\|_{L^\infty} \leq c_0$ and

$$\int_E \phi(x,y)\bar{\rho}(y)\,\mathrm{d}y = 0, \forall x, \quad \int_E \phi(x,y)\bar{\rho}(x)\,\mathrm{d}x = 0, \forall y,$$

we have

$$\sup_{N\geq 2}\int_{\Pi^{dN}} \bar{f}^{\otimes N}\exp\left(\frac{1}{N}\sum_{i,j=1}^{N}\phi(x_i,x_j)\right)\mathrm{d}X^N \leq C < \infty.$$

The Landau-Maxwellian Equation

Recall the spatially homogeneous Landau equation reads

$$\partial_t f = Q_L(f, f), \quad t \geq 0, v \in \mathbb{R}^d,$$

where $f = f(t, v)$ denotes the probability density function now only for the velocity variable.

We first consider the case with Maxwellian molecules, i.e. when $a(z) = |z|^2 I_d - z \otimes z$.

The Landau system conserves the total mass, momentum and kinetic energy of the gas. We normalize the quantities to

$$\int_{\mathbb{R}^d} f(v)\,\mathrm{d}v = 1, \quad \int_{\mathbb{R}^d} v f(v)\,\mathrm{d}v = 0, \quad \int_{\mathbb{R}^d} |v|^2 f(v)\,\mathrm{d}v = d, \text{or}$$

We can also rewrite the Landau equation in a compact formulation as

$$\partial_t f = \nabla \cdot [(a * f)\nabla f - (b * f) f], \quad \text{or} \quad \partial_t f = (a * f) : \nabla^2 f - (c * f) f,$$

with the vector field $b = \nabla \cdot a$ and the scalar $c = \nabla \cdot b$, namely

$$b_\alpha(v) = \partial_\beta a_{\alpha\beta}(v) = -(d-1)v_\alpha, \quad c(v) = \partial_\alpha b_\alpha(v) = -d(d-1).$$

Particle Systems for the Landau-Maxwellian

We are interested in deriving the Landau equation as the mean-field limit of many particle system. We consider the N indistinguishable interacting particle system for instance in [Fournier '09] that

$$\mathrm{d}V_t^i = \frac{2}{N}\sum_{j=1}^N b(V_t^i - V_t^j)\,\mathrm{d}t + \sqrt{2}\Big(\frac{1}{N}\sum_{j=1}^N a(V_t^i - V_t^j)\Big)^{\frac{1}{2}}\,\mathrm{d}B_t^i, \quad i = 1, ..., N.$$

We use the convention that $a(0) = 0$ and $b(0) = 0$ to omit the notation $i \neq j$. Applying Itô's formula and the relation $\nabla \cdot a = b$, we can derive the Liouville equation of N-particle joint distribution $F_N(t, V)$, $V = (v^1, \dots, v^N)$ on $\mathbb{R}^{dN}$ as

$$\partial_t F_N = \sum_{i=1}^N \mathrm{div}_{v_i}\Big[\frac{1}{N}\sum_{j=1}^N a(v^i - v^j)\nabla_{v^i} F_N - \frac{1}{N}\sum_{j=1}^N b(v^i - v^j) F_N\Big],$$

where the initial value is fully tensorized as $F_N(0) = f_0^{\otimes N}$.

Entropic Propagation of Chaos

Theorem (Carrillo, Feng, Guo, Jabin and W. ('26))

For any $T > 0$, we assume that classical solution $f \in \mathcal{C}^1([0, T], \mathcal{C}^2(\mathbb{R}^d))$ solves the Landau equation with $f \geq 0$ and the conservation laws. Assume that the initial data $f_0 \in \mathcal{C}^2(\mathbb{R}^d)$ and there exists some positive constants C_1, C_2 and C_3 such that f_0 satisfies the Gaussian-type bound from above that

$$f_0(v) \leq C_1 \exp(-C_1^{-1}|v|^2),$$

and the logarithm growth conditions

$$|\nabla \log f_0(v)| \leq C_2(1 + |v|), \quad |\nabla^2 \log f_0(v)| \leq C_3(1 + |v|^2).$$

Then for any entropy solution of the Liouville equation, we have the following estimate

$$\mathcal{H}_N(F_N|f_N)(t) \leq \mathcal{H}_N(F_N|f_N)(0) + \frac{C_T}{\sqrt{N}}.$$

Sequential Interacting Diffusions

We investigate a non-exchangeable system where particle i interacts only with its predecessors $\{1, \ldots, i-1\}$. This model was firstly introduced by Du-Jiang-Li ('23+) .

