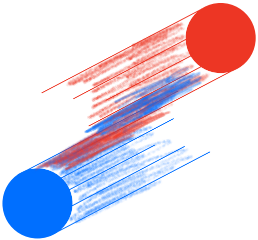
Kinetic equations and the Fisher information



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ICERM Topical Workshop
Partial Differential Equations for Many Particles Systems.

Providence, August 2026

The general Cauchy problem for a kinetic equation

Problem.
For any given density $f_{\text{in}}(x,v)$ find $f = f(t,x,v)$ solving

$$\partial_t f + v \cdot \nabla_x f = q(f) \text{ in } (0,T) \times \mathbb{R}^{2d},$$
$$f\,|_{t=0} = f_{\text{in}} \quad \text{in } \mathbb{R}^{2d}$$

where either $T = \infty$ or $T < \infty$ with blow up as $t \to T$.

This problem has been the concern of mathematical analysis for about a century, going back to Carleman.

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As hydrodynamic limits take us from this equation to fluids equations, it is widely (but not uniformly) believed that finite time breakdown is a typical ocurrence.

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This question has seen considerable progress in the 21st century: we highlight Alexandre-Villani (2002), Guo (2002, 2003), Gressman-Strain (2011), and Imbert-Silvestre (2022).

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Since the breakthrough work of Deng-Hani-Ma (2024) we know particle dynamics converge to the kinetic equation (hard spheres case) as long as f remains smooth, i.e. at least up to time T.

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where either $T = \infty$ or $T < \infty$ with blow up as $t \to T$.

We did not know of any collisional operator q where an initially smooth f broke down in finite time – until earlier this year: for $q \sim$ Landau for "very hard potentials" (see Jincheng's talk!).

In this talk

Consider the **homogeneous Landau equation**

$$\partial_t f = \bar{a}_{ij}\partial^2_{ij} f + f^2 \text{ where } \bar{a} := -D^2(-\Delta)^{-2} f.$$

for the Coulomb potential, $f = f(t, v)$, in $\mathbb{R}^3$.

In work with Silvestre, we find the Fisher information of f decreases in time. This fact rules out finite time blow up.

Roughly speaking, our result is made possible by

- a "lifted" equation in $\mathbb{R}^6$ (involving $\Delta_{\mathbb{S}^2}$!)
- the symmetries of the lifted equation
- a functional inequality in $\mathbb{S}^2$

The Boltzmann and Landau equations

Around 1870 Boltzmann (and Maxwell!) arrived at the equation

$$\partial_t f + v \cdot \nabla_x f = q(f) \qquad \text{in } (0,T) \times \mathbb{R}^3 \times \mathbb{R}^3$$

where

$$q(f)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \left(f(v_\sigma)f(w_\sigma) - f(v)f(w)\right) B \, d\sigma dw$$

Here, $B = \alpha b$ with $\alpha = \alpha(|v-w|)$ and $b = b(\frac{v-w}{|v-w|} \cdot \sigma)$ and

$$\begin{aligned} v_\sigma &= \tfrac{1}{2}(v+w) + \tfrac{1}{2}|v-w|\sigma \\ w_\sigma &= \tfrac{1}{2}(v+w) - \tfrac{1}{2}|v-w|\sigma \end{aligned}$$

The Boltzmann and Landau equations

We thus have the Boltzmann collisional integral

$$q(f)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \left(f(v_\sigma) f(w_\sigma) - f(v) f(w) \right) B \; d\sigma dw$$

For particles interacting by power laws it is classical that

$$b(\sigma_0 \cdot \sigma) \approx |\sigma - \sigma_0|^{-2-\frac{2}{s-1}}$$

This creates too strong a singularity when $s = 2$ (Coulomb!)...

The Boltzmann and Landau equations

...and so one may consider $b(\cdot)$ concentrating on $\sigma \approx \sigma_0$,

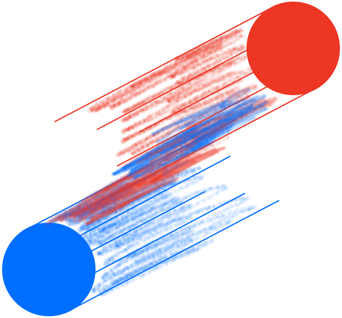
$$q(f)(v) = \int_{\mathbb{R}^3} \alpha \int_{\mathbb{S}^2} (f(v_\sigma)f(w_\sigma) - f(v)f(w))\, b\, d\sigma dw$$

A second order expansion (for f smooth) shows this converges to

$$q_L(f)(v) = \int_{\mathbb{R}^3} \alpha \operatorname{div}_{v-w}(a\nabla_{v-w}(f(v)f(w)))dw$$

where $a = a(v-w)$, $a(z)$ being the matrix $a_{ij}(z) = |z|^2\delta_{ij} - z_i z_j$.

