

Locally convex spaces and classical distributions in Mathlib

Anatole Dedecker

IMJ-PRG, Université Paris Cité

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Ethics statement

The work presented in this talk did *not* use:

- insane amounts of energy and water compared to the expected benefits
- stolen intellectual work.

It does *not* encourage people to:

- stop learning
- stop asking questions
- flood the publication system or open source effort
- stop improving Lean / Mathlib.

In short: no generative AI was used.

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Locally convex analysis

```
variable [NormedField  $\mathbb{k}$ ] [AddCommGroup E] [Module  $\mathbb{k}$  E]
/-- The proposition that the topology of `E` is induced by a
family of seminorms `p`. -/
structure WithSeminorms (p : SeminormFamily  $\mathbb{k}$  E  $\iota$ )
  [topology : TopologicalSpace E] : Prop where
  topology_eq_withSeminorms :
    topology = p.moduleFilterBasis.topology
```

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Between such spaces, we have a very convenient continuity criterion : for each generating seminorm p on the codomain, the seminorm $p \circ f$ is controlled by finitely many generating seminorms on the domain.

```
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/-- A topological vector space `E` is polynormable over
if its topology is induced by some family of seminorms.
Equivalently, its topology is induced by all its continuous
seminorms. -/
class PolynormableSpace [topology : TopologicalSpace E] where
  withSeminorms' : WithSeminorms
    (fun p : {p : Seminorm  $\mathbb{k}$  E // Continuous p}  $\mapsto$  p.1)
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```

Convenient to generalize :

- analytical form of the Hahn-Banach theorem (ongoing work by Aaron Lin)
- Banach-Steinhaus theorem once we defined equicontinuity

```
class LocallyConvexSpace (k E : Type*) [Semiring k]
  [PartialOrder k] [AddCommMonoid E] [Module k E]
  [TopologicalSpace E] : Prop where
convex_basis : ∀ x : E, (N x).HasBasis
  (fun s : Set E → s ∈ N x ∧ Convex k s) id
```

```
theorem locallyConvexSpace_iff_exists_convex_subset_zero :
  LocallyConvexSpace k E ↔ ∀ U ∈ N (0 : E),
    ∃ S ∈ N (0 : E), Convex k S ∧ S ⊆ U := ...
```

```
class LocallyConvexSpace (ℓ E : Type*) [Semiring ℓ]
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Krein-Milman theorem...

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Convenient to generalize geometric form of the Hahn-Banach theorem, Krein-Milman theorem...

Moritz Doll showed the equivalence between the two approaches over \mathbb{R} .

Subspaces and products of locally convex spaces are locally convex.

Some stability properties

Subspaces and products of locally convex spaces are locally convex.

theorem `LocallyConvexSpace.iInf`

```
{ts' : ι → TopologicalSpace E}
(h' : ∀ i, @LocallyConvexSpace ℚ E _ _ _ _ (ts' i)) :
@LocallyConvexSpace ℚ E _ _ _ _ (∏ i, ts' i) := ...
```

theorem `LocallyConvexSpace.induced`

```
{t : TopologicalSpace F} [LocallyConvexSpace ℚ F]
(f : E →1 [ℚ] F) :
@LocallyConvexSpace ℚ E _ _ _ _ (t.induced f) := ...
```

Recall $\mathcal{T}_1 \leq \mathcal{T}_2$ if the identity is continuous from \mathcal{T}_1 to \mathcal{T}_2 , i.e if $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

A quotient of a locally convex space is locally convex.

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This is completely missing from Mathlib. For the geometric definition, this should be very easy.

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For the analytical definition, one missing tool is `Seminorm.map` (see draft PR #34106) : given $f : E \twoheadrightarrow F$ and p a seminorm on E , we get a seminorm f_*p on F defined by:

$$f_*p(x) := \inf_{y \in f^{-1}(x)} p(y)$$

Then we need to show that the topology induced by the family $(f_*p_i)_i$ is the quotient of the topology induced by $(p_i)_i$.

