Implosion in kinetic equations

Partial Differential Equations for Many Particles Systems

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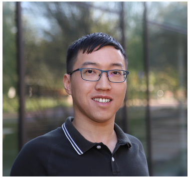

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	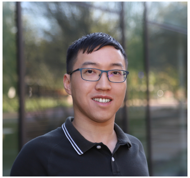			
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Table of contents

Introduction

Introduction

- The Boltzmann equation describes the behavior of a rarefied gas. At the *grazing collision limit*, it can be approximated by the **Landau equation**

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon_0} Q(f,f), \qquad (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3, f(t,x,v) \in \mathbb{R} \tag{1}$$

- The Landau collision operator is given by

$$\begin{aligned} Q(f,g)(v) &= \operatorname{div}_v \int_{\mathbb{R}^3} \mathbf{\Phi}(v-w)\left(f(w)\nabla_v g(v) - g(v)\nabla_w f(w)\right) \mathrm{d}w \\ &= \operatorname{div}_v (A[f]\nabla_v g - \operatorname{div}_v A[f] g)(v) \end{aligned}$$

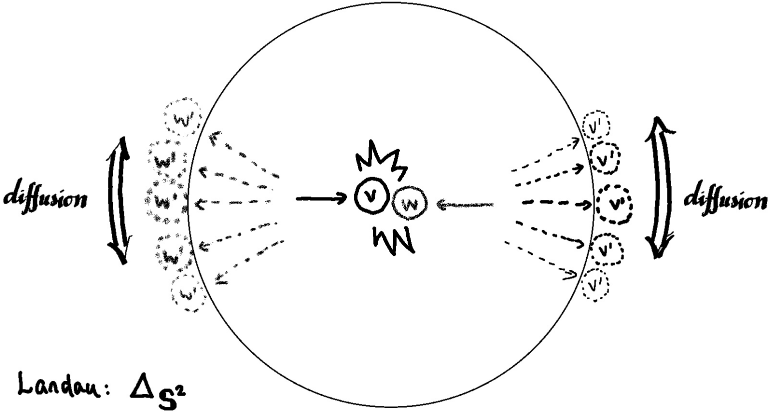
- The collision kernel $\mathbf{\Phi}$ is given by some power law

$$\mathbf{\Phi}(v) := \frac{1}{8\pi}|v|^{\gamma+2}(\operatorname{Id} - \mathbf{\Pi}_v), \qquad A[f](v) := \mathbf{\Phi} * f(v)$$

where $\mathbf{\Pi}_v = \frac{v}{|v|} \otimes \frac{v}{|v|}$ is the projection along the v direction.

- The physically meaningful potential is $\gamma = -3$ (Coulomb). But we also consider other potentials.

$f(v) \longleftrightarrow g(w)$



Landau: Δ_{S^2}

Boltzmann: $\Delta^s_{S^2}$ $\quad 0<s<1$

$v+w = v'+w'$

$|v|^2+|w|^2 = |v'|^2+|w'|^2$.

strength of diffusion: $|v-w|^{\gamma+2}$

diffusion operator: $\mathrm{div}_{v-w}\left(\Phi(v-w) \, \nabla_{v-w} \, f(v) g(w) \right)$

Well-posedness for the homogeneous Landau equation

$$\partial_t f + \cancel{v \cdot \nabla_x f} = \frac{1}{\varepsilon_0} Q(f,f), \qquad (\cancel{x}, v) \in \mathbb{R}^3 \times \mathbb{R}^3, f(t, \cancel{x}, v) \in \mathbb{R}$$