The Boltzmann and Landau equations



$$q_L(f)(v) = \int_{\mathbb{R}^3} \alpha \mathrm{div}_{v-w}(a\nabla_{v-w}(f(v)f(w)))dw$$

This is is the Landau operator and it models **grazing collisions**

The Boltzmann and Landau equations

The **Landau equation** looks just like the Boltzmann equation,

$$\partial_t f + v \cdot \nabla_x f = q(f)$$

except now $q(f) = q_L(f)$ is the Landau collision operator.

It is difficult to overstate the richness of these equations – all classical fluid models emerge from them as hydrodynamic limits.

We focus on the **space homogeneous regime**, where $\nabla_x f \equiv 0$

The Landau equation

The flow of the equation preserves the following quantities

$$\int_{\mathbb{R}^3} f \, dv, \ \int_{\mathbb{R}^3} fv \, dv, \ \int_{\mathbb{R}^3} f|v|^2 \, dv$$

Moreover, the entropy is monotone

$$\frac{d}{dt}\left(-\int_{\mathbb{R}^3} f \log f \, dv\right) \geq 0.$$

Outside of Maxwell molecules ($\alpha \equiv 1$) no other conserved or monotone quantities were known for these equations.

The Landau equation

Problem

Show that for any initial density there is an evolution of the Landau equation that is smooth and well defined for all time, or find there are densities that lead to **blow up** in finite time

The Landau equation

The equation can be written in a "divergence form"

$$\partial_t f = \mathrm{div}(A_f \nabla f - f \mathrm{div}(A_f))$$

and in a "non-divergence form"

$$\partial_t f = \mathrm{tr}(A_f D^2 f) + h_f f,\ h_f := -\mathrm{div}(\mathrm{div}(A_f)).$$

where $A_f(v)$ is the positive matrix given by

$$A_f(v) := \int_{\mathbb{R}^3} \alpha(|v-w|) a(v-w) f(w)\, dw.$$

The Landau equation

It is worth noting these are instances of what is known as a **McKean-Vlasov equation**. Now, focusing on

$$\partial_t f = \mathrm{tr}(A_f D^2 f) + h_f f,\; h_f := -\mathrm{div}(\mathrm{div}(A_f)).$$

one can see that for $\alpha(r) = r^\gamma$, $\gamma = -3$ we have

$$\partial_t f = \mathrm{tr}(A_f D^2 f) + f^2$$

This is very similar to the quadratic heat equation...

Why may blow up occur?

…and so it is good news and bad news

$$\partial_t f = \Delta f + f^2$$
$$\text{VS.}$$
$$\partial_t f = \mathrm{tr}(A_f D^2 f) + f^2$$

It is **good news** because solutions to nonlinear parabolic equations remain smooth as long as f remains bounded.

It is **bad news**, because the quadratic heat equation blows up, the quadratic term drives $\sup\limits_x |f(t,x)| \to \infty$ in finite time.

Why may blow up not occur?

The question of smoothness for the Landau equation was first understood for $\alpha(r) = r^\gamma$ with γ in the range $[-2, 1]$.

In fact, heuristically

$$A_f \sim (-\Delta)^{\frac{5+\gamma}{2}} f \text{ and } h_f = (-\Delta)^{\frac{3+\gamma}{2}} f$$

Conserved quantities yield (Desvillettes-Villani 2000)

$$A_f \gtrsim \langle v \rangle^\gamma$$

For $\gamma < -2$ ("very soft potentials") a challenge is the "reaction" term h_f is not well controlled by known conserved quantities.

Why may blow up not occur?

McKean's 1966 paper

Consider Kac's 1D "caricature",

$$\partial_t f = \int_{\mathbb{R}} \int_0^{2\pi} (f(v_\theta)f(w_\theta) - f(v)f(w))d\theta dw$$

McKean showed that

$$\frac{d}{dt} \int_{\mathbb{R}} (\partial_v \log f)^2 f \; dv \leq 0$$

Why may blow up not occur?