Topologies on $\mathcal{L}(E, F)$

When E, F are normed spaces over a nondiscrete normed field and $u \in \mathcal{L}(E, F)$, we let $\|u\|$ be the best constant $C \geq 0$ such that

$$\forall x, \|u(x)\| \leq C\|x\|$$

In other words, $u_i \rightarrow u$ in operator norm if and only if $u_i \rightarrow u$ uniformly on bounded subsets of E .

Topologies on $\mathcal{L}(E, F)$

General setup (Bourbaki): X is a set, (Y, \mathcal{U}_Y) is a **uniform space**, and \mathfrak{S} is a family of subsets of X .

Topology of uniform convergence : $f_i \longrightarrow g$ when, for any $U \in \mathcal{U}_Y$, for all i large enough, we have $\forall x, (f_i(x), g(x)) \in U$.

Topology of \mathfrak{S} -convergence : $f_i \longrightarrow g$ when, for all $S \in \mathfrak{S}$, we have $f_i|_S \longrightarrow g|_S$ uniformly.

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Type aliases:

$X \rightarrow_u Y$ $X \rightarrow_u [\mathfrak{S}] Y$

$E \rightarrow_{L_u} [\mathbb{k}, \mathfrak{S}] F$ $E \rightarrow_{L_c} [\mathbb{k}] F$ $E \rightarrow_{L_{p,t}} [\mathbb{k}] F$

Boilerplate heavy. Should we have `WithUniform` ?

Definition (Bounded sets in a TVS)

Let E be a TVS over a nondiscrete normed field \mathbb{K} . A subset S of E is (von Neumann) **bounded** if, for any neighborhood U of zero in E , there is a neighborhood of zero V in \mathbb{K} such that $VS \subseteq U$.

Since the operator norm is an instance, we put the topology of uniform convergence on bounded sets “by default” on $E \rightarrow L[\mathbb{K}] F$.

Hence, when we define the operator norm, we need to check that the topologies match (“forgetful inheritance”). Bourbaki only explains this for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$...

The space of distributions

Joint work with Luigi Massaci

Textbook definition

Let Ω be an open subset of \mathbb{R}^n . Denote by $\mathcal{D}(\Omega)$ the space of **test functions**, that is, smooth compactly supported functions, on Ω .

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Let Ω be an open subset of \mathbb{R}^n . Denote by $\mathcal{D}(\Omega)$ the space of **test functions**, that is, smooth compactly supported functions, on Ω .

Definition (Distribution)

A **distribution** on Ω is a linear form on $\mathcal{D}(\Omega)$ which is “continuous” in the following sense :

For any compact set $K \subseteq \Omega$, there exists a natural number n and a constant $C > 0$ such that :

$$\forall \varphi \in \mathcal{D}(\Omega), \text{Supp } \varphi \subseteq K \Rightarrow |T(\varphi)| \leq C \sup_{|\alpha| \leq n} \|\partial^\alpha f\|_\infty$$

Textbook definition

- The space of distributions is denoted $\mathcal{D}'(\Omega)$.
- Any locally integrable function $f : \Omega \rightarrow \mathbb{R}$ defines a distribution $T_f : \varphi \mapsto \int_{\Omega} f\varphi \, d\lambda$. We have $T_f = T_g \Rightarrow f = g$ a.e.
- Any (signed) Radon measure μ on Ω defines a distribution $T_{\mu} : \varphi \mapsto \int \varphi \, d\mu$. We have $T_{\mu} = T_{\nu} \Rightarrow \mu = \nu$.
- For any $T \in \mathcal{D}'(\Omega)$, we can define $\partial_j T : \varphi \mapsto -T(\partial_j \varphi)$.
If f is C^1 , we have $\partial_j T_f = T_{\partial_j f}$.

The topology on test functions

For any compact K of \mathbb{R}^n , denote by \mathcal{D}_K the space of test functions with support in K . Endow it with the topology of **uniform convergence of f and all of its derivatives**, i.e. view it as a subspace of $C(K)^{\mathbb{N}^n}$.

