

# Towards the formalisation of PDEs in analysis

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# Outline of today's talk

- 1 Challenges formalising analysis
- 2 ODEs and PDEs in mathlib
- 3 A glimpse of PDEs
- 4 Sobolev spaces

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Sociological reason: fewer experts means less material

# Challenges formalising analysis

- techniques, not just theorems → several versions; abstraction can help
- invisible mathematics: different kinds of integrals (e.g. Lebesgue vs. Bochner)
- invisible mathematics: coercions and working with real numbers
- more domain-specific tactics needed

# Three kinds of real numbers under the sky

`Real`: real numbers

good tactic support (`ring`, `linarith`, `field_simp`, `norm_num`)

`NNReal`: non-negative real numbers — encodes positivity condition

worse tactic support: `ring` and `norm_num` work less well

`ENNReal`: extended non-negative real numbers

natural for taking Lebesgue integrals or suprema; sometimes necessary  
even worse tactic support (no `linarith` or `field_simp`, `ring` and  
`norm_num` worse)

Conversions are painful: `norm_cast` helps for coercions (e.g. to `Real`);  
other direction requires additional side conditions

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Project ideas for this week: two tactics for mortal men

- 1 numerical expressions in `ENNRReal`, e.g.  
 $\frac{3}{5} * \frac{5}{3} = 1, 2 * \infty + 3 = \infty, \frac{1}{2} + \frac{1}{4} + \frac{1}{2} = 1, 8^{-1} * (\frac{1}{4} + \frac{1}{4}) = 16^{-1}$
- 2 take an `ENNRReal` calculation (with variables) and lift to  $\mathbb{R}_{\geq 0}$  or  $\mathbb{R}$

# ODEs in mathlib

Grönwall's lemma (in forward time)

Picard–Lindelöf theorem: existence and uniqueness of solutions to an initial value problem (under Lipschitz conditions)

smooth dependence on coefficients and initial conditions: in progress

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manifold theory (integral curves, flows of vector fields)

mostly done/under review

global existence (under suitable conditions) in progress

see Winston Yin's talk

# ODEs in mathlib: what's missing

clean up and merge PRs on manifold theory;  
complete smooth dependence on initial conditions

Cauchy–Kovalevskaya problem (existence and uniqueness of analytic solutions to an analytic PDE)

study concrete ODEs

numerical analysis and computations

## PDEs

$$F(D^k u(x), D^{k-1} u(x), \dots, u(x), x) = 0$$

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No.

But it could, and we are possibly not far from it.

# What we have

No PDE file in Mathlib, but...

- Differential calculus: derivatives, Hessian, Taylor series,  $C^n$ , etc.
- Probably all you need to state a PDE
- Integrals
- Sobolev inequality
- classical and tempered distributions

No Sobolev spaces (it's being worked on).

Easily doable: verify the fundamental solution of the Heat equation

Verifying the wave equation needs some form of Stokes' theorem

# PDE theory: what do we want?

PDEs are a bit like in Tolstoi's *Anna Karenina*

*All happy families are alike; each unhappy family is unhappy in its own way.*

There is no general theory of solvability of all PDEs: instead focus on families of PDEs and particular PDEs

Good first target: elliptic PDEs and their regularity theory

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well-posedness depends on your notion of solutions

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**weak solution**: relaxed regularity requirements;  
PDE is re-interpreted using weak derivatives

regularity estimates: must weak solutions in fact be e.g. classical solutions?

goal: find a suitable function space to study weak solutions

# An idea of elliptic regularity estimates

elliptic PDEs include the Laplace, Poisson and Cauchy–Riemann equation

Laplace equation  $\Delta u = \sum_i \frac{\partial^2 u}{\partial x_i^2} = 0$  for  $u: E \rightarrow \mathbb{R}$  with  $\dim E = n$

Poisson equation  $\Delta u = f$

**Fact:** weak solutions to  $\Delta u = 0$  (*harmonic functions*) are analytic!

**Fact:** if  $f \in C^k$  ( $k \geq 2$ ), then  $\Delta u = f$  implies  $u \in C^k$

A similar principle applies to e.g. the Cauchy–Riemann equation.

