The Sobolev embedding theorem via Fourier analysis

Moritz Doll
University of Melbourne

11th May 2026

Motivation: the Schrödinger equation

Consider the initial value problem:

$$\begin{cases} (i\partial_t + \Delta)u(t,x) = 0\,, \\ \qquad\qquad u(0,x) = u_0(x)\,. \end{cases}$$

The equation models a free quantum particle. The quantity $|u(t,\cdot)|^2$ is the probability density at time $t \in \mathbb{R}$.

If u_0 is compactly supported, then $u(t)$ is smooth for all $t \neq 0$. Hence, IVP is not well-posed in C^k for any k.

Solving the Schrödinger equation using functional analysis

The Laplacian $\Delta : \mathcal{C}_c^\infty(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is an unbounded essentially self-adjoint operator on $L^2(\mathbb{R}^n)$.

The spectral theorem implies that

$$U(t) = e^{-it\Delta}$$

is a strongly continuous group of unitary operators and in particular, $U(t) : L^2 \to L^2$.

But what about higher (or lower) regularity?

Sobolev spaces via the spectral theorem

Using the spectral theorem again, we can define for any $s \in \mathbb{R}$,

$$(1-\Delta)^{s/2}$$

noting that $\sigma(-\Delta) \subset [0, \infty)$.

Sobolev spaces are the domain of this operator:

$$H^s(\mathbb{R}^n) = D((1-\Delta)^{s/2})\,.$$

Constant coefficient PDEs and Fourier transform

The Fourier transform is initially defined for integrable functions $f \in L^1$ via

$$(\mathcal{F}f)(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\xi} f(x)\, dx\,.$$

It follows that $\mathcal{F}f$ is continuous and vanishes at infinity (Riemann–Lebesgue).

Note that for $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, we have

$$\mathcal{F}(\partial_{x_j} f) = 2\pi i \xi_j \mathcal{F}(f)\,,$$
$$\mathcal{F}(2\pi i x_j f) = -\partial_{x_j} \mathcal{F}(f)\,.$$

In particular,

$$(1-\Delta)^{s/2} = \mathcal{F}^{-1}(1+|2\pi\xi|^2)^{s/2}\mathcal{F}\,.$$

Schwartz functions

Using the previous observation we build a space that incorporates both regularity and decay:

Definition

A smooth function $u : \mathbb{R}^n \to \mathbb{C}$ is called a *Schwartz function* if for all $\alpha, \beta \in \mathbb{N}_0^n$, there exist $C_{\alpha,\beta} > 0$ such that

$$\left| x^\alpha \partial_x^\beta u(x) \right| \leq C_{\alpha,\beta}$$

for all $x \in \mathbb{R}^n$. We then write $u \in \mathcal{S}(\mathbb{R}^n)$.

Facts:

- $\mathcal{S}(\mathbb{R}^n)$ is a topological vector space.
- $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is a continuous linear equivalence.

Schwartz functions in mathlib

Let E, F be normed $\mathbb{R}$-vector spaces. Then we define

```
structure SchwartzMap (E F) where
  toFun : E → F
  smooth' : ContDiff ℝ ∞ toFun
  decay' (k n : ℕ) : ∃ (C : ℝ), ∀ (x : E),
    ‖x‖ ^ k * ‖iteratedFDeriv ℝ n toFun x‖ ≤ C
```

Note: no basis in E required, vector-valued Schwartz functions without any changes.

Fourier transform:

```
def fourierTransformCLM : 𝓢(E, F) →L[𝕂] 𝓢(E, F) :=
  mkCLM ((𝓕 : (E → F) → (E → F)) ·) ?_ ?_ ?_ ?_
```

Use notation typeclasses for $\mathcal{F}$ (and Δ and ∂_v)

Tempered distributions

The topological dual of the Schwartz space is called the space of *tempered distributions*,

```
abbrev 𝒮'(E, F) := 𝒮(E, ℂ) →L[ℂ] F
```

There is a question about the topology on this space, we chose the topology of pointwise convergence.