Endow $\mathcal{D}(\Omega)$ with the “inductive limit **locally convex** topology”, that is, the finest locally convex topology such that the inclusions $\mathcal{D}_K \rightarrow \mathcal{D}(\Omega)$ are continuous.

The topology on test functions

Of course, we want to replace \mathbb{R}^n with an arbitrary finite dimensional real vector space E . Instead of partial derivatives, we have :

- `lineDeriv` : ∂_u for an arbitrary vector u .
- `fderiv` : the Fréchet / total derivative.

We also allow a normed space F as codomain.

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- `fderiv` : the Fréchet / total derivative.

We also allow a normed space F as codomain.

For each $i \in \mathbb{N}$, we define a linear map `structureMapLM` :

$$\begin{aligned} \mathcal{D}_K(E, F) &\rightarrow C_b(E, \mathcal{L}^i(E, F)) \\ f &\mapsto D^i f \end{aligned}$$

We endow $\mathcal{D}_K(E, F)$ with the initial topology for these maps. This is locally convex by the general theory.

```
def seminorm (i : ℕ) : Seminorm ℚ  $\mathcal{D}_{\{K\}}$ (E, F) :=  
  (normSeminorm ℚ (E →b (E [×i] → L[ℝ] F))).comp  
  (structureMapLM ℚ T i)
```

```
theorem withSeminorms :  
  WithSeminorms  
  (ContDiffMapSupportedIn.seminorm ℚ E F T K) := ...
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```
theorem withSeminorms :  
  WithSeminorms  
    (ContDiffMapSupportedIn.seminorm ℚ E F T K) := ...
```

```
theorem seminorm_fderivLM_le {i : ℕ} (f :  $\mathcal{D}_{\{K\}}$ (E, F)) :  
  N[ℚ]_{K, i} (fderivLM ℚ T T f) ≤ N[ℚ]_{K, i + 1} f := ...
```

```
noncomputable def fderivCLM :  
   $\mathcal{D}_{\{K\}}$ (E, F)  $\rightarrow L[\mathbb{Q}] \mathcal{D}_{\{K\}}$ (E, E  $\rightarrow L[\mathbb{R}] F$ ) where  
  ...
```

The topology on test functions

For any compact set $K \subseteq \Omega$, we call `TestFunction.ofSupportedIn` the inclusion $\mathcal{D}_K(E, F) \rightarrow \mathcal{D}(\Omega, F)$.

```
variable (Ω F n) in
noncomputable def originalTop :
  TopologicalSpace  $\mathcal{D}(\Omega, F) :=$ 
  ⊔ (K : Compacts E) (K_sub_Ω : (K : Set E) ⊆ Ω),
  coinduced (ofSupportedIn K_sub_Ω)
  ContDiffMapSupportedIn.topologicalSpace
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```

This is the inductive limit topology **in the category of topological spaces** : it has no reason to be a LCTVS topology though in that case it is.

The topology on test functions

```

variable (Ω F n) in
instance topologicalSpace : TopologicalSpace  $\mathcal{D}(\Omega, F) :=
  sInf {t : TopologicalSpace  $\mathcal{D}(\Omega, F) |
    originalTop Ω F n ≤ t ∧
    @IsTopologicalAddGroup  $\mathcal{D}(\Omega, F) t _ ^
    @ContinuousSMul ℝ  $\mathcal{D}(\Omega, F) _ _ t ^
    @LocallyConvexSpace ℝ  $\mathcal{D}(\Omega, F) _ _ _ _ t}$$$$$ 
```

This is automatically locally convex by general theory !

The topology on test functions

Universal property !

variable [LocallyConvexSpace \mathbb{R} V]

theorem `continuous_iff_continuous_comp`

$(f : \mathcal{D}(\Omega, F) \rightarrow_1 [\mathbb{R}] V) :$

Continuous $f \leftrightarrow$

$\forall (K : \text{Compacts } E) (K_sub_Omega : (K : \text{Set } E) \subseteq \Omega),$

Continuous $(f \circ \text{ofSupportedIn } K_sub_Omega) := \dots$

The proof is just abstract nonsense !

More practical variant :

