ODE to Joy

And a new library management tool for humans and AI

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Axiomatic_AI


ODEs
curves on manifolds
Me
Physics PhD
Lean 10 hrs a day

ODE API

We should be able to express...

- Particles on a trajectory through a vector field
- Hamiltonian problems
- Schrödinger equation in finite dimensions
- Exponential curve on a Lie group
- Geodesic equations on a Riemannian manifold

Solutions to first-order ODEs

Basic definitions

```
variable {E : Type*} [NormedAddCommGroup E] [NormedSpace ℝ E]

def IsIntegralCurveOn (γ : ℝ → E) (v : ℝ → E → E) (s : Set ℝ) : Prop :=
  ∀ t ∈ s, HasDerivWithinAt γ (v t (γ t)) s t
```

"Is solution on a set"

```
def IsIntegralCurveAt (γ : ℝ → E) (v : ℝ → E → E) (t₀ : ℝ) : Prop :=
  ∀ᶠ t in 𝓝 t₀, HasDerivAt γ (v t (γ t)) t
```

"Is solution on a neighbourhood"

```
def IsIntegralCurve (γ : ℝ → E) (v : ℝ → E → E) : Prop :=
  ∀ t : ℝ, HasDerivAt γ (v t (γ t)) t
```

"Is global solution"

Solutions to first-order ODEs

Local existence (Picard-Lindelöf)

Conditions on the vector field

```
structure IsPicardLindelof {E : Type*} [NormedAddCommGroup E]
    (f : ℝ → E → E) {tmin tmax : ℝ} (t₀ : Icc tmin tmax) (x₀ : E) (a r L K : ℝ≥0) : Prop where
  lipschitzOnWith : ∀ t ∈ Icc tmin tmax, LipschitzOnWith K (f t) (closedBall x₀ a)
  continuousOn : ∀ x ∈ closedBall x₀ a, ContinuousOn (f · x) (Icc tmin tmax)
  norm_le : ∀ t ∈ Icc tmin tmax, ∀ x ∈ closedBall x₀ a, ‖f t x‖ ≤ L
  mul_max_le : L * max (tmax - t₀) (t₀ - tmin) ≤ a - r

variable {E : Type*} [NormedAddCommGroup E] [NormedSpace ℝ E] [CompleteSpace E]
```

Existence on a finite interval

```
theorem exists_eq_isIntegralCurveOn
    (hf : IsPicardLindelof f t₀ x₀ a r L K) (hx : x ∈ closedBall x₀ r) :
    ∃ α : ℝ → E, α t₀ = x ∧ IsIntegralCurveOn α f (Icc tmin tmax)
```

Existence of local flow

```
theorem exists_forall_mem_closedBall_eq_isIntegralCurveOn_lipschitzOnWith
    (hf : IsPicardLindelof f t₀ x₀ a r L K) :
    ∃ α : E → ℝ → E, (∀ x ∈ closedBall x₀ r, α x t₀ = x ∧
      IsIntegralCurveOn (α x) f (Icc tmin tmax)) ∧
      ∃ L' : ℝ≥0, ∀ t ∈ Icc tmin tmax, LipschitzOnWith L' (α · t) (closedBall x₀ r)
```

Refactored from Yury Kudryashov's formalisation

PR #35043

Solutions to first-order ODEs

Local existence (Picard-Lindelöf)

```
structure IsPicardLindelof {E : Type*} [NormedAddCommGroup E]
    (f : ℝ → E → E) {tmin tmax : ℝ} (t₀ : Icc tmin tmax) (x₀ : E) (a r L K : ℝ≥0) : Prop where
  lipschitzOnWith : ∀ t ∈ Icc tmin tmax, LipschitzOnWith K (f t) (closedBall x₀ a)
  continuousOn : ∀ x ∈ closedBall x₀ a, ContinuousOn (f · x) (Icc tmin tmax)
  norm_le : ∀ t ∈ Icc tmin tmax, ∀ x ∈ closedBall x₀ a, ‖f t x‖ ≤ L
  mul_max_le : L * max (tmax - t₀) (t₀ - tmin) ≤ a - r

variable {E : Type*} [NormedAddCommGroup E] [NormedSpace ℝ E] [CompleteSpace E]

theorem exists_eq_isIntegralCurveOn
    (hf : IsPicardLindelof f t₀ x₀ a r L K) (hx : x ∈ closedBall x₀ r) :
    ∃ α : ℝ → E, α t₀ = x ∧ IsIntegralCurveOn α f (Icc tmin tmax)

theorem exists_forall_mem_closedBall_eq_isIntegralCurveOn_lipschitzOnWith
    (hf : IsPicardLindelof f t₀ x₀ a r L K) :
    ∃ α : E → ℝ → E, (∀ x ∈ closedBall x₀ r, α x t₀ = x ∧
      IsIntegralCurveOn (α x) f (Icc tmin tmax)) ∧
      ∃ L' : ℝ≥0, ∀ t ∈ Icc tmin tmax, LipschitzOnWith L' (α · t) (closedBall x₀ r)
```