Fourier transform $\mathcal{F}$:

```
instance : FourierTransform 𝒮'(E, F) 𝒮'(E, F) where
  fourier := PointwiseConvergenceCLM.precomp F
    (fourierCLM ℂ 𝒮(E, ℂ))
```

Directional derivative ∂_v:

```
instance : LineDeriv E 𝒮'(E, F) 𝒮'(E, F) where
  lineDerivOp v := PointwiseConvergenceCLM.precomp F
    (−lineDerivOpCLM ℂ 𝒮(E, ℂ) v)
```

Support of distributions

A tempered distribution $u \in \mathcal{S}'(E, F)$ *vanishes on a set* $U \subset E$ if for all $f \in \mathcal{S}$ with $\operatorname{supp} f \subset U$ it follows that $u(f) = 0$.

A point $x \in E$ is *not* in the support of u if there exists a neighborhood of x on which u vanishes.

Support of distributions

A tempered distribution $u \in \mathcal{S}'(E, F)$ *vanishes on a set* $U \subset E$ if for all $f \in \mathcal{S}$ with $\operatorname{supp} f \subset U$ it follows that $u(f) = 0$.

A point $x \in E$ is *not* in the support of u if there exists a neighborhood of x on which u vanishes.

```
variable [FunLike F α β] [TopologicalSpace α] [Zero V]
def IsVanishingOn (f : F → V) (s : Set α) : Prop :=
  ∀ (u : F), tsupport u ⊆ s → f u = 0

def dsupport (f : F → V) : Set α :=
  ⋃₀ { s | IsVanishingOn f sᶜ ∧ IsClosed s}

theorem notMem_dsupport_iff_eventually {x : α} :
    x ∉ dsupport f ↔ ∀ᶠ u in (𝓝 x).smallSets, IsVanishingOn f u
```

Upshot: can unify support for different kinds of distributions

Bessel potential spaces

For $s \in \mathbb{R}$, let $\mathcal{J}^s : \mathcal{S}' \to \mathcal{S}'$ be defined via

$$\mathcal{J}^s := \mathcal{F}^{-1}(1 + |\xi|^2)^{s/2}\mathcal{F}$$

denote the *Bessel potential operator*. Fourier-based Sobolev spaces:

$$H^{s,p}(\mathbb{R}^n) := \{u \in \mathcal{S}' \colon \mathcal{J}^s u \in L^p\}.$$

Issues: incompatible types $\mathcal{S}'$ and L^p, dependent types, ..

Bessel potential spaces formalized

- Predicate

```
def MemSobolev (s : ℝ) (p : ℝ≥0∞) [Fact (1 ≤ p)] (f : 𝓢'(E, F)) :=
  ∃ (f' : Lp F p (volume : Measure E)), besselPotential E F s f = f'
```

- Structure

```
structure Sobolev (s : ℝ) (p : ℝ≥0∞) [Fact (1 ≤ p)] where
  toDistr : 𝓢'(E, F)
  sobFn : Lp F p (volume : Measure E)
  bessel_toDistr_eq_sobFn :
    besselPotential E F s toDistr = sobFn
```

Bessel potential spaces formalized

- Predicate

```
def MemSobolev (s : ℝ) (p : ℝ≥0∞) [Fact (1 ≤ p)] (f : 𝒮′(E, F)) :=
  ∃ (f′ : Lp F p (volume : Measure E)), besselPotential E F s f = f′
```

- Structure

```
structure Sobolev (s : ℝ) (p : ℝ≥0∞) [Fact (1 ≤ p)] where
  toDistr : 𝒮′(E, F)
  sobFn : Lp F p (volume : Measure E)
  bessel_toDistr_eq_sobFn :
    besselPotential E F s toDistr = sobFn
```

- ~~Type alias~~

```
    def Sobolev′ (_s : ℝ) (p : ℝ≥0∞) := Lp F p (volume : Measure E)
```