```
def limitCLM (toFun :  $\mathcal{D}(\Omega, F) \rightarrow V$ )  
  (T :  $\prod (K : \text{Compacts } E), (K : \text{Set } E) \subseteq \Omega \rightarrow$   
     $\mathcal{D}_{\{K\}}(E, F) \rightarrow L[\mathbb{R}] V$ )  
  (toFun_eq_T :  $\forall K K\_sub\_Omega f,$   
    toFun (ofSupportedIn K_sub_Omega f) = T K K_sub_Omega f) :  
   $\mathcal{D}(\Omega, F) \rightarrow L[\mathbb{R}] V := \dots$ 
```

We get `fderiv` and `lineDeriv` as continuous linear maps in just a few lines.

At this point, we can define $\mathcal{D}'(\Omega) := \mathcal{L}(\mathcal{D}(\Omega, \mathbb{R}), \mathbb{R})$, and define the line derivative by $\partial_u T(\varphi) := -T(\partial_u \varphi)$.

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But Mathlib loves total derivatives. Note that, for a function $f : E \rightarrow \mathbb{R}$, the function $Df : E \rightarrow \mathcal{L}(E, \mathbb{R})$ is naturally **vector-valued**.

Definition (Vector-valued distributions)

Let E be a finite-dimensional real normed space, Ω an open in E , and F an arbitrary LCTVS. An **F -valued distribution** on Ω is a continuous linear map $\mathcal{D}(\Omega, \mathbb{R}) \rightarrow F$.

Denote by $\mathcal{D}'(\Omega, F)$ the space of such distributions but it's no longer a dual space...

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Denote by $\mathcal{D}'(\Omega, F)$ the space of such distributions but it's no longer a dual space...

If $f : \Omega \rightarrow F$ is locally integrable, we still have $T_f : \varphi \mapsto \int \varphi \cdot f$.

For $T \in \mathcal{D}'(\Omega, F)$ and $u \in E$, we still have $\partial_u T \in \mathcal{D}'(\Omega, F)$.

Let $T \in \mathcal{D}'(\Omega, F)$. We want to build $DT \in \mathcal{D}'(\Omega, \mathcal{L}(E, F))$ out of $\partial_{\bullet} T : E \rightarrow \mathcal{D}'(\Omega, F)$.

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$$DT(\varphi)(u) := \partial_u T(\varphi) = -T(\partial_u \varphi)$$

Continuity ?

Let $T \in \mathcal{D}'(\Omega, F)$. We want to build $DT \in \mathcal{D}'(\Omega, \mathcal{L}(E, F))$ out of $\partial_\bullet T : E \rightarrow \mathcal{D}'(\Omega, F)$.

$$DT(\varphi)(u) := \partial_u T(\varphi) = -T(\partial_u \varphi)$$

Continuity?