Toscani's 1992 paper

Toscani shows that for 2D Boltzmann with Maxwell molecules

$$\frac{d}{dt}\int_{\mathbb{R}^2} |\nabla \log f|^2 f \; dv \leq 0$$

Villani (late 90's) developed an approach for the Landau equation with Maxwell molecules yielding monotonicity of the Fisher information for the equation in **any dimension**.

Numerical simulations by Filbet
Later, Filbet reports on numerical evidence for the monotonicity of the Fisher information even for the case of Coulomb potentials.

Why may blow up not occur?

In the 2010's, Krieger and Strain introduced the model

$$\partial_t f = ((-\Delta)^{-1} f)\Delta f + \alpha f^2$$

Showing for $\alpha < \frac{2}{3}$ that radial solutions do not blow up.
(Later improved to $\alpha < \frac{74}{75}$ by Gressman, Krieger, and Strain)

In work with Gualdani (2016) we rule out blow ups for $\alpha = 1$.

Main result

Theorem (with Luis Silvestre)
For f a classical solution to Landau with $\gamma = -3$ we have

$$\frac{d}{dt} \int_{\mathbb{R}^3} \frac{|\nabla f|^2}{f} \, dv \leq 0.$$

As a consequence, solutions starting from smooth initial data f_{in} don't blow up and remain smooth for all times.

This includes other potentials α, e.g. $\alpha(r) = r^\gamma$ when

$$|\gamma| \leq \sqrt{19}$$

$$\left(\text{improved to } |\gamma| \leq \sqrt{22} \text{ by Sehyun Ji}\right)$$

Main result

Our result has no smallness assumption but we do work with classical solutions with some decay at infinity.

Later works have improved the requirements on the initial data.

• Golding, Gualdani, and Loher (ARMA 2025) GWP $f_0 \in L^{3/2}$
• Desvillettes, Golding, Gualdani, and Loher (2024 preprint) existence $f_0 \in L^1_{|v|^{18}}$
• Ji (2024 preprint) existence $f_0 \in L^1_{2-\gamma}$

Interesting Q: rate of regularization at $t = 0$? (joint work with Cabrera and Gualdani suggests a very fast regularization rate)

Main result

Blow up is ruled out as the Fisher information controls $\|f\|_{L^3}$

$$\int_{\mathbb{R}^3} \frac{|\nabla f_{\rm in}|^2}{f_{\rm in}}\, dv \geq \int_{\mathbb{R}^3} \frac{|\nabla f|^2}{f}\, dv \geq c_3 \left(\int_{\mathbb{R}^3} f^3 \, dv \right)^{1/3}$$

(it's known blow up at $t = T$ requires $\|f(t)\|_{L^{3/2+}} \to \infty$ as $t \to T$)

Outline of the proof

The proof has three broad ingredients

- a "lifted" equation in $\mathbb{R}^6$ (involving $\Delta_{\mathbb{S}^2}$!)
- the symmetries of the lifted equation
- a functional inequality in $\mathbb{S}^2$

Interlude: Diffusions and Fisher

If f is a probability density in $\mathbb{R}^N$ evolving by $\partial_t f = \Delta f$ then

$$\frac{d}{dt}\int_{\mathbb{R}^N} f \log f \; dx = -\int_{\mathbb{R}^N} |\nabla \log f|^2 f \; dx \geq 0$$

The identity follows from a straightforward integration by parts.

Interlude: Diffusions and Fisher

An even stronger property holds: if f, g are two probability densities evolving by the heat equation, then

$$\frac{d}{dt}\int_{\mathbb{R}^N} f\log(f/g)\,dx = -\int_{\mathbb{R}^N} |\nabla \log(f/g)|^2 f\,dx$$

This identity also follows easily via integration by parts.

On the left we have the **relative entropy** of f and g, $H(f \mid g)$.

We see that $H(f \mid g)$ contracts along the diffusion.

Interlude: Diffusions and Fisher

The Fisher information is itself monotone

$$\frac{d}{dt}\int_{\mathbb{R}^N} |\nabla \log f|^2 f \; dx = -2\int_{\mathbb{R}^N} |\nabla^2 \log f|^2 f \; dx$$

This identity also follows from integration by parts, but this computation is more complicated and difficult to generalize.