- **The Sequential System:** For $i \geq 2$:

$$dX_t^i = \left(b(t, X_t^i) + (K \star \mu_t^{i-1})(X_t^i)\right) dt + \sigma dB_t^i,$$

where $\mu_t^{i-1} := \frac{1}{i-1} \sum_{j<i} \delta_{X_t^j}$ is the empirical measure of predecessors.

- **The Limit (McKean-Vlasov):**

$$d\overline{X}_t = \left(b(t, \overline{X}_t) + (K \star \overline{\rho}_t)(\overline{X}_t)\right) dt + \sigma dB_t,$$

where $\overline{\rho}_t = \mathcal{L}(\overline{X}_t)$ and $(K \star \nu)(x) := \int_{\mathbb{R}^d} K(x, y)\nu(dy)$.

Goal: Quantify the convergence of the i-th particle to the limit as $i \to \infty$.

Path-space Formulation and Incremental Relative Entropy

For $t \in [0, T]$, set $C_t := C([0,t]; \mathbb{R}^d)$ and define path laws

$$\bar{P}_{[0,t]} := \mathcal{L}(\bar{X}_{[0,t]}) \in \mathcal{P}(C_t), \qquad P^{1:i}_{[0,t]} := \mathcal{L}\big((X^1, \ldots, X^i)_{[0,t]}\big) \in \mathcal{P}(C^i_t).$$

Theorem (W. & Zhao ('26+))

For the sequential interacting diffusion before, we assume that the drift term and the interaction term are both bounded and the diffusion coefficient σ is a non-degenerate constant matrix. Then we have the estimates for incremental relative entropy

$$R_i(t) := \mathrm{H}\left(P^{1:i}_{[0,t]} \,\middle|\, P^{1:i-1}_{[0,t]} \otimes \bar{P}_{[0,t]}\right) \lesssim \frac{1}{i-1}, \qquad i \geq 2,$$

and the global entropy

$$S_N(t) := \mathrm{H}\left(P^{1:N}_{[0,t]} \,\middle|\, \bar{P}^{\otimes N}_{[0,t]}\right) \lesssim \log N.$$

Modulated (Free) Energy

Now we focus on gradient flows, i.e. $K = -\nabla V$. To show the mean field limit/propagation of chaos for (IPS) with more singular interactions, *Relative Entropy Method* itself is not enough.
The modulated energy method introduced by Serfaty (with an appendix with Duerinckx) ('20)) can treat more singular (Coulomb, Riesz potentials+ possible perturbation!) interactions for deterministic flows.
In Bresch-Jabin-W. ('23), we put relative entropy and modulated energy together to form a modulated free energy as

$$E_N(F_N|f^{\otimes N}) = \mathcal{H}_N(F_N|f^{\otimes N}) + \mathcal{K}_N(F_N|f^{\otimes N}),$$

with

$$\mathcal{K}_N(F_N|f^{\otimes N}) = \frac{1}{\sigma}\mathbb{E}_{F_N}\Big\{\frac{1}{2}\int_{x\neq y} V(x-y)(\,\mathrm{d}\mu_N(x) - \,\mathrm{d}f(x))(\,\mathrm{d}\mu_N(y) - \,\mathrm{d}f(y))\Big\}.$$

See also the 2D Coulomb gas with additive noises on the whole space in Cai-Feng-Gong-W. ('24+).

Derivation of the Patlak-Keller-Segel System

Take $K = -\nabla V$ and $V = \lambda \log |x| + V_e(x)$ where $\lambda > 0$. Then (MFD) is the famous Patlak-Keller-Segel (PKS) model, which is one of the first models of chemotaxis for micro-organisms. Note that in 2D, V is the **attractive Poisson** potential.