- Maxwellian molecules ($\gamma = 0$): global well-posedness Villani (1998).
- Hard potentials ($\gamma > 0$): global well-posedness Desvillettes and Villani (2000a); Desvillettes and Villani (2000b).
- Moderately soft potentials ($-2 \leq \gamma < 0$): Wu (2014); Silvestre (2017).
- Very soft potentials ($-3 < \gamma < -2$) and Coulomb potential ($\gamma = -3$):
  - conditional regularity: Silvestre (2017); Gualdani and Guillen (2019); Alonso et al. (2024)
  - partial regularity: Golse, Gualdani, et al. (2022); Golse, Imbert, et al. (2024)
  - global well-posedness: Guillen and Silvestre (2025); Golding, Gualdani, and Loher (2025); S. Ji (2024); Desvillettes, Golding, et al. (2024); He, J. Ji, and Luo (2025)

Inhomogeneous Landau equation

- Global well-posedness near global Maxwellian: Guo (2002); Strain and Guo (2006); Strain and Guo (2008); Carrapatoso and Mischler (2017).
- Global well-posedness near vacuum: Luk (2019); Chaturvedi (2022).
- Conditional regularity: Cameron, Silvestre, and Snelson (2018); Henderson and Snelson (2020); Snelson (2020); Henderson, Snelson, and Tarfulea (2019); Henderson, Snelson, and Tarfulea (2020); Snelson and Solomon (2023); Golding and Henderson (2025); Golding, Henderson, and Silvestre (2026).

In this talk, we show

Landau equation *can* blowup in finite time
for very hard potentials $\gamma \in (\sqrt{3}, 2]$
by lifting a self-similar hydrodynamic implosion

Main theorem

We define the mass ϱ, momentum m, and energy density e associated to f by

$$(\varrho, \mathrm{m}, \mathrm{e})(t,x) := \int_{\mathbb{R}^3} f(t,x,v)(1, v, |v|^2)\, \mathrm{d}v. \tag{2}$$

Theorem (Bedrossian–Chen–Gualdani–Ji–Vicol–Y. (2026), arXiv preprint)

Fix $\gamma \in (\sqrt{3}, 2]$. Let $(\bar{\rho}, \bar{\mathbf{U}}, \bar{\Theta}, r)$ denote the imploding profile for the 3D compressible Euler in Shao et al. (2025), with a blowup speed $r > (\gamma+3)/(\gamma+2)$.

- *For small ε_0, there exists a nice[1] initial data f_{in}, such that the solution f to (1) is positive, smooth, and develops a finite time singularity at a time $T = 1$.*
- *As $t \to 1^-$, the hydrodynamic fields $(\varrho, \mathrm{m}, \mathrm{e})$ blow up at $x = 0$, and the solution converges to the local Maxwellian associated with Euler in self-similar variables.*
- *$f(t)$ remains $L^\infty_{t,x}$, but C^α_x norm blows up for any $\alpha > 0$.*

[1] smooth, positive, Gaussian decay in v; $\varrho, \varrho^{-1}, \mathrm{m}, \mathrm{e} \le C$ all bounded

Main theorem (cont'd)

Define self-similar exponents $\bar{c}_x = 1/r > 0$, $\bar{c}_v = -1 + 1/r < 0$.

$$X = \frac{x}{(1-t)^{\bar{c}_x}}, \quad V = \frac{v}{(1-t)^{\bar{c}_v}}, \quad s = \log\left(\frac{1}{1-t}\right) \in (0,\infty) \text{ when } t \in (0,1)$$

In SS variables, f converges to the *local Maxwellian* associated with the Euler profile:

$$\lim_{t\to 1^-} f(t, (1-t)^{\bar{c}_x} X, (1-t)^{\bar{c}_v} V) = \mathcal{M}_{\bar{\rho},\bar{\mathbf{U}},\bar{\Theta}}(X, V), \tag{3}$$

for any fixed $X, V \in \mathbb{R}^3$, where

$$\mathcal{M}_{\bar{\rho},\bar{\mathbf{U}},\bar{\Theta}}(X, V) = \frac{\bar{\rho}(X)}{(2\pi\bar{\Theta}(X))^{3/2}} \exp\left(-\frac{|V - \bar{\mathbf{U}}(X)|^2}{2\bar{\Theta}(X)}\right).$$