The point of view taken here is that we owe this identity to

1) the symmetries of the PDE $\partial_t f = \Delta f$
2) diffusions producing a contraction in the relative entropy

Interlude: Diffusions and Fisher

The actual Fisher information

If we are given a 1-parameter family of probability densities

$$f_\theta(x),\ x \in \mathbb{R}^N,\ \theta \in (-\delta, \delta)$$

Then their **Fisher information** is the integral

$$I(f_\theta, \theta) = \int_{\mathbb{R}^N} (\partial_\theta \log f_\theta)^2 f_\theta \, dx$$

Fisher arrived at it in his foundational work, via the limit

$$\lim_{h \to 0} \frac{2}{h^2} H(f_\theta \mid f_{\theta+h}) = I(f, \theta)$$

Interlude: Diffusions and Fisher

The actual Fisher information

The Fisher information assigns to any smooth vector field b a functional over probability densities f, via

$$I_b(f) := I(f_{b,\theta}, 0)$$

where $f_{b,\theta}$ is the 1-parameter family of densities obtained by flowing f along the 1-parameter flow Φ^t generated by b, e.g.

$$f_{b,\theta}(x) = f(\Phi^\theta(x)) \text{ when } \mathrm{div}(b) = 0.$$

In this case $(\partial_\theta f)_{|\theta=0}$ is just $b \cdot \nabla f$, and so

$$I_b(f) = \int_{\mathbb{R}^N} (b \cdot \nabla \log f)^2 f \, dx$$

Interlude: Diffusions and Fisher

Fix a constant vector e. For $f(t,x)$ solving $\partial_t f = \Delta f$ set

$$f_h(t,x) := f(t, x+he)$$

For each h the function f_h **solves the same PDE** as f

Recalling the diffusion **contracts** the relative entropy...

$$\frac{d}{dt}\int_{\mathbb{R}^N} f \log(f/f_h)\, dx = -\int_{\mathbb{R}^N} |\nabla \log(f/f_h)|^2 f \, dx.$$

Interlude: Diffusions and Fisher

Multiply both sides of the identity by $2/h^2$,

$$\frac{d}{dt}\left(\frac{2}{h^2}H(f \mid f_h)\right) = -2\int_{\mathbb{R}^N}\left|\frac{\nabla \log(f/f_h)}{h}\right|^2 f\,dx$$

Taking $h \to 0$ on both sides, and using the limit relation between Fisher information and relative entropy we arrive at

$$\frac{d}{dt}\int_{\mathbb{R}^N}(e\cdot\nabla\log f)^2 f\,dx = -2\int_{\mathbb{R}^N}\left|(\nabla^2\log f)e\right|^2 f\,dx$$

This is valid for every vector e – taking the average over all unit directions we obtain the rate of change of the (full) Fisher information along the heat equation.

Interlude: Diffusions and Fisher

The big lesson:
The monotonicity of the Fisher info is related to symmetries!

One can argue similarly with rotations instead of translations:

$$\frac{d}{dt}\int_{\mathbb{R}^N} |\nabla_{\mathbb{S}^{N-1}} \log f|^2 f \, dx \leq 0$$

This mixing of symmetry and entropy will be an important ingredient in what follows.

The lifted equation

The collision operator has the form

$$q(f) = \int_{\mathbb{R}^3} \alpha \mathrm{div}_{v-w}(a\nabla_{v-w}[f \otimes f])dw$$

That is, it is the combination of the operations

$$f \to f \otimes f \to Q(f \otimes f) \to \mathrm{Pr}(Q(f \otimes f)) = q(f)$$

Where, for a function $F(v,w)$, Q denotes the operator

$$Q(F)(v,w) := \alpha(|v-w|)\mathrm{div}_{v-w}\left(a(v-w)\nabla_{v-w}F\right)$$

and Pr denotes the projection on the first marginal.

The lifted equation

Given $f(v)$, consider the linear Cauchy problem

$$\begin{cases} \partial_t F = Q(F) & \text{for } t > 0 \\ F = f \otimes f & \text{at } t = 0 \end{cases}$$

Then,

$$q(f) = \partial_t(\mathrm{Pr}(F))_{|t=0}$$

This linear PDE will be called **the lifted equation**.