Elementary properties

- $H^{s,p}$ is a Banach space, $H^{s,2}$ is a Hilbert space
- directional derivatives and Laplacian

```
theorem lineDerivOp_toDistr (v : E) {s : ℝ} (f : Sobolev E F s 2) :
  (∂_{v} f).toDistr = ∂_{v} f.toDistr := by sorry

theorem laplacian_toDistr {s : ℝ} (f : Sobolev E F s 2) :
  (Δ f).toDistr = Δ f.toDistr := by sorry
```

- delta distribution

```
theorem delta_toDistr (hs : 2 * s < -finrank ℝ E) :
  (delta E s).toDistr = TemperedDistribution.delta (0 : E) := by
 sorry
```

Localization

For $\Omega \subset E$ open, define

$$H^s(\Omega) := \{u \in \mathcal{D}'(\Omega, F) \colon \exists U \in H^s(\mathbb{R}^n), U|_\Omega = u\}.$$

```
structure SobolevRestrict (Ω : Opens E) (s : ℝ) where
  toFun : 𝒟'(Ω, F)
  exists_memSobolev : ∃ u : 𝒮'(E, F),
    toFun = u.restrict Ω ∧ MemSobolev s 2 u
```

Here, u is not unique, so it can't be part of the structure!

The space $H^s(\Omega)$ is a Hilbert space by identifying it with a closed subspace of $H^s(E, F)$.

Sobolev embedding

Theorem

Let $s > n/2$, then $u \in H^s$ implies $u \in \dot{C}$ and the embedding is continuous.

```
def toZeroAtInfty (s : ℝ) : Sobolev E F s 2 →L[ℂ] C₀(E, F) := sorry

theorem toZeroAtInfty_apply_toTemperedDistribution
  (hs : finrank ℝ E < 2 * s) (f : Sobolev E F s 2) :
  (toZeroAtInfty E F s f).toTemperedDistribution = f.toDistr := by
 sorry
```

Trace theorem

Theorem

For $s > 1/2$ the restriction $\gamma : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^{n-1})$ extends to a bounded linear operator

$$\gamma : H^s(\mathbb{R}^n) \to H^{s-1/2}(\mathbb{R}^{n-1}) .$$

Use abstract extension theorems together with the estimate

$$\|\gamma(u)\|_{H^{s-1/2}} \leq C \|u\|_{H^s}$$

for $u \in \mathcal{S}$.

```
def restrictFst (a : E′) :
  Sobolev (WithLp 2 (E × E′)) F s 2 →L[ℂ]
  Sobolev E F (s - (finrank ℝ E′)/2) 2
```

Conclusions

The good:

- `fun_prop` and `positivity` save a lot of time
- The combination of `calc` and `grw`/`gcongr` makes calculations more robust and readable

Conclusions

The good:

- `fun_prop` and `positivity` save a lot of time
- The combination of `calc` and `grw`/`gcongr` makes calculations more robust and readable

The bad:

- Somewhat repetitive calculations for showing that certain operators are bounded on Schwartz functions
  (e.g., calculating iterated derivatives of products)
- Coercions between $\mathcal{S}$, L^p, H^s, and $\mathcal{S}'$
- Lack of `simp` normal forms for composition of operators

Conclusions

The good:

- `fun_prop` and `positivity` save a lot of time
- The combination of `calc` and `grw`/`gcongr` makes calculations more robust and readable

The bad:

- Somewhat repetitive calculations for showing that certain operators are bounded on Schwartz functions
  (e.g., calculating iterated derivatives of products)
- Coercions between $\mathcal{S}$, L^p, H^s, and $\mathcal{S}'$
- Lack of `simp` normal forms for composition of operators

The ugly:

- Coercions between $\mathbb{R}$ and $\mathbb{C}$
- Simple calculations can still be tedious (real vs natural powers)

Thank you for your attention!

Code can be found at `github.com/mcdoll/DirichletProblem`