```

variable {A ι : Type*} (C : Set (Set A)) (X : ι → Type*)
  [∀ i, UniformSpace (X i)]
/-- The natural bijection between `A → Π i, X i` and
`Π i, A → X i`, upgraded to a uniform isomorphism between
`A →_u [C] (Π i, X i)` and `Π i, A →_u [C] X i`. -/
protected def uniformEquivPiComm :
  (A →_u [C] ((i : ι) → X i)) ≈_u ((i : ι) → A →_u [C] X i) :=
  ...

```

Definition (Order of a distribution)

A distribution T has order $\leq n$ if, for any compact set $K \subseteq \Omega$, there exists a constant $C > 0$ such that :

$$\forall \varphi \in \mathcal{D}_K, |T(\varphi)| \leq C \sup_{|\alpha| \leq n} \|\partial^\alpha f\|_\infty$$

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Equivalently, T can be evaluated on functions which are only n -times differentiable.

Properties : differentiation may increase the order by (at most) one, T_f has order zero, ...

For any $n \in \mathbb{N} \cup \{\infty\}$, we define the spaces $\mathcal{D}_K^{(n)}$, $\mathcal{D}^{(n)}(\Omega)$ by looking only at the derivatives up to order n , and we let $\mathcal{D}'^{(n)}(\Omega, F) := \mathcal{L}(\mathcal{D}^{(n)}(\Omega), F)$.

We have $D : \mathcal{D}^{(n+1)} \rightarrow \mathcal{D}^{(n)}$, hence $D : \mathcal{D}'^{(n)} \rightarrow \mathcal{D}'^{(n+1)}$.

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In fact, we define $D : \mathcal{D}^{(k)} \rightarrow \mathcal{D}^{(n)}$, hence $D : \mathcal{D}'^{(n)} \rightarrow \mathcal{D}'^{(k)}$, which are zero unless $k > n$. This becomes a bit annoying because of all the extra parameters.

Topology on distributions

We endow $\mathcal{D}'^{(n)}(\Omega, F)$ with the topology of uniform convergence of compact subsets of $\mathcal{D}^{(n)}(\Omega, \mathbb{R})$.

Because $\mathcal{D}(\Omega, \mathbb{R})$ is Montel (bounded closed sets are compact), this is the same as the topology of bounded convergence.

For **sequences**, this is the same as pointwise convergence. But it is not the case for a general filter...

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Abstract theory tells us that all the operations defined by transposition are continuous. We can also show that $T \mapsto DT$ is continuous.

Future work

- Prove that $T_{Df} = DT_f$ (should be very easy)
- Multiplication by a smooth function (should be very easy)
- Singular support
- Localization principle: $\mathcal{D}'(-; F)$ is a sheaf
- Structure of point-supported distributions
- Tensor product of distributions

Less elementary : convolution

You know everything !

We use the larger space $\mathcal{E}(\Omega; \mathbb{R})$ of smooth functions on Ω as our test functions. Note that these can't be smoothly extended outside of Ω .

Here, the space $\mathcal{E}(\Omega; \mathbb{R})$ has the “weak topology” : uniform convergence on compact sets of all derivatives. Making this an instance will make geometers angry but this is what we already do in the continuous case...

Topological properties of \mathcal{D} and \mathcal{D}'

Future work

- \mathcal{D} and \mathcal{D}' are complete
- Description of bounded sets $\implies \mathcal{D}$ is Montel
- \mathcal{D} is bornological and barrelled
- \mathcal{D} is dense in \mathcal{D}'
- $\mathcal{D}'' = \mathcal{D}$
- \mathcal{D} and \mathcal{D}' are nuclear

Two main approaches :

- gluing of distributions in charts
- dual of compactly supported 1-densities (or $\frac{1}{2}$ -densities)

In any case, we will want to be able to test against densities, so we need :

- the bundle of α -densities
- a convenient way to construct topologies on spaces of sections : jet bundles ?

To go to currents, we need a working API for differential forms

Theorem (Schwartz kernel theorem)

Let E_1, E_2 be finite dimensional spaces and Ω_i an open subset of E_i for $i \in \{1, 2\}$.

The natural map $\mathcal{D}'(\Omega_1 \times \Omega_2; \mathbb{R}) \rightarrow \mathcal{D}'(\Omega_1, \mathcal{D}'(\Omega_2; \mathbb{R}))$ is an isomorphism of topological vector spaces.

This extends to manifolds.

N. BOURBAKI

ÉLÉMENTS DE MATHÉMATIQUE

Espaces vectoriels topologiques