Theorem (Bresch, Jabin & W. ('23))

Given the potential V and $K = -\nabla V$. Assume that $\rho_N \in L^\infty(0, T; L^1(\Pi^{dN}))$ is an entropy solution to the Liouville equation, with initial condition that $\rho_N(0) = \bar{\rho}^{\otimes N}(0)$. Assume that $\bar{\rho} \in L^\infty(0, T; W^{2,\infty}(\Pi^d))$ solves (MFD) with $\inf \bar{\rho} > 0$. Assume further that $\lambda < 2d\sigma$. Then there exists a constant $C > 0$ and an exponent $\theta > 0$, independent of N, such that for any fixed k,

$$\|\rho_{N,k} - \bar{\rho}^{\otimes k}\|_{L^\infty(0,T;L^1(\Pi^{kd}))} \leq \frac{Ck^{1/2}}{N^\theta}.$$

The optimal constant 4σ in 2D corresponds to the critical mass $8\pi\sigma$ for which we have blow-up in finite time for PKS. (Blanchet,Dolbeault and Perthame ('06)).

Strategies

- The modulated free energy E_N "effectively" control the distance between ρ_N and $\bar{\rho}_N$. **Goal:**

$$E_N(\rho_N|\bar{\rho}_N) \geq \frac{1}{C}\mathcal{H}_N(\rho_N|\bar{\rho}_N) - \frac{C}{N^\theta}.$$

- Control $\frac{\mathrm{d}}{\mathrm{d}t}E_N$ above by E_N or $\mathcal{H}_N$ or $\mathcal{K}_N + C/N^\theta$. The PKS case is okay by the previous large deviation estimates, since now for $|\nabla V(x)| \leq C/|x|$.
- For more singular kernels, we need to establish that

$$-\frac{1}{2\sigma}\int_{\Pi^{dN}} \mathrm{d}\rho_N \int_{x\neq y} \nabla V(x-y)(\psi(x)-\psi(y))(\mathrm{d}\mu_N - \mathrm{d}\bar{\rho})^{\otimes 2}$$
$$\leq C\mathcal{K}_N(\rho_N|\bar{\rho}_N) + C\mathcal{H}_N(\rho_N|\bar{\rho}_N) + \varepsilon(N),$$

where $\varepsilon(N) \to 0$ as $N \to \infty$.

The Fluctuation Measures

We define the fluctuation measure as $\eta^N(t) = \sqrt{N}(\mu_N - \bar{\rho})$. Consider (IPS) with $\sigma_N = \sigma > 0$. Then by Ito's formula, one has

$$\mathrm{d}\langle \eta_t^N, \varphi \rangle = \langle \sigma \Delta \varphi, \eta_t^N \rangle \,\mathrm{d}t + \mathcal{K}_t^N(\varphi)\,\mathrm{d}t + \frac{\sqrt{2\sigma}}{\sqrt{N}} \sum_{i=1}^N \nabla \varphi(X_i^t) \cdot \mathrm{d}B_t^i,$$

where the interaction term is given by

$$\mathcal{K}_t^N(\varphi) = \sqrt{N} \langle \nabla \varphi, \mu_N K * \mu_N - \bar{\rho} K * \bar{\rho} \rangle.$$

By direct computations, one has

$$\begin{aligned} &\sqrt{N}(\mu_N K * \mu_N - \bar{\rho} K * \bar{\rho}) \\ =& \bar{\rho} K * \eta^N + \eta^N K * \bar{\rho} + \frac{1}{\sqrt{N}} \eta^N K * \eta^N. \end{aligned}$$

The Limiting SPDE

Formal computations indicates that the expected limit equation for the fluctuation measures should be

$$\partial_t \eta = \sigma \Delta \eta - \nabla \cdot (\bar{\rho} K * \eta) - \nabla \cdot (\eta K * \bar{\rho}) - \sqrt{2\sigma} \nabla \cdot \left(\sqrt{\bar{\rho}} \xi \right), \ (SPDE)$$

where ξ is a space-time noise on $\mathbb{R}^+ \times \mathbb{T}^d$, i.e. a family of centered Gaussian random variables.

