Formalizing de Rham cohomology

Progress report

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Definition

Given a C^∞ real manifold M, let $\Omega^k(M)$ be the space of differential k-forms on M. Let $d_k : \Omega^k(M) \to \Omega^{k+1}(M)$ be the exterior derivative. Then $H^k(M) = \mathrm{Ker}(d_k) \big/ \mathrm{Im}(d_{k-1})$ is called *k-th de Rham cohomology group* of M.

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- Definition of a differential form.
- Definition of the exterior derivative.
- Proof of $d^2 = 0$.
- Definition of de Rham cohomologies.
- Basic properties.

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- The space of bundled r-smooth k-forms on a manifold.
- Transfer of properties from normed spaces to manifolds.
- Definition of de Rham cohomologies.
- Basic properties.

Continuous alternating maps

Differential forms on a normed vector space are defined as functions `ω : E → E [⋀^ι]→L[𝕜] F`, where `E [⋀^ι]→L[𝕜] F` is the space of *continuous alternating maps*.

```
structure ContinuousAlternatingMap (R M N ι : Type*) [Semiring R]
    [AddCommMonoid M] [Module R M] [TopologicalSpace M] [AddCommMonoid N]
    [Module R N] [TopologicalSpace N] where
  toFun : (ι → M) →
  /-- Additive in each argument. -/
  map_update_add' :
    ∀ [DecidableEq ι] (m : ι → M) (i : ι) (x y : M),
      toFun (update m i (x + y)) = toFun (update m i x) + toFun (update m i y)
  /-- Linear in each argument -/
  map_update_smul' :
    ∀ [DecidableEq ι] (m : ι → M) (i : ι) (c : R) (x : M),
      toFun (update m i (c • x)) = c • toFun (update m i x)
  /-- Antisymmetric (alternating) -/
  map_eq_zero_of_eq' : ∀ (v : ι → M) (i j : ι), v i = v j → i ≠ j → toFun v = 0
  /-- Continuous -/
  cont : Continuous toFun
```

Exterior derivative: definition

- We define the exterior derivative in two steps.

```
def ContinuousAlternatingMap.alternatizeUncurryFin
    (ω : E →L[𝕜] E [⋀^Fin n]→L[𝕜] F) :
    E [⋀^Fin (n + 1)]→L[𝕜] F where
  toFun := ∑ i : Fin (n + 1), (-1) ^ (i : ℕ) • f (v i) (Fin.removeNth i v)
  ...

def extDeriv (ω : E → E [⋀^Fin n]→L[𝕜] F) (x : E) :
    E [⋀^Fin (n + 1)]→L[𝕜] F :=
  .alternatizeUncurryFin (fderiv 𝕜 ω x)
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- This way we can separate calculus from algebra in the proofs.

Properties of the exterior derivative

```
theorem extDeriv_extDeriv (h : ContDiff 𝕜 r ω) (hr : minSmoothness 𝕜 2 ≤ r) :
    extDeriv (extDeriv ω) = 0 :=

theorem extDeriv_pullback (hω : DifferentiableAt 𝕜 ω (f x))
    (hf : ContDiffAt 𝕜 r f x) (hr : minSmoothness 𝕜 2 ≤ r) :
    extDeriv (fun x ↦ (ω (f x)).compContinuousLinearMap (fderiv 𝕜 f x)) x =
      (extDeriv ω (f x)).compContinuousLinearMap (fderiv 𝕜 f x) := by

theorem extDeriv_apply_vectorField {ω : E → E [⋀^Fin (n + 1)]→L[𝕜] F}
    {V : Fin (n + 2) → E → E}
    (hω : DifferentiableAt 𝕜 ω x) (hV : ∀ i, DifferentiableAt 𝕜 (V i) x) :
    extDeriv ω x (V · x) =
      (∑ i : Fin (n + 2), (-1) ^ (i : ℕ) •
        fderiv 𝕜 (fun x ↦ ω x (i.removeNth (V · x))) x (V i x)) -
        ∑ i : Fin (n + 1), ∑ j ≥ i,
          (-1) ^ (i + j : ℕ) •
            ω x (Matrix.vecCons (lieBracket 𝕜 (V i.castSucc) (V j.succ) x)
              (j.removeNth ◁ i.castSucc.removeNth (V · x))) := by
```

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- The current file closely follows the existing file about the bundle of continuous linear maps.

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- Link between naive cohomology definition and Mathlib homology framework.
- I have little time for this, collaborators are very welcome!