The lifted equation

Lifting the Fisher information

The Fisher information works well with the lifted equation.
For now, we will use $I(f)$ to denote the Fisher information

$$I(f) = \int_{\mathbb{R}^N} |\nabla \log f|^2 f \, dx$$

for f a probability density in $\mathbb{R}^N$ (same I regardless of N)
(in this context it has been known as *Linnik's functional*)

The lifted equation

Lifting the Fisher information

The Fisher information is *additive* under independence

$$I(f^{(1)} \otimes f^{(2)}) = I(f^{(1)}) + I(f^{(2)})$$

The information from two independent experiments **equals** the sum of their information considered individually

The lifted equation

Lifting the Fisher information

…and the Fisher information is *superadditive* in general

$$I(F) \geq I(f^{(1)}) + I(f^{(2)})$$

($f^{(1)}(v)$ and $f^{(2)}(w)$ being marginals of $F(v, w)$)

The total information from two experiments is **no smaller** than the sum of their information considered individually

The lifted equation

Lifting the Fisher information

These two properties follow easily by Fubini and by an argument involving Jensen's inequality. They can be summarized as

If $F(v,w)$ is a probability density in $\mathbb{R}^N \times \mathbb{R}^N$ whose marginals are both equal to a density f in $\mathbb{R}^N$, we have

$$2I(f) = I(f \otimes f) \leq I(F)$$

This observation while elementary has notable consequences. In 1991 Carlen showed how it yields simple proofs of the log-Sobolev inequality and the Blachman-Stam inequality.

The lifted equation

Lifting the Fisher information

Lemma

For any f solving the Landau equation we have

$$2\langle i'(f), q(f)\rangle = \langle i'(f\otimes f), Q(f\otimes f)\rangle$$

This is a property not only of the Fisher information, but of any functional over distributions that is superadditive and additive under an independence assumption (tensor products).

Examples: Boltzmann entropy (in fact, Boltzmann implicitly used this in the proof of the H-theorem), quadratic transport distance (see Caja-Lopez, Delgadino, Gualdani, and Tasković)

The lifted equation

Lifting the Fisher information

This is a **substantial simplification** of the original question

$$\text{Is it the case that } \partial_t f = q(f) \text{ implies } \tfrac{d}{dt} i(f) \leq 0?$$

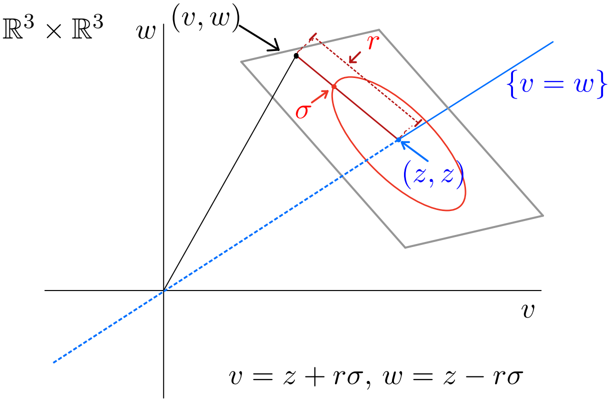
We are now asking a question about a linear PDE:

$$\text{Is it the case that } \partial_t F = Q(F) \text{ implies } \tfrac{d}{dt} i(F) \leq 0?$$

(and we only ask this for F such that $F(t,v,w) = F(t,w,v)$)

The lifted equation

Structure of the lifted equation



To distinguish between coordinates systems we will use a "bar":

$$\bar{F}(t, \sigma, r, z) := F(t, z + r\sigma, z - r\sigma)$$

The lifted equation is related to a "foliation" of $\mathbb{R}^6$ by $\mathbb{S}^2$.

Lemma

The function $F(t, v, w)$ solves

$$\partial_t F = Q(F) \text{ in } (0, \infty) \times \mathbb{R}^6$$

if and only if $\bar{F}(t, \sigma, r, z)$ solves

$$\partial_t \bar{F} = \alpha(r) \Delta_{\mathbb{S}^2} \bar{F} \text{ in } \mathbb{R}_+ \times \mathbb{S}^2 \times \mathbb{R}_+ \times \mathbb{R}^3$$

Estimating the Fisher information over time

For $F:\mathbb{R}^6 \to \mathbb{R}$ we have the following decomposition

$$|\nabla_{v,w}F|^2 = \frac{1}{2}\left(|\nabla_z \bar{F}|^2 + r^{-2}|\nabla_\sigma \bar{F}|^2 + (\partial_r \bar{F})^2\right)$$