We use the martingale solution to the previous SPDE. In short, we call $(\eta_\cdot)$ a martingale solution to the SPDE on some stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$, if η is a continuous $(\mathcal{F}_t)-$adapted process in $C([0,T], H^{-\alpha-2}) \cap L^2([0,T], H^{-\alpha})$, $\forall \alpha > d/2$, a.s. and for any test function $\varphi \in C^\infty(\Pi^d)$ and for any $t \in [0,T]$,

$$\langle \eta_t, \varphi \rangle - \langle \eta_0, \varphi \rangle = \int_0^t \langle \sigma \Delta \varphi, \eta \rangle \,\mathrm{d}s + \int_0^t \langle \nabla \varphi, \bar{\rho} K * \eta + \eta K * \bar{\rho} \rangle + \mathcal{M}_t(\varphi),$$

where $M_t(\varphi)$ is a continuous adapted centered Gaussian process with values in $H^{-\alpha-1}$ for each $\alpha > d/2$.

Assumptions

Our analysis build on our previous work (Jabin & W. ('18)). We make the following assumptions.

- **(A1)-CLT for initial values.** There exists η_0, which belongs to the space of tempered distributions, such that the sequence $\{\eta_0^N\}_{N\geq 1}$ converges in distribution to η_0 in $\mathcal{S}'(\Pi^d)$.
- **(A2)-Regularity for the kernel.** The kernel K is either bounded or can be made bounded upon symmetrization. (Include the Biot-Savart law.)
- **(A3) -The optimal convergence rate in Jabin and W. ('18)**. That is $\mathcal{H}_N(\rho_N|\bar{\rho}_N) \leq \frac{C_T}{N}$.
- **(A4) - The regularity of the mean-field PDE.** There exists $\beta > d/2$, s.t. $\bar{\rho} \in C([0,T], C^\beta(\Pi^d))$

Gaussian Fluctuations

Theorem (W., Zhao, Zhu ('23))

Under assumptions **(A1)-(A4)**, *the sequence* η^N *converges in distribution to* η *in the space* $L^2([0,T], H^{-\alpha}) \cap C([0,T], H^{-\alpha-2})$ *for every* $\alpha > d/2$, *where* η *is the unique (in distribution) martingale solution to the SPDE.*

Note that here the regularity $\alpha > d/2$ is optimal (rewriting the SPDE in mild form and study the stochastic part by Kolmogorov's theorem, then applying Schauder estimates).

A general result before requires that $K \in C^{d/2+2}$ by Fernandez and Méléard ('97).

Our limit $(\eta_\cdot)$ is characterized by a generalized Ornstein-Uhlenbeck process.

We can treat the general case $\sigma_N \to \sigma \geq 0$ but with $|\sigma_N - \sigma| = O(1/N)$ as well, but the assumption for the mean-field law $\bar{\rho}$ should be modified.

New Large Deviation Type Estimate

Theorem (W., Zhao, Zhu ('23))

For any probability measure $\bar{\rho}$ on a Polish space $S (= \Pi^d)$. Assume further that $\phi \in L^\infty(S^2)$ with $\|\phi\|_{L^\infty}$ is small enough, and that

$$\int_{S^2} \bar{\rho}(x)\bar{\rho}(y)\phi(x,y)\,\mathrm{d}x\,\mathrm{d}y = 0.$$

Then

$$\sup_{N\geq 2} \int_{S^N} \bar{\rho}^{\otimes N} \exp\left(N|\langle \phi, \mu_N \otimes \mu_N\rangle|^2\right) \mathrm{d}X^N \leq C < \infty.$$

This estimate can be rewritten as the following form

$$\sup_{N\geq 2} \int_{S^N} \bar{\rho}^{\otimes N} \exp\left(\kappa N|\langle h, \mu_N^{\otimes 2} - \bar{\rho}^{\otimes 2}\rangle|^2\right) \mathrm{d}X^N < C < \infty.$$

Uniform Estimates for η^N

To go through the standard procedure (1. Tightness; 2. Identifying the limit;3. The well-posedness of the limiting SPDE), we need

1. For any $\alpha > d/2$, for any $t \in [0, T]$, one has

$$\mathbb{E}\|\mu_N(t) - \bar{\rho}_t\|_{H^{-\alpha}}^2 \lesssim \left(\mathcal{H}_N(\rho_N|\bar{\rho}_N)(t) + \frac{1}{N}\right).$$

2. Assuming that $|x|K \in L^\infty$ and K is anti-symmetric. Assume that $\alpha > 2 + d/2$, then

$$\mathbb{E}\|\nabla \cdot (\mu_N K * \mu_N - \bar{\rho} K * \bar{\rho})\|_{H^{-\alpha}}^2 \lesssim \left(\mathcal{H}_N(\rho_N|\bar{\rho}_N)(t) + \frac{1}{N}\right)$$

3. For any $\phi \in W^{1,\infty}$, one has

$$\mathbb{E}\Big|\langle \phi, (\mu_N - \bar{\rho}) K * (\mu_N - \bar{\rho})\rangle\Big| \lesssim \left(\mathcal{H}_N(\rho_N|\bar{\rho}_N) + \frac{1}{N}\right).$$

Entropy-dissipation Informed PDE Solver

This part is based on the following two papers.