Moreover,

$$\lim_{t\to 1^-} \frac{\varrho(t, (1-t)^{\bar{c}_x} X)}{[(1-t)^{\bar{c}_v}]^3} = \bar{\rho}(X).$$

Lifting Euler

Lifting fluid solution to kinetic equation

There is a mature program that constructs solution to the kinetic equation using fluid solution in high collision regime. Caflisch (1980) lifts Euler to Boltzmann, using series expansion: for $\varepsilon_0 \ll 1$

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon_0} Q_B(f, f), \qquad f = f^{(0)} + \varepsilon_0 f^{(1)} + \varepsilon_0^2 f^{(2)} + \dots$$

- Order ε_0^{-1}: $Q_B(f^{(0)}, f^{(0)}) = 0$, then $f^{(0)} = \mathcal{M}_{\varrho,\mathbf{u},\theta}$ for some $\varrho, \mathbf{u}, \theta$.
- Order ε_0^{0}: Solvability of equation for $f^{(1)}$ enforces that $\varrho, \mathbf{u}, \theta$ solve compressible Euler for monatomic gas, ...

◇ Idea: if we have a sufficiently regular solution $\varrho, \mathbf{u}, \theta$ in $[0, T]$, then $\mathcal{M}_{\varrho,\mathbf{u},\theta}$ solves the kinetic equation up to small error

◇ Guo (2002) studies the perturbation $\tilde{f} = f - \mathcal{M}_{\varrho,\mathbf{u},\theta}$ near a global Maxwellian $\varrho = \theta = 1$, $\mathbf{u} = 0$ for Landau

Isentropic Euler equation

$$\begin{cases} \partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) = 0 \\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\varrho} \nabla p = 0 \end{cases} \qquad \Longrightarrow$$

For isentropic Euler with monatomic gas, $p = \frac{1}{\kappa} \varrho^\kappa$ with $\kappa = 5/3$.

- Guderley (1942) constructed a self-similar imploding solution with shock
- Merle et al. (2022): smooth implosion, but missing the monatomic case $\kappa = 5/3$
- Shao et al. (2025): smooth implosion for monatomic gas
- Chen et al. (2026): smooth implosion for anisentropic Euler with explicit rate

c: local sound speed. For monatomic gas, $\varrho = \mathrm{c}^3$, $\theta = \frac{1}{\kappa}\mathrm{c}^2$, $p = \frac{1}{\kappa}\mathrm{c}^5$.

Isentropic Euler equation

$$\begin{cases} \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0 \\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\varrho} \nabla p = 0 \end{cases} \implies \begin{cases} (\partial_t + \mathbf{u} \cdot \nabla) \varrho + \varrho \operatorname{div} \mathbf{u} = 0 \\ (\partial_t + \mathbf{u} \cdot \nabla) \mathbf{u} + \varrho^{\kappa - 2} \nabla \varrho = 0 \end{cases}$$

For isentropic Euler with monatomic gas, $p = \frac{1}{\kappa} \varrho^{\kappa}$ with $\kappa = 5/3$.

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Isentropic Euler equation

$$\begin{cases} \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0 \\ \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\varrho} \nabla p = 0 \end{cases} \quad \Longrightarrow \quad \begin{cases} (\partial_t + \mathbf{u} \cdot \nabla) \mathrm{c} + \frac{1}{3} \mathrm{c} \operatorname{div} \mathbf{u} = 0 \\ (\partial_t + \mathbf{u} \cdot \nabla) \mathbf{u} + 3 \mathrm{c} \nabla \mathrm{c} = 0 \end{cases}$$

For isentropic Euler with monatomic gas, $p = \frac{1}{\kappa} \varrho^\kappa$ with $\kappa = 5/3$.