Accordingly one can decompose $I(F)$ as follows

$$I(F) = I_{\text{par}}(F) + I_{\text{sph}}(F) + I_{\text{rad}}(F)$$

Estimating the Fisher information over time

$$I(F) = I_{\text{par}}(F) + I_{\text{sph}}(F) + I_{\text{rad}}(F)$$

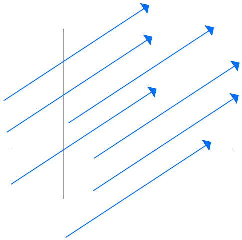
Here,

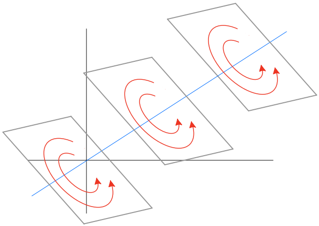
$$I_{\text{par}}(F) := \int_{\mathbb{R}_+\times\mathbb{R}^2}\int_{\mathbb{S}^2} \frac{|\nabla_z \bar{F}(\sigma, r, z)|^2}{\bar{F}(\sigma, r, z)} r^2 \, d\sigma dr dz$$

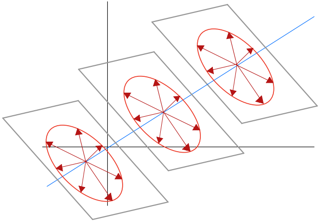
$$I_{\text{sph}}(F) := \int_{\mathbb{R}_+\times\mathbb{R}^2}\int_{\mathbb{S}^2} \frac{|\nabla_\sigma \bar{F}(\sigma, r, z)|^2}{\bar{F}(\sigma, r, z)} \, d\sigma dr dz$$

$$I_{\text{rad}}(F) := \int_{\mathbb{R}_+\times\mathbb{R}^2}\int_{\mathbb{S}^2} \frac{(\partial_r \bar{F}(\sigma, r, z))^2}{\bar{F}(\sigma, r, z)} r^2 \, d\sigma dr dz$$

Estimating the Fisher information over time







Estimating the Fisher information over time

Fix a vector $e \in \mathbb{R}^3$ and write

$$\bar{F}_{e,h}(t, \sigma, r, z) = \bar{F}(t, \sigma, r, z + he)$$

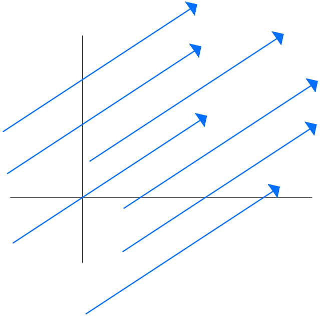
Then, $I_{\mathrm{par}}(F)$ is just the average over $e \in \mathbb{S}^2$ of

$$\lim_{h \to 0} \left\{ \frac{2}{h^2} H(\bar{F} \mid \bar{F}_{e,h}) \right\}$$

These are all decreasing in t, as we saw in the Interlude, **since the PDE for $\bar{F}$ is translation invariant in z.**

Estimating the Fisher information over time

$$\frac{d}{dt} I_{\text{par}}(F(t)) \leq 0$$



Estimating the Fisher information over time

The derivative $\frac{d}{dt} I_{\mathrm{sph}}(F)$ is (essentially) a classical computation.

In fact, for any fixed $r > 0, z \in \mathbb{R}^3$

$$\frac{d}{dt} \int_{\mathbb{S}^2} \frac{|\nabla_\sigma \bar{F}|^2}{\bar{F}} \, d\sigma = -2\alpha(r) \int_{\mathbb{S}^2} |\nabla_\sigma^2 \ln \bar{F}|^2 \bar{F} + |\nabla_\sigma \ln \bar{F}|^2 \bar{F} \, d\sigma$$

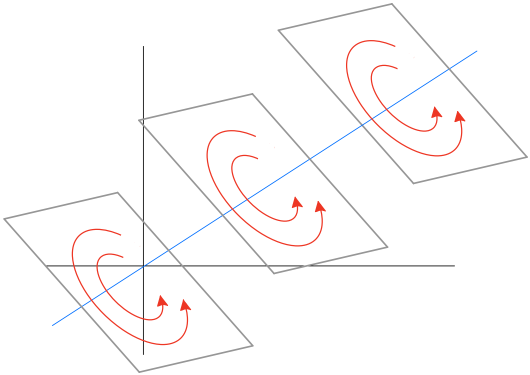
This has been known in the context of geometric analysis and it is a consequence of the Bochner formula.