- Shen, Z., Wang, Z., Kale, S., Ribeiro, A., Karbasi, A. and Hassani, H., 2022, June. Self-consistency of the fokker planck equation. In Conference on Learning Theory (pp. 817-841). PMLR.
- Shen, Z. and Wang, Z., 2023. Entropy-dissipation Informed Neural Network for McKean-Vlasov Type PDEs. arXiv preprint arXiv:2303.11205. NeurIPS 2023.

Our goal is to solve the Cauchy problem of the PDE of the continuity type, which describes the evolution of a density function $\bar{\rho}_t$. For instance let us assume that

$$\partial_t \bar{\rho}_t + \mathrm{div}_x(\bar{\rho}_t \mathcal{A}[\bar{\rho}_t]) = 0, \quad \bar{\rho}|_{t=0} = \bar{\rho}_0.$$

Initially, we can guess a velocity field $f = f(t, x; \theta)$. With this velocity field and initial data, one can compute the flow generated by f, say X_t^θ, and then the computed density ρ_t^θ.

Entropy-dissipation Informed PDE Solver continued

One can then cook up the following loss function on the time interval $[0, T]$ based on the self-consistency structure of the continuity equation, i.e.

$$R(\theta) = \int_0^T \int_{\mathbb{R}^d} |f(t, x; \theta) - \mathcal{A}[\rho_t^\theta]|^2 \, \mathrm{d}\rho_t^\theta(x) \, \mathrm{d}t.$$

Then for many important cases, for instance the linear Fokker-Planck case, 2D Navier-Stokes cases and the McKean-Vlasov equation case in general, one has the following estimate

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^d} \rho_t^\theta \log \frac{\rho_t^\theta}{\bar{\rho}_t} \leq CR(\theta) = C \int_0^T \mathrm{d}t \int_{\mathbb{R}^d} \mathrm{d}\bar{\rho}_0(x) \left| f(t, X(t,x); \theta) - \mathcal{A}[\bar{\rho}_t^\theta](X(t,x)) \right|^2.$$

The Kinetic Landau Equation

We recall the (full) kinetic (Fokker-Planck-) Landau equation as

$$\partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = Q_L(f, f), \quad t \geq 0, x \in \Omega \subset \mathbb{R}^d, v \in \mathbb{R}^d,$$

where $f = f(t, x, v)$ denotes the probability density, F is an exterior force and can be chosen to be self-consistent for instance as $K *_x \rho_f$, and the Landau collision operator is defined as

$$Q_L(f, f)(v) = \mathrm{div}_v \left(\int_{\mathbb{R}^d} a(v - v_*)[f(v_*) \nabla f(v) - f(v) \nabla_{v_*} f(v_*)] \, \mathrm{d}v_* \right),$$

where $d \geq 2$, and a is a (semi-positive-definite) matrix given by

$$a(z) = |z|^{\gamma+2} \left(I_d - \frac{z \otimes z}{|z|^2} \right), \quad \gamma \in [-d-1, 1].$$

Note that $d = 3, \gamma = -3$ corresponds to the Coulomb interactions. Maxwellian molecules: $\gamma = 0$; Hard potential: $\gamma \in (0, 1]$; Soft potential: $\gamma \in (-3, 0)$.

Derivation of the Landau Equation

The well-posedness of the kinetic Landau equation is quite less developed. But now the well-posedness of the spatially homogeneous Landau equation is quite well-understood, thanks to the recent breakthrough in Guillen-Silvestre (Acta, '25). For instance, given initial data with finite Fisher information, the spatially homogeneous Landau equation admits a unique classical solution.
There are several different approaches to derive (approximate) the Landau equation.