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Self-similar Euler equation and Landau equation

Self-similar variables:

$$X = \frac{x}{(1-t)^{\bar{c}_x}}, \quad V = \frac{v}{(1-t)^{\bar{c}_v}}, \quad s = \log\left(\frac{1}{1-t}\right), \quad \bar{c}_x > 0, \quad \bar{c}_v < 0.$$

If $\mathrm{c}(t,x) = (1-t)^{\bar{c}_v}\mathrm{C}(s,X)$, $\mathbf{u}(t,x) = (1-t)^{\bar{c}_v}\mathbf{U}(s,X)$, then

$$\begin{cases} (\partial_t + \mathbf{u}\cdot\nabla)\mathrm{c} + \frac{1}{3}\mathrm{c}\,\mathrm{div}\,\mathbf{u} = 0 \\ (\partial_t + \mathbf{u}\cdot\nabla)\mathbf{u} + 3\mathrm{c}\nabla\mathrm{c} = 0 \end{cases} \Longrightarrow \begin{cases} [\partial_s + (\bar{c}_x X + \mathbf{U})\cdot\nabla - \bar{c}_v]\mathrm{C} + \frac{1}{3}\mathrm{C}\,\mathrm{div}\,\mathbf{U} = 0 \\ [\partial_s + (\bar{c}_x X + \mathbf{U})\cdot\nabla - \bar{c}_v]\mathbf{U} + 3\mathrm{C}\nabla\mathrm{C} = 0 \end{cases}$$

Shao et al. (2025) constructed smooth stationary solutions $(\bar{\mathrm{C}}, \bar{\mathbf{U}})$ for a sequence of $r_n \nearrow 3-\sqrt{3}$.

Self-similar Euler equation and Landau equation

Self-similar variables:

$$X = \frac{x}{(1-t)^{\bar{c}_x}}, \quad V = \frac{v}{(1-t)^{\bar{c}_v}}, \quad s = \log\left(\frac{1}{1-t}\right), \quad \bar{c}_x > 0, \quad \bar{c}_v < 0.$$

If $\mathrm{c}(t,x) = (1-t)^{\bar{c}_v} \mathrm{C}(s,X)$, $\mathbf{u}(t,x) = (1-t)^{\bar{c}_v} \mathbf{U}(s,X)$, then

$$\begin{cases} (\partial_t + \mathbf{u}\cdot\nabla)\mathrm{c} + \frac{1}{3}\mathrm{c}\,\mathrm{div}\,\mathbf{u} = 0 \\ (\partial_t + \mathbf{u}\cdot\nabla)\mathbf{u} + 3\mathrm{c}\nabla\mathrm{c} = 0 \end{cases} \implies \begin{cases} [\partial_s + (\bar{c}_x X + \mathbf{U})\cdot\nabla - \bar{c}_v]\mathrm{C} + \frac{1}{3}\mathrm{C}\,\mathrm{div}\,\mathbf{U} = 0 \\ [\partial_s + (\bar{c}_x X + \mathbf{U})\cdot\nabla - \bar{c}_v]\mathbf{U} + 3\mathrm{C}\nabla\mathrm{C} = 0 \end{cases}$$

If $f(t,x,v) = (1-t)^{\bar{c}_f} F(s,X,V)$ with $\bar{c}_f = 0$, then

$$(\partial_t + v\cdot\nabla_x) f = \frac{1}{\varepsilon_0} Q(f,f) \quad \implies \quad [\partial_s + (\bar{c}_x X + V)\cdot\nabla_X + \bar{c}_v V\cdot\nabla_V] F = \frac{1}{\varepsilon_s} Q(F,F)$$

Here $\varepsilon_s = \varepsilon_0\, \mathrm{e}^{-\omega s}$ with $\omega = -\bar{c}_f - (\gamma+3)\bar{c}_v - 1 > 0$ provided $r > (\gamma+3)/(\gamma+2)$.