State vector field property around x_0 but show existence of curve starting at $x \neq x_0$.

PR #35043

Solutions to first-order ODEs

Uniqueness (Grönwall)

```
theorem IsIntegralCurveOn.eqOn_Icc_right
    (hv : ∀ t ∈ Ico a b, LipschitzOnWith K (v t) (s t))
    (hf : ContinuousOn f (Icc a b)) (hf' : IsIntegralCurveOn f v (Ico a b))
    (hfs : ∀ t ∈ Ico a b, f t ∈ s t)
    (hg : ContinuousOn g (Icc a b)) (hg' : IsIntegralCurveOn g v (Ico a b))
    (hgs : ∀ t ∈ Ico a b, g t ∈ s t) (ha : f a = g a) :
    EqOn f g (Icc a b)

theorem IsIntegralCurveAt.eventuallyEq
    (hv : ∀ᶠ t in 𝓝 t₀, LipschitzOnWith K (v t) (s t))
    (hf : IsIntegralCurveAt f v t₀) (hfs : ∀ᶠ t in 𝓝 t₀, f t ∈ s t)
    (hg : IsIntegralCurveAt g v t₀) (hgs : ∀ᶠ t in 𝓝 t₀, g t ∈ s t)
    (heq : f t₀ = g t₀) : f =ᶠ[𝓝 t₀] g
```

Formalised by Yury Kudryashov

PR #35043

Integral curves on manifolds

Basic definitions

```
variable
  {E : Type*} [NormedAddCommGroup E] [NormedSpace ℝ E]
  {H : Type*} [TopologicalSpace H] {I : ModelWithCorners ℝ E H}
  {M : Type*} [TopologicalSpace M] [ChartedSpace H M]

def IsMIntegralCurveOn (γ : ℝ → M) (v : (x : M) → TangentSpace I x) (s : Set ℝ) : Prop :=
  ∀ t ∈ s, HasMFDerivAt[s] γ t ((1 : ℝ →L[ℝ] ℝ).smulRight <| v (γ t))

def IsMIntegralCurveAt (γ : ℝ → M) (v : (x : M) → TangentSpace I x) (t₀ : ℝ) : Prop :=
  ∀ᶠ t in 𝓝 t₀, HasMFDerivAt% γ t ((1 : ℝ →L[ℝ] ℝ).smulRight <| v (γ t))

def IsMIntegralCurve (γ : ℝ → M) (v : (x : M) → TangentSpace I x) : Prop :=
  ∀ t : ℝ, HasMFDerivAt% γ t ((1 : ℝ →L[ℝ] ℝ).smulRight (v (γ t)))
```