- Taking the grazing limit of the corresponding Boltzmann equation. See Ling-Bing He & Yu-Long Zhou (ARMA, '21).
- Taking the weakly coupling limit of from a Newton's dynamics. In this direction, the result is quite limited. See Boblylev-Pulvirenti-Saffirio (CMP '13). There is no short time convergence result as Lanford's theorem for Boltzmann, not even the long time one as Deng-Hani-Ma.
- From Kac's $N-$particle model to the spatially homogeneous Landau equation, i.e. Kac's program. See Feng-Wang ('25 +) , Tabary ('25+) .

Goal and Setting

- We consider N-particle stochastic systems with Coulomb interactions leading, in the mean-field limit, to the spatially homogeneous Landau equation.
- Compared to classical mean-field limits (first-/second-order McKean–Vlasov SDEs), the Landau case is substantially harder:
  - the diffusion matrix is *distribution-dependent*;
  - it is also *degenerate* and *singular* (Coulomb singularity).
- The standard BBGKY hierarchy approach is ill-adapted: the degenerate dissipation does not control higher-order derivatives in the hierarchy.
- Our strategy: use the **duality approach** of Bresch–Duerinckx–Jabin (BDJ)('24+) and adapt it to the Landau master equation with Coulomb interactions.

The Landau Equation with Coulomb Interactions

Recall that the spatially homogeneous Landau equation in $\mathbb{R}^3$ with Coulomb interactions

$$\frac{\partial f}{\partial t} = Q(f,f) = \frac{\partial}{\partial v_\alpha} \int_{\mathbb{R}^3} a_{\alpha\beta}(v - v_*) \left(f(v_*) \frac{\partial f(v)}{\partial v_\beta} - f(v) \frac{\partial f(v_*)}{\partial v_{*\beta}} \right) \mathrm{d}v_*,$$

where $t \geq 0$ and $v \in \mathbb{R}^d$. The coefficient matrix is given by

$$a(z) = |z|^{-1} (\Pi_{\alpha,\beta}(z))_{\alpha,\beta}, \quad \text{with } \Pi_{\alpha\beta}(z) = \delta_{\alpha\beta} - \frac{z_\alpha z_\beta}{|z|^2}.$$

The Landau-Coulomb equation also conserves the total mass, momentum and kinetic energy of the gas. We can also rewrite the Landau equation in a compact formulation as

$$\partial_t f = \nabla \cdot [(a * f) \nabla f - (b * f) f],$$

with the vector field $b = \nabla \cdot a(z) = -2z/|z|^3$.

Kac's Model for the Landau-Coulomb Equation

Consider the Kac's model given by the following SDEs

$$\mathrm{d}V_t^i = \frac{2}{N}\sum_{j=1}^{N} b(V_t^i - V_t^j)\,\mathrm{d}t + \frac{\sqrt{2}}{\sqrt{N}}\sum_{j=1}^{N}\sigma(V_t^i - V_t^j)\,\mathrm{d}Z_t^{i,j}, \quad i = 1, \ldots, N,$$

where $Z_t^{i,j} = B_t^{i,j} = -Z_t^{j,i}$, for $i < j$ and $\sigma\sigma^\top = a$.
We use the convention that $a(0) = 0$ and $b(0) = 0$ to omit the notation $i \neq j$.
Applying Itô's formula, we can derive the Liouville equation of N-particle joint distribution $F_N(t, V)$, $V = (v^1, \ldots, v^N)$ on $\mathbb{R}^{dN}$ as

$$\partial_t F_N = \frac{1}{2N}\sum_{i=1}^{N}(\nabla_{v_i} - \nabla_{v_j}) \cdot \Big(a(v^i - v^j) \cdot (\nabla_{v_i} - \nabla_{v_j})F_N\Big),$$

where the initial value is fully tensorized as $F_N(0) = f_0^{\otimes N}$.
The equation for F_N is exactly the lifted one of f to dimension N, in the same spirit as in Guillen-Silvestre ('25).