These type of identities are a chief focus of Bakry-Émery theory.

Estimating the Fisher information over time

Integrating this identity in r, z,

$$\frac{d}{dt} I_{\mathrm{sph}}(F) = -2 \int_{\mathbb{R}_+ \times \mathbb{R}^3} \int_{\mathbb{S}^2} |\nabla_\sigma^2 \ln \bar{F}|^2 F + |\nabla_\sigma \ln \bar{F}|^2 \bar{F} \; d\sigma dr dz$$



Estimating the Fisher information over time

It remains to study $I_{\mathrm{rad}}(f)$. We argue as with z, considering

$$\bar{F}_h(t, \sigma, r + h, z)$$

If $\alpha(r) \neq \alpha(r + h)$ then $\bar{F}$ and $\bar{F}_h$ don't solve the same PDE, and the argument from the Interlude must be modified. Observe:

$$\partial_t \bar{F}_h = \alpha(r)\Delta_\sigma \bar{F}_h + (\delta_h \alpha)\Delta_\sigma \bar{F}_h$$

where $\delta_h \alpha := \alpha(r + h) - \alpha(r)$.

Estimating the Fisher information over time

Taking this extra term into account, we have

$$\begin{aligned}\frac{d}{dt}H_{z,r}(\bar{F}\mid\bar{F}_h) &= -\alpha\int_{\mathbb{S}^2}|\nabla_\sigma \ln(\bar{F}/\bar{F}_h)|^2\bar{F}d\sigma \\ &\quad -(\delta_h\alpha)\int_{\mathbb{S}^2}\ln(\bar{F}/\bar{F}_h)\Delta_\sigma\bar{F}_h d\sigma\end{aligned}$$

Ultimately, this leads us to

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{S}^2}\frac{(\partial_r\bar{F})^2}{\bar{F}}d\sigma = -\alpha\int_{\mathbb{S}^2}|\nabla_\sigma\partial_r\ln\bar{F}|^2\bar{F}d\sigma+\alpha'\int_{\mathbb{S}^2}\partial_r\ln\bar{F}\Delta_\sigma\bar{F}d\sigma$$

Estimating the Fisher information over time

There is a hidden square that can be completed, and so

$$I_{\mathrm{rad}}(F) \leq \int_{\mathbb{R}_+ \times \mathbb{R}^3} \int_{\mathbb{S}^2} ((\sqrt{\alpha})')^2 |\nabla_\sigma \ln \bar{F}|^2 \bar{F} r^2 d\sigma dr dz$$

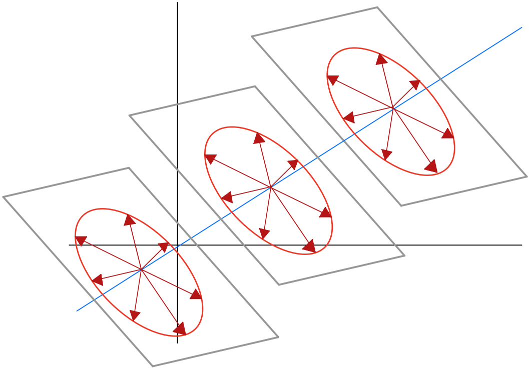
In our whole computation, this is the one term with a bad sign.

Observe that it vanishes for Maxwell molecules (this already gives a new proof of the results of Toscani and Villani).

Estimating the Fisher information over time

Thus we have for $I_{\text{rad}}(F)$

$$\frac{d}{dt} I_{\text{rad}}(F) \leq \int_{\mathbb{R}_+ \times \mathbb{R}^3} \int_{\mathbb{S}^2} ((\sqrt{\alpha})')^2 |\nabla_\sigma \ln \bar{F}|^2 \bar{F} r^2 d\sigma dr dz$$



Estimating the Fisher information over time

Combining the estimates for $\frac{d}{dt}$ of $I_{\text{par}}, I_{\text{sph}}$, and I_{rad} we have

Lemma

$$\frac{d}{dt} I(F) \leq -\int_{\mathbb{R}_+ \times \mathbb{R}^2} \int_{\mathbb{S}^2} \alpha \Gamma_2(\ln \bar{F}) \bar{F} d\sigma dr dz + \int_{\mathbb{R}_+ \times \mathbb{R}^2} \int_{\mathbb{S}^2} ((\sqrt{\alpha})')^2 |\nabla_\sigma \ln \bar{F}|^2 \bar{F} r^2 d\sigma dr dz$$