Parallel to API for solutions to first-order ODEs

Manifold notation elaborator courtesy of Michael Rothgang

Integral curves on manifolds

Local existence (for C^1 vector fields)

```
variable
  {E : Type*} [NormedAddCommGroup E] [CompleteSpace E] [NormedSpace ℝ E]
  {H : Type*} [TopologicalSpace H] {I : ModelWithCorners ℝ E H}
  {M : Type*} [TopologicalSpace M] [ChartedSpace H M] [IsManifold I 1 M]
  {v : (x : M) → TangentSpace I x} (t₀ : ℝ) {x₀ : M}
  (hv : CMDiffAt 1 (fun x ↦ (⟨x, v x⟩ : TangentBundle I M)) x₀)
  (hx : I.IsInteriorPoint x₀)

theorem exists_isMIntegralCurveAt_of_contMDiffAt :
    ∃ γ : ℝ → M, γ t₀ = x₀ ∧ IsMIntegralCurveAt γ v t₀

theorem exists_mem_nhds_isMIntegralCurveOn_Ioo_of_contMDiffAt :
    ∃ u ∈ 𝓝 x₀, ∃ ε > (0 : ℝ), ∃ γ : M × ℝ → M, ∀ x ∈ u, γ ⟨x, t₀⟩ = x ∧
      IsMIntegralCurveOn (γ ⟨x, ·⟩) v (Ioo (t₀ - ε) (t₀ + ε)) ∧
      ContinuousOn γ (u ×ˢ Ioo (t₀ - ε) (t₀ + ε))
```

Local existence of integral curve

Existence of local flow
(continuous in initial condition)

PR #26394

Integral curves on manifolds

Local existence (for C^1 vector fields)

```
variable
  {E : Type*} [NormedAddCommGroup E] [CompleteSpace E] [NormedSpace ℝ E]
  {H : Type*} [TopologicalSpace H] {I : ModelWithCorners ℝ E H}
  {M : Type*} [TopologicalSpace M] [ChartedSpace H M] [IsManifold I 1 M]
  {v : (x : M) → TangentSpace I x} (t₀ : ℝ) {x₀ : M}
  (hv : CMDiffAt 1 (fun x ↦ (⟨x, v x⟩ : TangentBundle I M)) x₀)
  (hx : I.IsInteriorPoint x₀)

theorem exists_isMIntegralCurveAt_of_contMDiffAt :
    ∃ γ : ℝ → M, γ t₀ = x₀ ∧ IsMIntegralCurveAt γ v t₀

theorem exists_mem_nhds_isMIntegralCurveOn_Ioo_of_contMDiffAt :
    ∃ u ∈ 𝓝 x₀, ∃ ε > (0 : ℝ), ∃ γ : M × ℝ → M, ∀ x ∈ u, γ ⟨x, t₀⟩ = x ∧
      IsMIntegralCurveOn (γ ⟨x, ·⟩) v (Ioo (t₀ - ε) (t₀ + ε)) ∧
      ContinuousOn γ (u ×ˢ Ioo (t₀ - ε) (t₀ + ε))
```

Let's stay away from the boundary...

PR #26394

Integral curves on manifolds

Global existence (for C^1 vector fields)

```
variable
  {E : Type*} [NormedAddCommGroup E] [CompleteSpace E] [NormedSpace ℝ E]
  {H : Type*} [TopologicalSpace H] {I : ModelWithCorners ℝ E H}
  {M : Type*} [TopologicalSpace M] [ChartedSpace H M] [IsManifold I 1 M]
  [BoundarylessManifold I M]
  {v : (x : M) → TangentSpace I x} (hv : CMDiff 1 (fun x ↦ (⟨x, v x⟩ : TangentBundle I M)))

theorem exists_isMIntegralCurve_of_isMIntegralCurveOn
    {ε : ℝ} (hε : 0 < ε) (h : ∀ x : M, ∃ γ : ℝ → M, γ 0 = x ∧ IsMIntegralCurveOn γ v (Ioo (-ε) ε))
    (x : M) :
    ∃ γ : ℝ → M, γ 0 = x ∧ IsMIntegralCurve γ v
```

Global existence from uniform local existence

```
variable
  [T2Space M] [Nonempty M] [CompactSpace M]

theorem exists_global_flow_of_compactSpace :
    ∃ γ : ℝ → M → M, ∀ x, γ 0 x = x ∧ IsMIntegralCurve (γ · x) v
```

Global flow on compact manifolds

PR #26395

Smoothness of the flow

Solutions to first-order ODEs

Local smoothness with respect to initial condition

Given a C^k vector field f, where $k \geq 1$, and a point x_0, there exist $\epsilon > 0$ and a function $\alpha : E \to C((t_0 - \epsilon, t_0 + \epsilon), E)$ such that $(x, t) \mapsto \alpha(x)(t)$ is a local flow of f at (x_0, t_0), and that α is C^k.

```
lemma exists_localFlow {f : E → E} {x₀ : E} {k : ℕ} (hk : 1 ≤ k)
    (hf : ContDiffAt ℝ k f x₀) (t₀ : ℝ) :
    ∃ (ε : ℝ) (hε : 0 < ε) (α : E → C(Icc (t₀ - ε) (t₀ + ε), E)),
        "α is a local flow of f around x₀ on (t₀ - ε, t₀ + ε)" ∧
        ContDiffAt ℝ k α x₀
```