# Case study: elliptic regularity in differential geometry and global analysis

**holomorphic curves** are a major technical tool to study symplectic manifolds

smooth map  $u: \Sigma \rightarrow M$  with  $du$  complex-linear, where  $(M, \omega)$  symplectic manifold,  $\Sigma$  Riemann surface,  $J$  (almost) complex structure on  $M$

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**Trouble:**  $C^\infty(\Sigma, M)$  is only a Fréchet manifold, not Banach

**Solution:** consider weak solutions in  $C^k(\Sigma, M)$  or Sobolev spaces  $W^{k,p}(\Sigma, M)$  (for  $kp > 2$ )

Elliptic regularity proves the solution sets are the same.

# Sobolev spaces: a suitable space for studying weak solutions

$X$  measure space,  $E$  finite-dimensional real normed space,  $1 \leq p \leq \infty$

## Lebesgue space

$L^p(X) := \{f: X \rightarrow E \text{ measurable} \mid \int_X |f(x)|^p dx < \infty\}$  / a.e. equivalence  
is Banach, reflexive, dual space  $(L^p(X))^* \cong L^q(X)$  where  $\frac{1}{p} + \frac{1}{q} = 1$   
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$W^{k,p}(X) := \{f \in L^p(X) \mid f \text{ has } k \text{ weak derivatives, each in } L^p(X)\}$

# Two kinds of Sobolev spaces

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## Sobolev–Slobodeckij spaces

$W^{s,p}(\Omega) := \{f \in W^{\lfloor s \rfloor, p}(\Omega) \mid \sup_{|\alpha|=\lfloor s \rfloor} \|D^{\alpha} f\|_{s-\lfloor s \rfloor, p, \Omega} < \infty\}$ , where  $\|\cdot\|_{\theta, p, \Omega}$  is the **Slobodeckij seminorm** (analogous to Hölder seminorm)

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**Fact:** for  $1 < p < \infty$ ,  $k \in \mathbb{N}$

$$W^{k,p}(\mathbb{R}^n) = \{f: \mathbb{R}^n \rightarrow E \mid \mathcal{F}^{-1} \left[ (1 + |\xi|^2)^{k/2} \mathcal{F}f \right] \in L^p(\Omega)\}$$

**Bessel potential spaces** are defined using Fourier multipliers

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**Fact.** If  $\Omega$  has Lipschitz boundary,  $W^{s,2}(\Omega) = H^{s,2}(\Omega)$  (otherwise only  $\supset$ )

# Classical distributions

space of test functions

$$C_0^\infty(\Omega) = \{f \in C^\infty(\Omega, \mathbb{R}) \mid f|_\omega \text{ is smooth, } \text{supp } f \text{ is compact}\}$$

distributions allow integrating test functions:  $\mathcal{D}'(\Omega) = (C_0^\infty(\Omega))^*$

Examples:

$f \in L^1_{\text{loc}}(\Omega)$  induces  $[f] \in \mathcal{D}'(\Omega, F)$  by  $\phi \mapsto \int_\Omega f(x)\phi(x) dx$  (regular)

Dirac delta distribution  $\delta: \phi \mapsto \phi(0)$  (not regular)

tempered distributions are similar: test against all Schwartz functions

vector-valued classical distributions: details omitted (see Dedecker's talk)

# Weak derivatives

$X$  measure space,  $\Omega \subset X$ ,  $E$  real normed space;  $\alpha$  multi-index

If  $f$  has classical derivative  $f'$ , then  $\int f D^\alpha \phi = (-1)^{|\alpha|} \int D^\alpha f \phi$  for all  $f \in C_0^\infty(\Omega)$

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The above formula makes sense for distributions also:  $T \in \mathcal{D}'(\Omega)$  has weak derivative  $D^\alpha T$  if  $T(D^\alpha \phi) = (-1)^{|\alpha|} (D^\alpha T)\phi$  for all  $\phi \in C_0^\infty(\Omega)$