Lifting self-similar Euler solution

Now we know that $(\bar{\mathsf{C}}, \bar{\mathbf{U}})$ is a stationary solution to SS Euler, we get the density $\bar{\rho} = \bar{\mathsf{C}}^3$ and temperature $\bar{\Theta} = \frac{1}{\kappa}\bar{\mathsf{C}}^2$. We lift it to a local Maxwellian [3]

$$\mathcal{M}(t, X, V) = \frac{\bar{\rho}(X)}{(2\pi\bar{\Theta}(X))^{3/2}} \exp\left(-\frac{|V - \bar{\mathbf{U}}(X)|^2}{2\bar{\Theta}(X)}\right) = C \exp\left(-\frac{\kappa}{2}\left|\frac{V - \bar{\mathbf{U}}}{\bar{\mathsf{C}}}\right|^2\right).$$

We are going to seek a solution to

$$[\partial_s + (\bar{c}_x X + V)\cdot\nabla_X + \bar{c}_v V \cdot \nabla_V]F = \frac{1}{\varepsilon_s} Q(F, F)$$

of the form

$$F = \mathcal{M} + \mathcal{M}_1^{1/2}\tilde{F},$$

where $\mathcal{M}_1 = \mathcal{M}/\bar{\rho}$ is normalized in $L^2(V)$.

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[3] To avoid vacuum at large x, we actually modify $\bar{\mathsf{C}}$ so that it tends to a positive number instead of 0. It creates a small error but it is negligible in SS variable.

The perturbation equation

Equation of $\tilde{F}$

Recall $F = \mathcal{M} + \mathcal{M}_1^{1/2}\tilde{F}$ is supposed to solve

$$(\partial_s + \mathcal{T})F = \frac{1}{\varepsilon_s} Q(F, F)$$

with $\mathcal{T} = (\bar{c}_x X + V)\cdot\nabla_X + \bar{c}_v V\cdot\nabla_V$ being the transport. Then $\tilde{F}$ solves

$$(\partial_s + \mathcal{T} + (\partial_s + \mathcal{T})\log \mathcal{M}_1^{1/2})\tilde{F} = \frac{1}{\varepsilon_s}\mathcal{L}_{\mathcal{M}}\tilde{F} + \frac{1}{\varepsilon_s}\mathcal{N}(\tilde{F}, \tilde{F}) - \mathcal{M}_1^{-1/2}(\partial_s + \mathcal{T})\mathcal{M},$$

with

$$\begin{aligned} \text{linear collision op} \qquad & \mathcal{L}_{\mathcal{M}}\tilde{F} = \mathcal{M}_1^{-1/2}\left[Q(\mathcal{M}, \mathcal{M}_1^{1/2}\tilde{F}) + Q(\mathcal{M}_1^{1/2}\tilde{F}, \mathcal{M})\right], \\ \text{nonlinear collision op} \qquad & \mathcal{N}(\tilde{F}, \tilde{G}) = \mathcal{M}_1^{-1/2} Q(\mathcal{M}_1^{1/2}\tilde{F}, \mathcal{M}_1^{1/2}\tilde{G}). \end{aligned}$$

Macro-micro decomposition

- As a self adjoint operator in $L^2(V)$,

$$\ker(\mathcal{L}_{\mathcal{M}}) = \mathrm{Span}\{1, V_i, |V|^2\}\mathcal{M}_1^{1/2}$$

  and it is coersive on $\ker(\mathcal{L}_{\mathcal{M}})^{\perp} = \mathrm{ran}(\mathcal{L}_{\mathcal{M}})$ (Guo (2002); Mouhot (2006); Mouhot and Strain (2007)).