Estimating the Fisher information over time

Combining the estimates for $\frac{d}{dt}$ of $I_{\text{par}}, I_{\text{sph}}$, and I_{rad} we have

Lemma

$$\begin{aligned}\frac{d}{dt}I(F) \leq & -\int_{\mathbb{R}_+\times\mathbb{R}^2}\int_{\mathbb{S}^2}\alpha\Gamma_2(\ln\bar{F})\bar{F}d\sigma drdz \\ & +\int_{\mathbb{R}_+\times\mathbb{R}^2}\int_{\mathbb{S}^2}((\sqrt{\alpha})')^2|\nabla_\sigma\ln\bar{F}|^2\bar{F}r^2d\sigma drdz\end{aligned}$$

An integral inequality in $\mathbb{S}^2$

To finish it suffices to show that for each fixed r, z we have

$$r^2((\sqrt{\alpha})')^2 \int_{\mathbb{S}^2} |\nabla_\sigma \ln \bar{F}|^2 \bar{F} \, d\sigma \leq \alpha(r) \int_{\mathbb{S}^2} \Gamma_2(\ln \bar{F}) \bar{F} \, d\sigma$$

Equivalently

$$\left(\frac{r\alpha'(r)}{2\alpha(r)} \right)^2 \int_{\mathbb{S}^2} |\nabla_\sigma \ln \bar{F}|^2 \bar{F} \, d\sigma \leq \int_{\mathbb{S}^2} \Gamma_2(\ln \bar{F}) \bar{F} \, d\sigma$$

An integral inequality in $\mathbb{S}^2$

Lemma
If $f : \mathbb{S}^2 \to \mathbb{R}$ is positive, smooth, and **even**, then

$$\lambda \int_{\mathbb{S}^2} |\nabla_\sigma \ln f|^2 f \, d\sigma \leq \int_{\mathbb{S}^2} \Gamma_2(\ln f) f \, d\sigma$$

for some constant $\lambda \geq \frac{19}{4}$.

In conclusion:
The Landau equation admits $i(f)$ as a Lyapunov functional if

$$\sup_r \left| \frac{r\alpha'(r)}{\alpha(r)} \right| \leq \sqrt{19}$$

An integral inequality in $\mathbb{S}^1$

Exercise

If $f : \mathbb{S}^1 \to \mathbb{R}$, $f = f(\theta)$, is positive, smooth, and even, then

$$4 \int_{\mathbb{S}^1} (\partial_\theta \ln f)^2 f \; d\sigma \leq \int_{\mathbb{S}^1} (\partial_\theta^2 \ln f)^2 f \; d\sigma$$

Hint. Apply the Poincaré inequality in $\mathbb{S}^1$ to $\partial_\theta \sqrt{f}$.

The Boltzmann equation

Theorem (Imbert, Silvestre, and Villani, 2024)
For f a classical solution to the homogeneous Boltzmann equation (for minimal cross-section assumptions) we have

$$\frac{d}{dt}\int_{\mathbb{R}^3} \frac{|\nabla f|^2}{f}\, dv \leq 0.$$

As a consequence, solutions starting from smooth initial data f_{in} don't blow up and remain smooth for all times.

This result covers the case of very soft potentials ($\gamma \in (-3, -2)$).

Recent developments

A few more recent developments I want to mention (since I am out of time!)

- Implosion solutions for Landau built from compressible Euler implosions (Bedrossian, Chen, Gualdani, Ji, Vicol, and Yang) (Jinchang Yang's talk this afternoon)
- Existence for the Landau hierarchy (Carrillo and S. Guo)
- Lifted equation-based scheme (Huang, Cheng, and Gamba)
- Existence and estimates for the fuzzy Landau equation (collisions happening at a distance) (with Gualdani, Pavlović, Tasković, and Zamponi)
- Singular kernels $\Rightarrow$ entropy production may increase (Silvestre)

Summary

- The Landau equation has global smooth solutions

- $i(f)$ is monotone for many potentials (including Coulomb)

- Associated linear PDE in double the variables

- Symmetries (spherical, translational) are important

- Motivation for integral inequalities in $\mathbb{S}^2$

- Q's: when exactly does a functional "lift"? What about the space inhomogenous problem? (see: fuzzy models) Can we find α's without I-monotonicity? Monotonicity in **numerical schemes**?