Propagation of Chaos

Theorem (Feng and W. ('26))

Let $f_0 \in L^1 \cap L^\infty(\mathbb{R}^3)$ be some probability density function satisfying the normalized conditions. Let $F_N \in L^\infty([0,T]; L^1 \cap L^\infty(\mathbb{R}^{3N}))$ be the unique bounded weak solution to the Liouville equation with initial data $f_0^{\otimes N}$. Let $f \in C^1((0,\infty); \mathcal{S}(\mathbb{R}^3)) \cap L^\infty([0,\infty); L^1 \cap L^\infty(\mathbb{R}^3))$ be the unique bounded smooth solution to the Landau equation with initial data f_0.
Assume that f_0 has finite weighted Fisher information and high-order L^1 moment:

$$\int_{\mathbb{R}^3} \langle v \rangle^3 |\nabla \log f_0|^2 f_0 \, \mathrm{d}v < \infty, \quad \|f_0\|_{L^1_m} < \infty \text{ for some } m > 6.$$

Then for any $T > 0$, we have the propagation of chaos: for any $k \geq 1$, the k-marginal $F_{N,k}$ converges weakly to $f^{\otimes k}$ as $N \to \infty$ on $[0,T] \times \mathbb{R}^{3k}$.

Remark

Weak convergence can be improved to be convergence in Wasserstein sense. One can also obtain entropic propagation of chaos and propagation of chaos in L^1.

Basics of the Duality Method

For any fixed time $T > 0$ and fixed $k \in \mathbb{Z}^+$, and for any test function $\varphi \in C_c^\infty(\mathbb{R}^3)$, we construct the corresponding backward dual Kolmogorov equation

$$\partial_t \Phi_N = -\frac{1}{2N} \sum_{i,j=1}^{N} (\nabla_{v^i} - \nabla_{v^j}) \cdot \Big(a(v^i - v^j) \cdot (\nabla_{v^i} - \nabla_{v^j}) \Phi_N \Big),$$

with the symmetric final data

$$\Phi_N(T, v^{[N]}) = (C_N^k)^{-1} \sum_{1 \le i_1 < \cdots < i_k \le N} \varphi(v^{i_1}) \cdots \varphi(v^{i_k}).$$

Motivation: Study the evolution of observables instead of the joint law. Now Φ_N is an observable along the particle trajectory. Propagation of Kac's chaos is encoded in appropriate limits of expectations involving Φ_N. See Bresch-Duerinckx-Jabin ('24+).

The Duality Method

By direct computations,

$$\begin{aligned}
&\int_{\mathbb{R}^{3k}} F_{N,k}(T)\varphi^{\otimes k} - \int_{\mathbb{R}^{3k}} f(T)^{\otimes k}\varphi^{\otimes k} \\
&= \int_{\mathbb{R}^{3N}} F_N(T)\Phi_N(T) - \int_{\mathbb{R}^{3N}} f(T)^{\otimes N}\Phi_N(T) \quad \text{(Symmetry)} \\
&= \int_{\mathbb{R}^{3N}} F_N(0)\Phi_N(0) - \int_{\mathbb{R}^{3N}} f(T)^{\otimes N}\Phi_N(T) = -\int_0^T \int_{\mathbb{R}^{3N}} \frac{\mathrm{d}}{\mathrm{d}t} f(t)^{\otimes N}\Phi_N(t).
\end{aligned}$$

Using the equations, the right-hand side can be re-written as (up to a small error)

$$\int_0^T \int_{\mathbb{R}^{3N}} N V_f(v^1, v^2) f^{\otimes N}\Phi_N \,\mathrm{d}v^{[M]},$$

where

$$\begin{aligned}
V_f(v,w) = &-a(v-w) : \left(\frac{\nabla^2 f}{f}(v) - \nabla \log f(v) \otimes \nabla \log f(w)\right) \\
&- b(v-w)\cdot(\nabla \log f(v) - \nabla \log f(w)) \\
&+ a * f(v) : \frac{\nabla^2 f}{f}(v) - c * f(v).
\end{aligned}$$

Further Discussion

1. Propagation of Chaos: Uniform-in-time estimates in particular for kinetic Vlasov equations (Gong, W. and Xie ('24+)); Derivation of Vlasov-Poisson/Landau. The underlying space for the velocity variable is on the whole space.
2. (Path/dynamical) Large deviation principles for kinetic theories (Boltzmann, Landau, Vlasov-Poisson, 2D Euler equation...).
3. Quantify the convergence from Kac's model to the (spatially homogeneous) Landau equation. This will be useful to numerical analysis (See in particular Kai Du and Lei Li ('24+)).

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Thank you!