Chapitres 6 et 7

Thanks for listening !

Theorem (Normed Banach-Steinhaus)

Let E, F be two normed spaces over \mathbb{R} or \mathbb{C} , with E complete, and $T : I \rightarrow \mathcal{L}(E, F)$ a family of continuous linear maps.

Assume that, for any $x \in E$, the set $\{T_i(x)\}_{i \in I}$ is bounded in E . Then the set $\{T_i\}_{i \in I}$ is bounded in $\mathcal{L}(E, F)$, for the operator norm.

Corollary

Under the same assumption, if $T : \mathbb{N} \rightarrow \mathcal{L}(E, F)$ converges *pointwise*, then the limit is still continuous !

Here is the generalization proven in Bourbaki.

Definition (Barrels and barrelled spaces)

Let E be a LCTVS. A subset S of E is a **barrel** if it is closed, convex, balanced, and absorbent. E is **barrelled** if any barrel in E is a neighborhood of zero.

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Definition (Barrels and barrelled spaces)

Let E be a LCTVS. A subset S of E is a **barrel** if it is closed, convex, balanced, and absorbent. E is **barrelled** if any barrel in E is a neighborhood of zero.

- If E is a *Baire space* (e.g. it is completely metrizable), then E is barrelled.
- The space $\mathcal{D}(\Omega)$ is barrelled.

Theorem (Locally-convex Banach-Steinhaus)

Let E, F be two LCTVSs over \mathbb{R} or \mathbb{C} , with E barrelled, and $T : I \rightarrow \mathcal{L}(E, F)$ a family of continuous linear maps.

Assume that, for any $x \in E$, the set $\{T_i(x)\}_{i \in I}$ is bounded in E . Then the family $(T_i)_i$ is **equicontinuous**.

Corollary

Consider a sequence $T : \mathbb{N} \rightarrow \mathcal{D}'(\Omega)$ of distributions. If, for all φ , $T_n(\varphi) \rightarrow S(\varphi)$, then the linear map S is in fact a distribution.

Definition (BarrelledSpace)

A \mathbb{K} -TVS E is **barrelled** if any **lower-semicontinuous seminorm** on E is in fact continuous.

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A \mathbb{K} -TVS E is **barrelled** if any **lower-semicontinuous seminorm** on E is in fact continuous.

Proposition (Continuity in polynormable spaces)

Let E, F be two polynormable TVSs. A linear map $T : E \rightarrow F$ is continuous if and only if, for any continuous seminorm q on F , $q \circ T$ is a continuous seminorm on E .

Definition (BarrelledSpace)

A \mathbb{K} -TVS E is (\mathbb{K} -)barrelled if any **lower-semicontinuous seminorm** on E is in fact continuous.

Proposition (Equicontinuity in polynormable spaces)

Let E, F be two polynormable TVSs. A **family** of linear map $T_i : E \rightarrow F$ is **equicontinuous** if and only if, for any continuous seminorm q on F , $\sup_i q \circ T_i$ is a well-defined continuous seminorm on E .

Theorem (`PolynormableSpace.banach_steinhaus`)

Let E, F be two polynormable TVSs, with E barrelled, and $T : I \rightarrow \mathcal{L}(E, F)$ a family of continuous linear maps.

Assume that, for any $x \in E$, the set $\{T_i(x)\}_{i \in I}$ is bounded in E . Then the family $(T_i)_i$ is equicontinuous.

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Proof. Let q be a continuous seminorm on F . By assumption, the seminorm $p : x \mapsto \sup_i q(T_i(x))$ is well defined. It is a supremum of continuous seminorms, hence it is lower-semicontinuous. Since E is barrelled, it is in fact continuous. □

Proposition (`BaireSpace.instBarrelledSpace`)

Let E be a polynormable space. If E is a Baire space, then it is barrelled.

Recall that a topological space is a **Baire space** if any countable union of closed sets with empty interior has empty interior

Bonus : Banach-Steinhaus theorem

Future work

Proof.



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This means that, for any y close enough to 0, we have $p(x + y) \leq N$.

But $p(y) \leq p(x) + p(x + y)$, so there is a neighborhood V of zero such that $\forall y \in V, p(y) \leq 2N$. □

Disclaimer

While all of this works for $\mathbb{K} = \mathbb{Q}_p$, this does **not** recover what people usually call the “non-archimedean Banach-Steinhaus theorem”.