PR #34288

Solutions to first-order ODEs

Local smoothness with respect to initial condition

- Follows Lang's *Fundamentals of Differential Geometry*
- Applies implicit function theorem in Banach spaces (Yury, agjftucker, myself)

$$T : E \times C(I, E) \to C(I, E)$$

$$T(x, \alpha)(t) := x + \int_0^t f(\alpha(u))\, du - \alpha(t)$$

- Picard-Lindelöf gives a solution, $T(x_0, \alpha_0) = 0$
- Implicit function theorem gives α as a C^k map in x, given T is C^k

PR #34288

Solutions to first-order ODEs

Local smoothness with respect to initial condition

$$T(x, \alpha)(t) := x + \int_0^t f(\alpha(u))\, du - \alpha(t)$$

- Need $T(x, \alpha)$ as a $C(I, E)$, not just $I \to E$
- f is only C^k on some set U; need to carry around $\operatorname{range} \alpha \subseteq U$
- Solution: set $T = 0$ as junk value if $\operatorname{range} \alpha \nsubseteq U$

PR #34288

Solutions to first-order ODEs

Local smoothness with respect to initial condition

$$T(x, \alpha)(t) := x + \int_0^t f(\alpha(u))\, du - \alpha(t)$$

- Need continuity of k-th derivative of $T(x, \alpha)$. For $k \geq 2$,

$$\left[D_\alpha^k T(x, \alpha)\,(\delta\alpha_1, \cdots, \delta\alpha_k)\right](t) = \int_0^t D^k f(\alpha(u))(\delta\alpha_1(u), \cdots, \delta\alpha_k(u))\, du$$

PR #34288

Solutions to first-order ODEs

Local smoothness with respect to initial condition

$$\left[D^k_\alpha T(x,\alpha)\,(\delta\alpha_1,\cdots,\delta\alpha_k)\right](t) = \int_0^t D^k f(\alpha(u))(\delta\alpha_1(u),\cdots,\delta\alpha_k(u))\,du$$

- Use `iteratedFDeriv` as a multilinear map (courtesy of Sébastien Gouëzel)

$$D^k f(x) : E\,[\times k] \to_L E$$

$$D^k_\alpha T(x,\alpha) : C(I,E)\,[\times k] \to_L C(I,E)$$

PR #34288

Solutions to first-order ODEs

Local smoothness with respect to initial condition

$$\left[D^k_\alpha T(x,\alpha)\,(\delta\alpha_1,\cdots,\delta\alpha_k)\right](t) = \int_0^t D^k f(\alpha(u))(\delta\alpha_1(u),\cdots,\delta\alpha_k(u))\,du$$

- Base case of `iteratedFDeriv`, using `continuousMultilinearCurryFin0`

$$E \simeq_{li} (E\,[\times 0] \to_L E) \qquad C(I,E) \simeq_{li} (C(I,E)\,[\times 0] \to_L C(I,E))$$

- Inductive case of `iteratedFDeriv`, using `continuousMultilinearCurryFin0`

$$(E \to_L (E\,[\times k] \to_L E)) \simeq_{li} (E\,[\times(k+1)] \to_L E) \qquad \cdots$$

PR #34288

Solutions to first-order ODEs

Local smoothness with respect to initial condition

$$T(x, \alpha)(t) := x + \int_0^t f(\alpha(u))\, du - \alpha(t)$$

- Need $T(x, \alpha)$ as a ~~$C(I, E)$~~ $C(I, E)\,[\times 0] \to_L C(I, E)$, not just $I \to E$
- Integral takes general $g : E\,[\times k] \to_L E$
- $D^k_\alpha T(x, \alpha)$ is *continuous* multilinear only if g is continuous
- Solution: set $T = 0$ as junk value if g, i.e. $D^k f$, is not continuous