- We will apply orthogonal projection $\mathcal{P}_M$, $\mathcal{P}_m$:

$$\tilde{F} = \tilde{F}_M + \tilde{F}_m \qquad \tilde{F}_M \in \ker(\mathcal{L}_{\mathcal{M}}) \qquad \tilde{F}_m \perp \ker(\mathcal{L}_{\mathcal{M}})$$

- The equation can be split into a macro equation and micro equation. Moreover, we can find a basis for the kernel $\Phi_{\mathbf{U}_i}, \Phi_P, \Phi_B$ $(i = 1, 2, 3)$ so that

$$\tilde{F}_M = \sum_{i=1,2,3} \tilde{\mathbf{U}}_i \Phi_{\mathbf{U}_i} + \tilde{P}\Phi_P + \tilde{B}\Phi_B$$

  so $\tilde{F}_M$ is completely determined by "macro quantities" $\widetilde{\mathbf{W}} := (\tilde{\mathbf{U}}, \tilde{P}, \tilde{B})$.

Micro equation

We have full stability on the equation of $\tilde{F}_m$:

$$(\partial_s + \mathcal{T} + (\partial_s + \mathcal{T}) \log \mathcal{M}_1^{1/2})\tilde{F}_m = \frac{1}{\varepsilon_s}\mathcal{L}_{\mathcal{M}}\tilde{F}_m + \frac{1}{\varepsilon_s}\mathcal{N}(\tilde{F}, \tilde{F}) + \dots,$$

using coercivity

$$\langle \mathcal{L}_{\mathcal{M}}\tilde{F}_m, \tilde{F}_m \rangle_{L^2(V)} \leq -C\|\tilde{F}_m\|_\sigma^2,$$

where σ norm is an anisotropically weighted H^1 norm.

$$\|\tilde{F}\|_\sigma^2 = \langle A[\mathcal{M}]\nabla_V \tilde{F}, \nabla_V \tilde{F}\rangle + \frac{\kappa^2}{4\bar{\mathsf{C}}^2}\langle A[\mathcal{M}\mathring{V} \otimes \mathring{V}]\tilde{F}, \tilde{F}\rangle \approx \|\boldsymbol{\Sigma}^{1/2}\nabla_V \tilde{F}\|_{L^2(V)}^2 + \|\Lambda^{1/2}\tilde{F}\|_{L^2(V)}^2.$$

Here the weight matrix $\boldsymbol{\Sigma}$ and weight Λ depend on sound speed $\bar{\mathsf{C}}$ and the normalized relative velocity

$$\mathring{V} = \frac{V - \bar{\mathbf{U}}}{\bar{\mathsf{C}}}, \quad \Lambda = \bar{\mathsf{C}}^{\gamma+3}\langle \mathring{V}\rangle^{\gamma+2}, \quad \boldsymbol{\Sigma} = \bar{\mathsf{C}}^{\gamma+5}\left(\langle \mathring{V}\rangle^{\gamma}\boldsymbol{\Pi}_{\mathring{V}} + \langle \mathring{V}\rangle^{\gamma+2}(\mathrm{Id} - \boldsymbol{\Pi}_{\mathring{V}})\right).$$

Micro equation

We have full stability on the equation of $\tilde{F}_m$:

$$(\partial_s + \mathcal{T} + (\partial_s + \mathcal{T}) \log \mathcal{M}_1^{1/2})\tilde{F}_m = \frac{1}{\varepsilon_s}\mathcal{L}_{\mathcal{M}}\tilde{F}_m + \frac{1}{\varepsilon_s}\mathcal{N}(\tilde{F}, \tilde{F}) + \dots,$$

using coercivity

$$\langle \mathcal{L}_{\mathcal{M}}\tilde{F}_m, \tilde{F}_m \rangle_{L^2(V)} \leq -C \|\tilde{F}_m\|_\sigma^2,$$

where σ norm is an anisotropically weighted H^1 norm.

Transport term gives decay in weighted space

$$\langle \mathcal{T} g, g \rangle_{L^2(\langle X \rangle^\eta \, \mathrm{d}X \, \mathrm{d}V)} \geq \left(\frac{\bar{c}_x}{2}(\bar{\eta} - \eta) + \frac{3\bar{c}_v}{2} \right) \|g\|^2_{L^2(\langle X \rangle^\eta \, \mathrm{d}X \, \mathrm{d}V)} + \dots$$

with critical weight $\bar{\eta} = -3 + 6(r-1)$.