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Sobolev space:  $f \in L^p(\Omega)$  lies in  $W^{k,p}(\Omega)$  if each weak derivative  $D^\alpha f$  of  $[f]$  on  $\Omega$  is regular and induced by an  $L^p$  function

# Properties of Sobolev spaces

$W^{k,p}(\Omega)$  is Banach for  $p \geq 1$ ;  $W^{0,p}(\Omega) = L^p(\Omega)$

$C_0^\infty(\Omega) \subset W^{k,p}(\Omega)$  is dense

natural inclusions  $W^{l,p} \hookrightarrow W^{k,p}$  for  $k \leq l$

intuition:  $f \in W^{k,p}$  has  $k - \frac{n}{p}$  continuous derivatives

Sobolev embedding theorem: if  $k \geq m$ ,  $p \leq q$  and  $k - \frac{n}{p} \geq m - \frac{n}{q}$ , there is a continuous inclusion  $W^{k,p}(\Omega) \hookrightarrow W^{m,q}(\Omega)$

Rellich–Kondrachov compactness: if  $kp > n$ ,  $k - \frac{n}{p} < 1$  and  $q \in [0, \infty)$ , the inclusion  $W^{k+d,p}(\Omega) \hookrightarrow W^{k,p}(\Omega)$  is compact

Banach algebra property:  $kp > n$  and e.g.  $\Omega$  open bounded with smooth boundary, then  $\|uv\|_{W^{k,p}} \leq \|u\|_{W^{k,p}} \|v\|_{W^{k,p}}$

# Formalisation of Sobolev spaces: general considerations

Joint with Filippo Nuccio and Floris van Doorn; building on formalisation of distributions

Lesson from Carleson project: prefer working with unbundled unquotiented functions `MemSobolev f` over `f : SobolevSpace` since

- $(f + g)(x) = f(x) + g(x)$  for *all*  $x$
- the former is more general (enormed spaces)

# Overview of formalisation status

- distributions prerequisites virtually merged now
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- basic definitions (weak derivatives, `MemSobolev`, Sobolev space)
- Sobolev norm; `MemSobolev` iff finite Sobolev norm
- monotonicity in  $k$  and (if  $\Omega$  is bounded) in 'p'
- relation  $W^{0,p}$  and  $L^p$ ; relation  $W^{k+1,p}$  and  $W^{k,p}$
- $C^k$  functions are Sobolev
- next steps:  
completeness, separability, uniform convexity  
smooth functions are dense  
deduce the Sobolev embedding theorem

# Formalisation considerations: Sobolev norm

Sobolev norm: want `MemSobolev` iff finite Sobolev norm

can we make this true for interesting reasons?

$\|f\|_{W^{k+1,p}} := \|f\|_{L^p} + \|Df\|_{W^{k,p}}$  does **not** work because of junk values

note that `MemLp f` is not equivalent to finite `eLpNorm` (fix in progress)

# Formalisation considerations: weak derivatives

`weakDeriv`: weak derivative of a function (as a distribution); junk value 0 if not locally integrable

`HasWeakDeriv` in analogy to `hasFDeriv`  
structure with local integrability condition included

`wderiv`: weak derivative as a function; 0 if not exists; 0 if one possible value

# Formalisation considerations: definition of Sobolev spaces

setting so far:  $W^{k,p}(\Omega, F)$  for  $E, F$  finite-dimensional real normed spaces,  
 $\Omega \subset E$  open,  $k \in \mathbb{N}$

different definitions have different trade-offs, one example:  
inductive definition for  $k = 0, 1, \geq 2$  — runs into universe issues

$E \rightarrow_{L[\mathbb{R}]} F$  has maximal universe of  $E$  and  $F$ ,  
while 0-ary function has universe of  $F$   
previously, `fderiv` had the same issue,  
composition of  $C^n$  functions restricted to same universe as first step

# Summary

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- 2 Elliptic PDEs and their regularity are a useful target
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Thanks for listening! Any questions?