PR #34288

Solutions to first-order ODEs

Local smoothness with respect to initial condition

$$T(x, \alpha)(t) := x + \int_0^t f(\alpha(u))\, du - \alpha(t)$$

- Finally, show the first derivative is invertible in the α-component

$$D_\alpha T(x, \alpha)\, \delta\alpha(t) = \int_0^t Df(\alpha(u))\, \delta\alpha(u)\, du - \delta\alpha(t)$$

- Standard analysis proof

Library wishlist

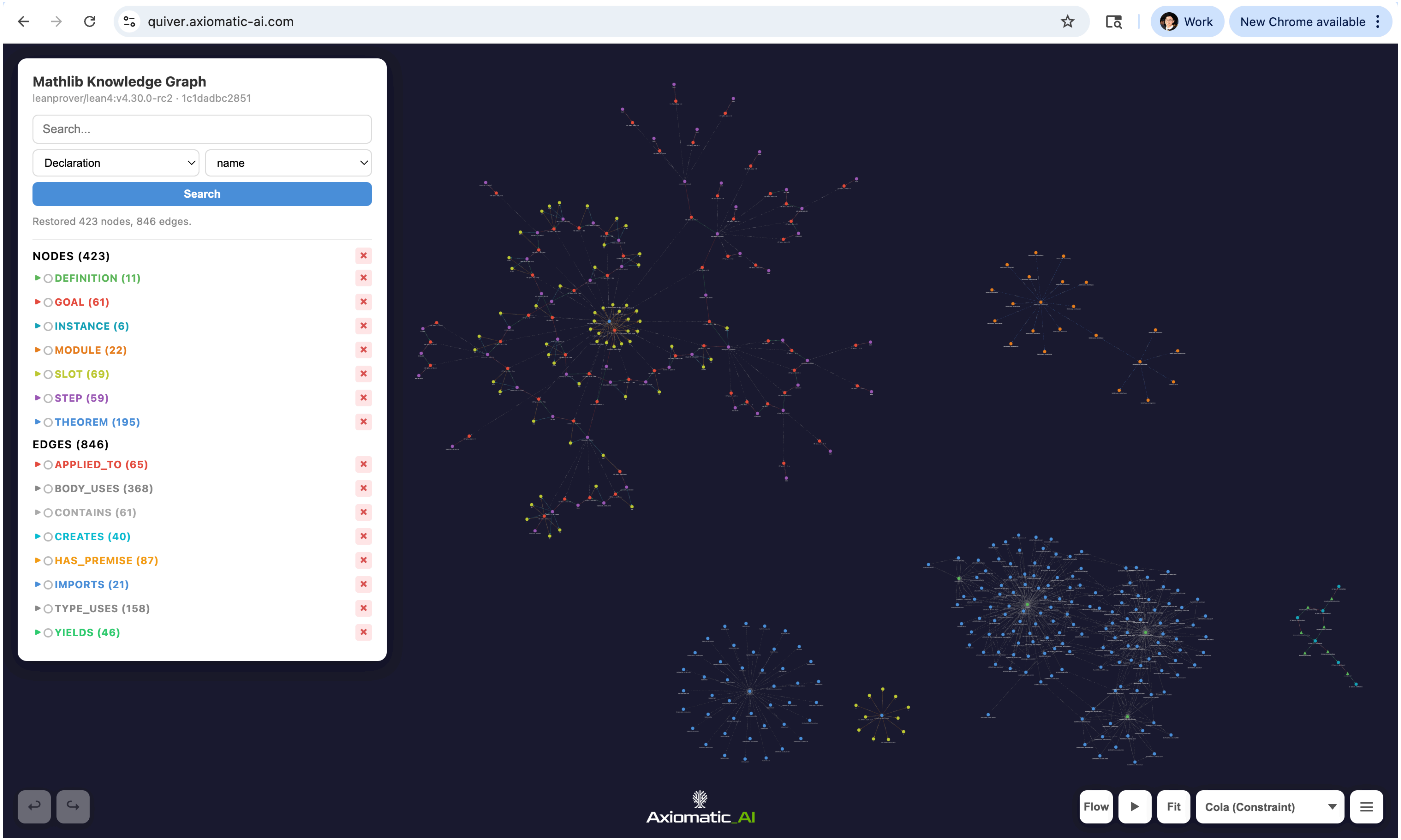
- Rest of the proof of smoothness of the flow (needs linear ODEs)
- Port to flows on manifolds
- Higher-order ODEs
- Start writing down concrete ODE problems! (Physlib?)