Micro equation (cont'd)

Energy inequality:

$$\frac{\mathrm{d}}{\mathrm{d}s}\|\tilde{F}_m\|_{L^2_w}^2 \leq -\frac{\bar{c}_x}{2}(\bar{\eta}-\eta)\|\tilde{F}_m\|_{L^2_w}^2 - \frac{C}{\varepsilon_s}\|\tilde{F}_m\|_{H^1_w}^2 + \text{nonlinear term}$$
$$+ \text{macro interaction}$$
$$+ \text{commutator term}$$
$$+ \text{error term} \ldots$$

- Interpolation between L^2_w and H^1_w controls the remaining terms.
- Similar estimates can be obtained for higher derivatives of $\tilde{F}_m$.

Macro equation

The macro quantity $\widetilde{\mathbf{W}} := (\tilde{\mathbf{U}}, \tilde{P}, \tilde{B})$ solves a equation that resembles linearized SS Euler equation:

$$\begin{aligned}
[\partial_s + (\bar{c}_x X + \bar{\mathbf{U}}) \cdot \nabla + \operatorname{div} \bar{\mathbf{U}} - 3\bar{c}_v]\tilde{\mathbf{U}} &= -\bar{\mathrm{C}} \nabla \tilde{P} && - (\nabla \bar{\mathrm{C}})\tilde{P} \\
& && + 3(\nabla \bar{\mathrm{C}})\tilde{B} \\
& && + \frac{1}{3}(\operatorname{div} \bar{\mathbf{U}})\tilde{\mathbf{U}} \\
& && - \tilde{\mathbf{U}} \cdot \nabla \bar{\mathbf{U}} && - \mathcal{I}_1(\tilde{F}_m), \\
[\partial_s + (\bar{c}_x X + \bar{\mathbf{U}}) \cdot \nabla + \operatorname{div} \bar{\mathbf{U}} - 3\bar{c}_v]\tilde{P} &= -\bar{\mathrm{C}} \operatorname{div} \tilde{\mathbf{U}} && - (\nabla \bar{\mathrm{C}}) \cdot \tilde{\mathbf{U}} && - \mathcal{I}_2(\tilde{F}_m), \\
[\partial_s + (\bar{c}_x X + \bar{\mathbf{U}}) \cdot \nabla + \operatorname{div} \bar{\mathbf{U}} - 3\bar{c}_v]\tilde{B} &= && && + \mathcal{I}_2(\tilde{F}_m).
\end{aligned}$$

Mimicking the ideas from Chen et al. (2024), we derive finite codimensional stability.

$$\partial_s \widetilde{\mathbf{W}} = \mathcal{L}_E \widetilde{\mathbf{W}} + (-\mathcal{I}_1, -\mathcal{I}_2, \mathcal{I}_2)\tilde{F}_m.$$

Macro equation (cont'd)

We have the following estimate:

$$\langle \mathcal{L}_E \mathbf{W}, \mathbf{W} \rangle_{L^2(\langle X \rangle^\eta \, dX)} \leq -\frac{\bar{c}_x}{4}(\bar{\eta} - \eta) \|\mathbf{W}\|^2_{L^2(\langle X \rangle^\eta \, dX)} + C \int_{B_R} |\mathbf{W}|^2 \, dX.$$

We can find a compact perturbation $\mathcal{K}$ so $\mathcal{L}_E - \mathcal{K}$ is fully dissipative:

$$\langle (\mathcal{L}_E - \mathcal{K}) \mathbf{W}, \mathbf{W} \rangle_{L^2(\langle X \rangle^\eta \, dX)} \leq -\frac{\bar{c}_x}{4}(\bar{\eta} - \eta) \|\mathbf{W}\|^2_{L^2(\langle X \rangle^\eta \, dX)}.$$

We prove finite codimension stability:

- the spectrum of $\mathcal{L}_E - \mathcal{K}$ is in $\{z \in \mathbb{C} : \mathrm{Re}(z) \leq -\frac{\bar{c}_x}{4}(\bar{\eta} - \eta)\}$
- the spectrum of $\mathcal{L}_E$ in $\{z \in \mathbb{C} : \mathrm{Re}(z) > \lambda_u\}$ is finite with finite multiplicity, and eigenfunctions are smooth
- semigroup $e^{\mathcal{L}_E}$ and $e^{\mathcal{L}_E - \mathcal{K}}$ can be bounded

Macro equation (cont'd)

To solve $\partial_s \widetilde{\mathbf{W}} = \mathcal{L}_E \widetilde{\mathbf{W}} + f$, we solve a coupled system and $\widetilde{\mathbf{W}} = \widetilde{\mathbf{W}}_1 + \widetilde{\mathbf{W}}_2$:

$$\partial_s \widetilde{\mathbf{W}}_1 = (\mathcal{L}_E - \mathcal{K}) \widetilde{\mathbf{W}}_1 + f$$
$$\partial_s \widetilde{\mathbf{W}}_2 = \mathcal{L}_E \widetilde{\mathbf{W}}_2 + \mathcal{K} \widetilde{\mathbf{W}}_1$$

Then

- With any initial value, $\widetilde{\mathbf{W}}_1$ can be solved with good energy estimate because of the full diffusivity of $\mathcal{L}_E - \mathcal{K}$
- $\widetilde{\mathbf{W}}_2$ only need to take care of a C_c^∞ forcing. We solve $\widetilde{\mathbf{W}}_2$ by Duhamel formula, by splitting $\mathcal{K}\widetilde{\mathbf{W}}_1$ using spectral projection, and then propagate stable forcing forward and unstable forcing backward (hence we cannot choose initial data)
- Therefore the solution map $\widetilde{\mathbf{W}}_1 \mapsto \widetilde{\mathbf{W}}_2$ is smooth and bounded
- Combined, we prove stability for a finite codimensional set of initial data

Summary

Summary

We take small initial data $\tilde{F}_m$ and $\widetilde{\mathbf{W}}_1$

- we find $\widetilde{\mathbf{W}}_2$, then $\widetilde{\mathbf{W}} = \widetilde{\mathbf{W}}_1 + \widetilde{\mathbf{W}}_2$
- $\widetilde{\mathbf{W}}$ determines $\tilde{F}_M$, then $\tilde{F} = \tilde{F}_m + \tilde{F}_M$
- then $F = \mathcal{M} + \mathcal{M}_1^{1/2}\tilde{F}$ solves SS Landau for all time
- then f solves the original Landau for $t \in (0,1)$, and blows up at $T = 1$

This is an oversimplification of the actual argument because we also need to

- modify the profile to avoid vacuum
- handle trilinear estimate and the error
- design the weight for L^2 and H^k norm
- couple macro/micro parts
- design fixed point argument
- verify global well-posedness ...

Summary

We take small initial data $\tilde{F}_m$ and $\widetilde{\mathbf{W}}_1$

- we find $\widetilde{\mathbf{W}}_2$, then $\widetilde{\mathbf{W}} = \widetilde{\mathbf{W}}_1 + \widetilde{\mathbf{W}}_2$
- $\widetilde{\mathbf{W}}$ determines $\tilde{F}_M$, then $\tilde{F} = \tilde{F}_m + \tilde{F}_M$
- then $F = \mathcal{M} + \mathcal{M}_1^{1/2}\tilde{F}$ solves SS Landau for all time
- then f solves the original Landau for $t \in (0,1)$, and blows up at $T = 1$

This is an oversimplification of the actual argument because we also need to

- modify the profile to avoid vacuum
- handle trilinear estimate and the error
- design the weight for L^2 and H^k norm
- couple macro/micro parts
- design fixed point argument
- verify global well-posedness ...

but nothing is impossible with great collaborators!

Thank you!

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