One more thing

Library management tasks

- List all theorems that share the premises A, B, and C
- List all theorems that transitively use theorem X up to depth N
- Identify declaration dependency clusters within a file
- List all theorems whose proof contains the goal $a \leq b$
- Typeclass synthesis chains
- List all goals rewritten by the lemma L

Demo


quiver.axiomatic-ai.com
Work
New Chrome available
Mathlib Knowledge Graph
leanprover/lean4:v4.30.0-rc2 · 1c1dadbc2851
Search...
Declaration
name
Search
Restored 423 nodes, 846 edges.
NODES (423)
DEFINITION (11)
GOAL (61)
INSTANCE (6)
MODULE (22)
SLOT (69)
STEP (59)
THEOREM (195)
EDGES (846)
APPLIED_TO (65)
BODY_USES (368)
CONTAINS (61)
CREATES (40)
HAS_PREMISE (87)
IMPORTS (21)
TYPE_USES (158)
YIELDS (46)
Axiomatic_AI
Flow
Fit
Cola (Constraint)

Find theorems about open... × Lean InfoView

Find theorems about open sets in compa...

In mathlib, when the goal is Continuous f, which tactics actually close it most often?

Quiver [resolve_conditions]

OUT {"result":"Showing 1–10 of 84 (call again with offset=10 for more):\ncondition_id

Thought for 0s

Quiver [tactic_distribution]

OUT {"result":"Tactic distribution (29 step applications):\n rwSeq

Thought for 0s

Quiver [tactic_distribution]

OUT {"result":"Tactic distribution (2560 step applications):\n funPropTacStx

Thought for 0s

Here's what the data shows. Two important caveats up front:

Graph stats

For Mathlib

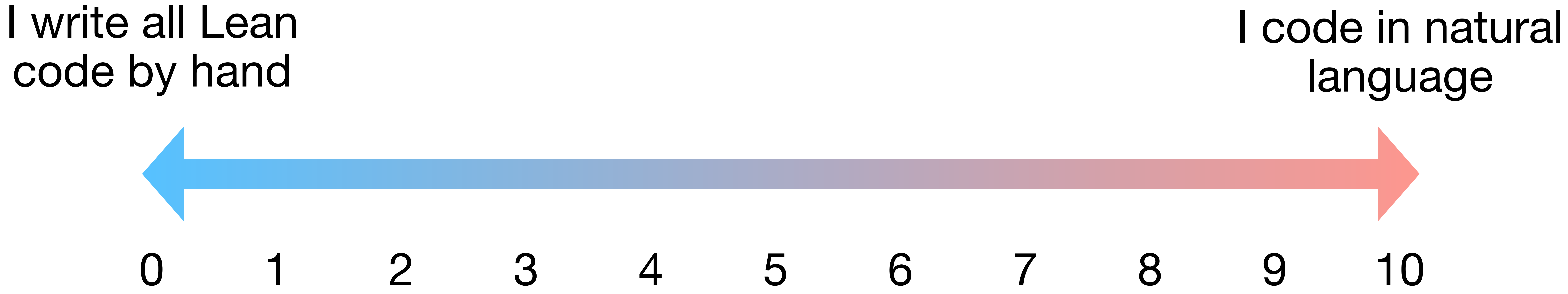
- 7.3M nodes, 50M edges
- 399K theorems + definitions
- 4.5M premises
- 720K proof steps
- 99K typeclass instances
- Natural language search

Axiomatic_AI

Features

- GitHub continuous integration for any Lean project
- Generate datasets via graph cross-cutting for neural training
- Web interface for humans
- MCP server for LLM agents

Where do you sit?



Try it out today!

- Web interface: quiver.axiomatic-ai.com
- MCP server:
  `claude mcp add --transport http quiver https://quiver.axiomatic-ai.com/mcp`
- Sign up via GitHub account (I will approve)
- Updated daily against Mathlib

Axiomatic_AI